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Multiple positive periodic solutions to a periodic predator-prey-chain model with harvesting terms

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Abstract—In this paper, a class of predator-prey-chain model with harvesting terms are studied. By using Mawhin's continuation theorem of coincidence degree theory and some skills of inequalities, some sufficient conditions are established for the existence of eight positive periodic solutions. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

Keywords—Positive periodic solutions; Predator-prey-chain model; Coincidence degree; Harvesting term.

I. Introduction

PREDATOR-PREY phenomena occur commonly in ecological systems, and they are always interesting topics of population dynamics. For autonomous predator-prey systems, i.e. all coefficients being constants, we usually pay much attention to the existence and stability of their equilibria, especially positive equilibria; but we investigate the existence and stability of periodic solutions for non-autonomous systems [1-3], whose coefficients are time dependent. When the seasonal effects, food supply, mating habits, etc., are considered, the non-autonomous systems are necessary.

In recently years, the existence of periodic solutions in biological models has been widely investigated by many researchers (see[4-6]). Models with harvesting terms are often considered(see[5-6]). Generally, the model with harvesting terms is described as follows:

$$\begin{cases} \dot{x}_1 = x_1 f(x_1, x_2, x_3) - h, \\ \dot{x}_2 = x_2 g(x_1, x_2, x_3) - k, \\ \dot{x}_3 = x_3 h(x_1, x_2, x_3) - l, \end{cases}$$

where x_1, x_2 and x_3 are functions of three species, respectively; h, k and l are harvesting terms standing for the harvest rate. Considering the inclusion of the effect of changing environment, Dong et al.(see[4]) considered the following model of ordinary differential equations with predator-prey-chain system impulsive perturbation:

$$\begin{cases} \dot{N}_1(t) = N_1(t)\big(b_1(t) - a_{11}(t)N_1(t) \\ \qquad\quad -a_{12}(t)N_2(t)\big), & t \neq t_k \\ \dot{N}_2(t) = N_2(t)\big(-b_2(t) + a_{21}(t)N_1(t) \\ \qquad\quad -a_{22}(t)N_2(t) - a_{23}(t)N_3(t)\big), & t \neq t_k \\ \dot{N}_3(t) = N_3(t)\big(-b_3(t) + a_{32}(t)N_2(t) \\ \qquad\quad -a_{33}(t)N_3(t)\big), & t \neq t_k \\ N_i(t_k^+) = (1 + G_{ik})N_i(t_k), & t = t_k, \end{cases}$$

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where $i = 1, 2, 3$.

To the author's knowledge, three-species periodic periodic predator-prey-chain model with harvesting terms have not been discussed in any generality, so we will consider the following with harvesting T-periodic predator-prey-chain system :

$$\begin{cases} \dot{N}_1(t) = N_1(t)\big(b_1(t) - a_{11}(t)N_1(t) \\ \qquad\quad -a_{12}(t)N_2(t)\big) - h_1(t), \\ \dot{N}_2(t) = N_2(t)\big(-b_2(t) + a_{21}(t)N_1(t) \\ \qquad\quad -a_{22}(t)N_2(t) - a_{23}(t)N_3(t)\big) - h_2(t), \\ \dot{N}_3(t) = N_3(t)\big(-b_3(t) + a_{32}(t)N_2(t) \\ \qquad\quad -a_{33}(t)N_3(t)\big) - h_3(t), \end{cases} \tag{1}$$

where $b_i(t)$, $a_{i2}(t)(i = 2, 3)$, $a_{i1}(t)(i = 1, 2)$, $a_{i3}(t)(i = 2, 3)$ are positive continuous ω-periodic functions.

Since a very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic solution, which plays a similar role as a globally stable equilibrium does in an autonomous model, also, on the existence of positive periodic solutions to system (1), few results are found in literatures. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (1). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. Therefore it is essential for us to investigate the existence of multiple positive periodic solutions for population models. Our main purpose of this paper is by using Mawhins continuation theorem of coincidence degree theory [8], to establish the existence of eight positive periodic solutions for system (1). For the work concerning the multiple existence of periodic solutions of periodic population models which was done by using coincidence degree theory, we refer to [5-6].

The organization of the rest of this paper is as follows. Next Section ,we will by employing the continuation theorem of coincidence degree theory, we establish the existence of eight positive periodic solutions of system (1). Finally, an example is given to illustrate the effectiveness of our results.

