International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:6, 2011

Projective Synchronization of a Class of Fractional-Order Chaotic Systems

Zahra Yaghoubi, Nooshin Bigdeli, Karim Afshar

***Abstract*—**This paper at first presents approximate analytical solutions for systems of fractional differential equations using the differential transform method. The application of differential transform method, developed for differential equations of integer order, is extended to derive approximate analytical solutions of systems of fractional differential equations. The solutions of our model equations are calculated in the form of convergent series with easily computable components. After that a drive-response synchronization method with linear output error feedback is presented for "generalized projective synchronization" for a class of fractional-order chaotic systems via a scalar transmitted signal. Genesio_Tesi and Duffing systems are used to illustrate the effectiveness of the proposed synchronization method.

***Keywords*—**Generalized projective synchronization; Fractional-order; Chaos; Caputo derivative; Differential transform method

I. Introduction

SYNCHRONIZATION of chaotic systems [1-4] has been focus of attention in recent literature owing to its applications in secure communications of analog and digital signals [5]. Fractional calculus deals with derivatives and integration of arbitrary order [6] and has deep and natural connections with many fields of applied mathematics, engineering and physics. Fractional calculus has wide range of applications in control theory [7], turbulence, electromagnetism, signal processing [8-9] and bioengineering.

In this paper, a synchronization called "generalized projective synchronization" is used and a drive-response synchronization method is developed, this method is used to "generalized projective synchronization" of a class of fractional-order chaotic systems via a scalar transmitted signal [10]. A new application of the differential transform method [11] is also introduced to provide approximate solutions for the system of fractional differential equations. There are several definitions of a fractional derivative of order α>0. The two most commonly used definitions are the Riemann–Liouville and Caputo. Each definition uses Riemann–Liouville fractional integration and derivatives of whole order. The differential transform method was first applied in the engineering domain in. In general, the differential transform method is applied to the solution of electric circuit problems. The differential transform method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial.

The remainder of the paper is organized as follows: section II provides a brief review of the fractional order derivative and the numerical algorithm of fractional-order differential equation. In section III, fractional differential equations using differential transform method and numerical examples are described. In section IV, the generalized projective synchronization is introduced. In section V, the synchronization criterion is given. In section VI, proposed method is applied to synchronize fractional-order Genesio_Tesi and Duffing chaotic systems. Finally, Section VII yields the conclusions.

Z. Yaghoubi is M. Sc. Student of Electrical Engineering Department, Imam Khomeini International University, Qazvin, Iran, (e-mail: z_yaghoubi@ikiu.ac.ir).

Nooshin Bigdeli, Electrical Engineering Department, Imam Khomeini International University, Qazvin, Iran (phone: +98 281 8371164, fax: +98 281 3787777, e-mail: bigdeli@ikiu.ac.ir)

Karim Afshar is with Electrical Engineering Department, Imam Khomeini International University, Qazvin, Iran (e-mail: afshar@ikiu.ac.ir).

II. Mathematical Background

Consider the system of fractional differential equations:

$$
\begin{aligned}
D^{\alpha_1}_{*}x_1(t) &= f_1(t, x_1, \dots, x_n) \\
D^{\alpha_2}_{*}x_2(t) &= f_2(t, x_1, \dots, x_n) \\
&\vdots \\
D^{\alpha_n}_{*}x_n(t) &= f_n(t, x_1, \dots, x_n)
\end{aligned}
\tag{1}
$$

Where, $D^{\alpha_i}_{*}$ is the derivative of xi of order α_1 in the sense of Caputo and $0 < \alpha_1 \leq 1$, subject to the initial conditions

$$x_1(0) = c_1, x_2(0) = c_2, \dots, x_n(0) = c_n. \tag{2}$$

The difference between the two definitions is in the order of evaluation. Riemann–Liouville fractional integration of order α is defined as:

$$J^{\alpha}_{*}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x_t}(x-t)^{\alpha-1}f(t)dt\,, \quad \alpha > 0, x > 0 \tag{3}$$

Where, x_0 and x_t are the initial and final state vectors, and:

$$\Gamma(z) = \int_0^{\infty} t^{z-1}\ e^{-t}\ dt \tag{4}$$

And the Riemann–Liouville and Caputo fractional derivatives of order α, are defined respectively, as:

$$D_{x0}^{\alpha} f(x) = \frac{d^m}{dx^m}[J^{m-\alpha} f(x)] \tag{5}$$

$$D_{*x0}^{\alpha} f(x) = J^{m-\alpha}[\frac{d^m}{dx^m} f(x)] \tag{6}$$

Where $m-1 < \alpha \le m$ and $m \in N$.