II. Preliminaries

For the readers' convenience, we first summarize a few concepts from [8].

Let $\mathbf{X}$ and $\mathbf{Z}$ be Banach spaces. Let $L : \operatorname{Dom} L \subset \mathbf{X} \to \mathbf{Z}$ be a linear mapping and $N : \mathbf{X} \to \mathbf{Z}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\operatorname{Im} L$ is a closed subspace of $\mathbf{Z}$ and

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty.$$

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : \mathbf{X} \to \mathbf{Z}$ and $Q : \mathbf{Z} \to \mathbf{Z}$ such that Im P=Ker L and Im L=Ker Q=Im $(I - Q)$. It follows that

$$L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)\mathbf{X} \to \text{Im } L$$

is invertible and its inverse is denoted by K_P. If Ω is a bounded open subset of $\mathbf{X}$, the mapping N is called L-compact on $\mathbf{X}$, if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbf{X}$ is compact. Because Im Q is isomorphic to Ker L, there exists an isomorphism $J : \text{Im } Q \to \text{Ker } L$.

In the proof of our existence result, we need the following continuation theorem from Gaines and Mawthin [8].

Lemma 1. *Let L be a Fredholm mapping of index zero and let N be L-compact on $\mathbf{X}$. Suppose:*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega$, $Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega$, $QNx \neq 0$;*
(c) *$deg\{JQN, \Omega \cap Ker\, L, 0\} \neq 0$.*

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap Dom\, L$.

Lemma 2. *Let $x > 0, y > 0, z > 0$ and $x > 2\sqrt{yz}$, for the functions $f(x, y, z) = \dfrac{x + \sqrt{x^2 - 4yz}}{2z}$ and $g(x, y, z) = \dfrac{x - \sqrt{x^2 - 4yz}}{2z}$, the following assertions hold.*

(1) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically increasing and monotonically decreasing on the variable $x \in (0, \infty)$, respectively.*
(2) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $y \in (0, \infty)$, respectively.*
(3) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $z \in (0, \infty)$, respectively.*

Proof: In fact, for all $x > 0, y > 0, z > 0$, we have

$$\frac{\partial f}{\partial x} = \frac{x + \sqrt{x^2 - 4yz}}{2z\sqrt{x^2 - 4yz}} > 0,\ \frac{\partial g}{\partial x} = \frac{\sqrt{x^2 - 4yz} - x}{2z\sqrt{x^2 - 4yz}} < 0,$$
$$\frac{\partial f}{\partial y} = \frac{-1}{\sqrt{x^2 - 4yz}} < 0,\ \frac{\partial g}{\partial y} = \frac{1}{\sqrt{x^2 - 4yz}} > 0,$$
$$\frac{\partial f}{\partial z} = \frac{-(x + \sqrt{x^2 - 4yz})^2}{4z^2\sqrt{x^2 - 4yz}} < 0,$$
$$\frac{\partial g}{\partial z} = \frac{(x - \sqrt{x^2 - 4yz})^2}{4z^2\sqrt{x^2 - 4yz}} > 0.$$

By the relationship of the derivative and the monotonicity, the above assertions obviously hold. The proof of Lemma 2 is complete. ■

For the sake of convenience, we denote by

$$f^l = \min_{t\in[0,\omega]} f(t), f^M = \max_{t\in[0,\omega]} f(t), \bar{f} = \frac{1}{\omega}\int_0^\omega f(t)\mathrm{d}t,$$

respectively, here $f(t)$ is a continuous ω-function.

Throughout this paper, we need the following assumptions:

$(T1)$ $b_1^L > 2\sqrt{a_{11}^M h_1^M}$;
$(T2)$ $a_{21}^M l_1^+ - b_2^L > 2\sqrt{a_{22}^L h_2^L}$;
$(T3)$ $a_{32}^M l_2^+ - b_3^L > 2\sqrt{a_{33}^L h_3^L}$.