III. Fractional differential transform method

In this section, we introduce the fractional differential transform method used in this paper to obtain approximate analytical solutions for the system of fractional differential equations. This method has been developed as follows:

$$D_{x0}^{q} f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m}\left[\int_{x0}^{x} \frac{f(t)}{(x-t)^{1+q-m}} dt\right] \tag{7}$$

$$f(x) = \sum_{k=0}^{\infty} F(k)(-x_0)^{k/\alpha} \tag{8}$$

Where α is the order of fraction and F(k) is the fractional differential transform of f (x).

Theorem1. If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.

Theorem2. If $f(x) = g(x)\ h(x)$, then $F(k) = \sum_{l=0}^{k} G(l)\ H(k-l)$

Theorem3. If $f(x) = g_1(x)g_2(x) \ldots g_{n-1}(x)g_n(x)$, then

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)\, G_2(k_2-k_1) \cdots G_{n-1}(k_{n-1}-k_{n-2}) G_n(k-k_{n-1}) \tag{9}$$

Theorem4. If $f(x) = D_{x0}^{q}[g(x)]$, then $F(k) = \frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)} G(k+\alpha q)$

Proofs of theorems were brought in [9].

IV. Definition of generalized projective synchronization

Let us at first introduce the projective synchronization and the generalized synchronization at first.

For two identical chaotic systems which are coupled through the variable z in the form:

$$\begin{cases} D_*^{\alpha} x_m = A(z) x_m \\ D_*^{\alpha} x_s = A(z) x_s \\ D_*^{\alpha} z = A(x_m, z) \end{cases} \tag{10}$$

$$\lim_{t\to\infty} \|\sigma x_m - x_s\| = 0$$

x_m is synchronized to x_s up to a scaling factor σ, via "projective synchronization". Besides, for the following two chaotic systems in unidirectional coupling form:

$$\begin{cases} D_*^{\alpha} x = f(x) \\ D_*^{\beta} y = g(y, u_{\mu}(x)) \end{cases} \tag{11}$$

If μ = 0, then y has no relation to x. When μ = 0, the two systems are said to be "generalized synchronization".

Consider the following chaotic systems:

$$\begin{cases} D_*^{\alpha} x = f(x) \\ D_*^{\alpha} y = g(y, h(x,y)) \end{cases} \tag{12}$$

If there exists a constant $\sigma \in R - \{0\}$ such that:

$$\lim_{t\to\infty} \|\sigma x - y\| = 0 \tag{13}$$

Then we regard the two systems are synchronized. Such synchronization is called "generalized projective synchronization".

V. The Proposed Method: Generalized Projective Synchronization of A Class of Fractional-Order Chaotic Systems

Assume the fractional order chaotic drive systems under study can be written as:

$$D_*^{\alpha} x = f(x) = Ax + BF(cx) + E \tag{14}$$

Since the adoption of a scalar transmitted signal is suitable and expected for synchronization and secure communication, we choose the output of the system like:

$$y = Cx \tag{15}$$

and use the scalar signal to drive the fractional-order response system which is constructed in the form:

$$D_*^{\alpha} \tilde{x} = A\tilde{x} + \sigma(BF(cx) + E) + K(\sigma y - \tilde{y}) \tag{16}$$

Where $\tilde{x} \in Rn$, $\tilde{y} = C\tilde{x}$, and σ is the synchronization scaling factor, $K \in Rn$ is a feedback gain matrix to be decided such that the synchronization error $\|\sigma x - \tilde{x}\|_2 \to 0$ as $t\to\infty$. Then, the error system can be expressed as:

$$D_*^{\alpha} e = (A - KC)e \tag{17}$$

$$D_*^{\alpha} \varepsilon = (A^T - C^T K^T)\varepsilon = A^T \varepsilon + C^T u \tag{18}$$

Where the input $u = -k^T \varepsilon$ of system (AT, CT) acts as a state feedback.

Lemma1. System $D_*^{\alpha} x = Ax$ is:

- Asymptotically stable if and only if $|\arg(\lambda i(A))| > \alpha\pi/2$, i = 1, 2, . . . , n, where arg(λi(A))

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:6, 2011

denotes the argument of the Eigen value λi of A. In this case, the components of the state decay towards 0 like t−α;

- Stable if and only if either it is asymptotically stable or those critical Eigen values which satisfy |arg(λi(A))| = απ/2 have geometric multiplicity one.

VI. SIMULATIONS

In this section, the proposed method is applied to synchronize fractional-order Genesio_Tesi and Duffing chaotic systems.

The fractional-order Genesio_Tesi chaotic systems can be expressed in this form:

$$\begin{pmatrix} D^{q1}_{*}x_1 \\ D^{q2}_{*}x_2 \\ D^{q3}_{*}x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.1 & -1.1 & -0.45 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_1^2 \tag{19}$$

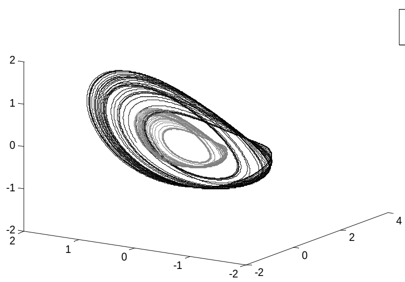
$$y = C(x_1, x_2, x_3)^T = x_1$$

q= [1, 1, 0.95].

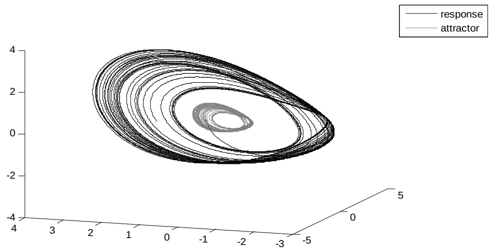
$$\begin{pmatrix} D^{q1}_{*}\tilde{x}_1 \\ D^{q2}_{*}\tilde{x}_2 \\ D^{q3}_{*}\tilde{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.1 & -1.1 & -0.45 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}$$

$$+\sigma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_1^2 + k(\sigma y - \tilde{y}) \tag{20}$$

We can set the Eigen values λi =−1 of A−KC with K= (2.55, 0.7525,-3.244). The phase diagrams of (19) and (20) are plotted together in Fig. 1(a) and (b) with scaling factor σ=2 and 4. The curves of synchronization errors are shown in Fig. 2 and 3 with σ =4, which indicate that the chaos synchronization between (19) and (20) is achieved.



(a)



(b)

Fig. 1 The attractors of drive system (19) and response system (20) with q = (1, 1, 0.95) (a) σ = 2 (b) σ = 4

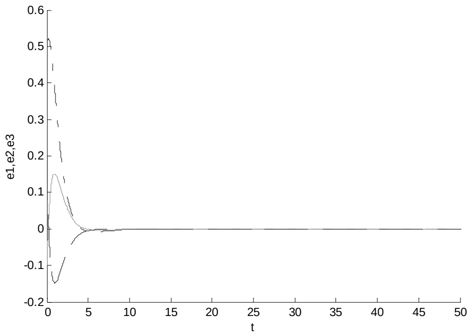


Fig.2 Synchronization errors of drive system (19) and response system (20)

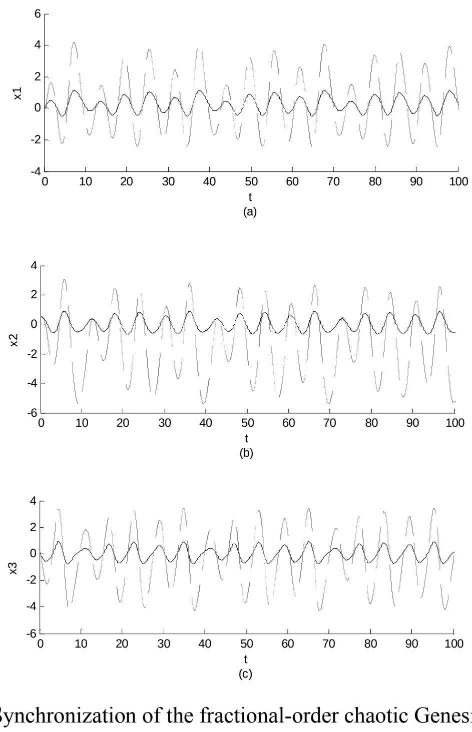


Fig.3 Synchronization of the fractional-order chaotic Genesio_Tesi systems (19) and (20) with

K= (2.55, 0.7525,-3.244),$\sigma = 4$. (a) $x1, \tilde{x}1$; (b) $x2, \tilde{x}2$; (c) $x3, \tilde{x}3$.

The fractional-order Duffing chaotic system, the equation will be defined as follows:

$$\begin{pmatrix} D_*^{q1} x_1 \\ D_*^{q2} x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -0.15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} x_1^3 + \begin{pmatrix} 0 \\ 0.3 \end{pmatrix} \cos(t) \tag{21}$$

$$y = C(x_1, x_2)^T = x_1$$

q=[0.9,1].

$$\begin{pmatrix} D_*^{q1} \tilde{x}_1 \\ D_*^{q2} \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -0.15 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \sigma[\begin{pmatrix} 0 \\ -1 \end{pmatrix} x_1^3 + \begin{pmatrix} 0 \\ 0.3 \end{pmatrix} \cos(t)] + k(\sigma y - \tilde{y}) \tag{22}$$

We can set the Eigen values λi =−1 of A−KC with K= (1.85, 1.7225). The phase diagrams of (21) and (22) are plotted together in Fig.4 (a) and (b) with scaling factor σ=2 and 4. The curves of synchronization errors are shown in Fig. 5 and 6 with σ =4, which indicate that the chaos synchronization between (21) and (22) is achieved.

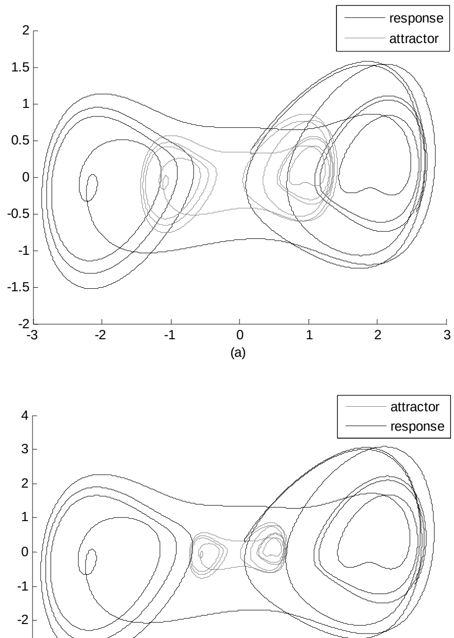


Fig.4 The attractors of drive system (21) and response system (22) with q = (0.9,1) (a) σ = 2 (b) σ = 4

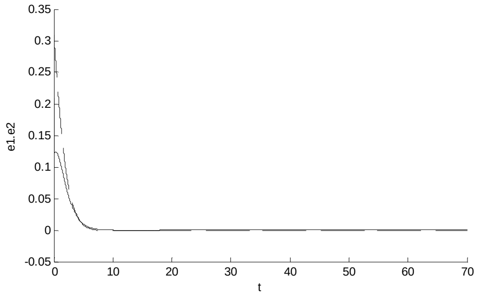


Fig.5 Synchronization errors of drive system (21) and response system (22)

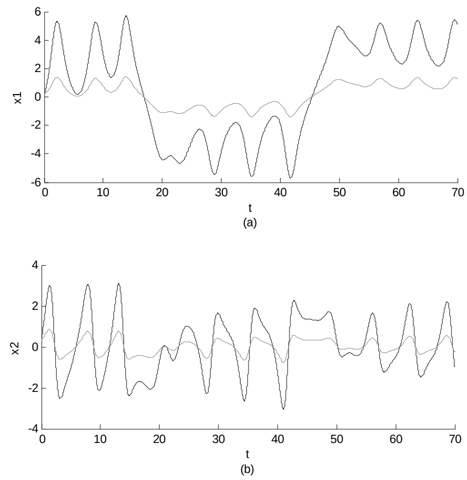


Fig.6 Synchronization of the fractional-order chaotic Duffing systems (21) and (22) with K= (1.85, 1.7225), $\sigma = 4$. (a) $x1, \tilde{x}1$; (b) $x2, \tilde{x}2$.

VII. Conclusions

In the present work we demonstrate that fractional order Genesio_Tesi and Duffing chaotic systems can be synchronized using a scalar transmitted signal It is further observed that the synchronization starts earlier for larger values of fractional order q.

References

[1] Pecora LM, Carroll TL. Synchronization in chaotic systems. Phys Rev Lett 1990;64(8):821.
[2] Pecora LM, Carroll TL. Driving systems with chaotic signals. Phys Rev A 1991;44:2374.
[3] Kocarev L, Parlitz U. General approach for chaotic synchronization with applications to communication. Phys Rev Lett 1995;74:5028.
[4] Carroll TL, Heagy JF, Pecora LM. Transforming signals with chaotic synchronization. Phys Rev E 1996;54(5):4676.
[5] Hilfer R. Applications of fractional calculus in physics. USA: World Scientific; 2001.
[6] Podlubny I. Fractional differential equations. San Diego: Academic Press; 1999.
[7] Sabatier J, Poullain S, Latteux P, Thomas J, Oustaloup A. Robust speed control of a low damped electromechanical system based on CRONE control: application to a four mass experimental test bench. Nonlinear Dyn 2004;38:383–400
[8] Anastasio TJ. The fractional-order dynamics of brainstem vestibulo-oculomotor neurons. Biol Cybern 1994;72:69–79.

[9] Ortigueira MD, Machado JAT. Fractional calculus applications in signals and systems. Signal Process 2006;86(10):2503–4.
[10] G.Peng , Y. Jiang. Generalized projective synchronization of a classof fractional-order chaotic systems via a scalar transmitted signal. 372 (2008) 3963–3970
[11] Arikoglu, I. Ozkol, Solution of fractional differential equations by using differential.