For simplicity, we also introduce some positive numbers as follows:

$$l_1^{\pm} = \frac{b_1^M \pm \sqrt{(b_1^M)^2 - 4a_{11}^L h_1^L}}{2a_{11}^L},$$
$$l_2^{\pm} = \frac{a_{21}^M l_1^+ - b_2^L \pm \sqrt{(a_{21}^M l_1^+ - b_2^L)^2 - 4a_{22}^L h_2^L}}{2a_{22}^L},$$
$$l_3^{\pm} = \frac{a_{32}^M l_2^+ - b_3^L \pm \sqrt{(a_{32}^M l_2^+ - b_3^L)^2 - 4a_{33}^L h_3^L}}{2a_{33}^L}.$$

III. Main result

In this section, by applying Mawhms continuation theorem, we shall show the existence of positive periodic solutions of (1).

Theorem 1. *Assume that $(T_1) - (T_3)$ hold, then system (1) has at least eight positive periodic solutions.*

Proof: By making the substitution $x_i(t) = \exp\{N_i(t)\}$, then system (1) is reformulated as

$$\begin{cases} \dot{x}_1(t) = b_1(t) - a_{11}(t)e^{x_1(t)} - a_{12}(t)e^{x_2(t)} \\ \qquad -h_1(t)e^{-x_1(t)}, \\ \dot{x}_2(t) = -b_2(t) + a_{21}(t)e^{x_1(t)} - a_{22}(t)e^{x_2(t)} \\ \qquad -a_{23}(t)e^{x_3(t)} - h_2(t)e^{-x_2(t)}, \\ \dot{x}_3(t) = -b_3(t) + a_{32}(t)e^{x_2(t)} - a_{33}(t)e^{x_3(t)} \\ \qquad -h_3(t)e^{-x_3(t)}. \end{cases} \tag{2}$$

Let

$$X = Z = \{x = (x_1, x_2, x_3)^T \in C(R, R^3) : x(t + \omega) = x(t)\}$$

and define

$$\|x\| = \sum_{i=1}^{3} \max_{t\in[0,\omega]} |x_i(t)|, \quad x \in \mathbf{X} \text{ or } \mathbf{Z}.$$

Equipped with the above norm $\|\cdot\|$, $\mathbf{X}$ and $\mathbf{Z}$ are Banach spaces. Let

$$N(x, \lambda) = \begin{pmatrix} b_1(\xi_1) - a_{11}(\xi_1)e^{x_1(\xi_1)} - \lambda a_{12}(\xi_1)e^{x_2(\xi_1)} \\ -b_2(\xi_2) + a_{21}(\xi_2)e^{x_1(\xi_2)} - a_{22}(\xi_2)e^{x_2(\xi_2)} \\ -b_3(\xi_3) + a_{32}(\xi_3)e^{x_2(\xi_3)} - a_{33}(\xi_3)e^{x_3(\xi_3)} \end{pmatrix.$$
$$\qquad \begin{matrix} -h_1(\xi_1)e^{-x_1(\xi_1)} \\ -\lambda a_{23}(\xi_2)e^{x_3(\xi_2)} - h_2(\xi_2)e^{-x_2(\xi_2)} \\ -h_3(\xi_3)e^{-x_3(\xi_3)} \end{matrix} \Bigg)$$

where $x \in \mathbf{X}, \lambda \in [0,1]$, $Lx = \dot{x} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}, Px = \frac{1}{\omega}\int_0^\omega x(t)\,\mathrm{d}t,\ x \in \mathbf{X};\ Qz = \frac{1}{\omega}\int_0^\omega z(t)\,\mathrm{d}t,\ z \in \mathbf{Z}$. Then it follows that Ker $L = R^3$, Im $L = \{z \in \mathbf{Z} : \int_0^\omega z(t)\,\mathrm{d}t = 0\}$ is closed in Z, $\dim \text{Ker } L = 3 = \text{codim Im } L$, and P, Q are continuous projectors such that

$$\text{Im } P = \text{Ker } L, \quad \text{Ker } Q = \text{Im } L = \text{Im } (I - Q).$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \text{Im } L \to \text{Ker } P \bigcap \text{Dom } L$ is given by

$$K_P(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^s z(s)\,\mathrm{d}s.$$

Then

$$QN(x,\lambda)=\begin{pmatrix}\frac{1}{\omega}\int_0^\omega F_1(s,\lambda)\,\mathrm{d}s\\ \frac{1}{\omega}\int_0^\omega F_2(s,\lambda)\,\mathrm{d}s\\ \frac{1}{\omega}\int_0^\omega F_3(s,\lambda)\,\mathrm{d}s\end{pmatrix}$$

and

$$\begin{aligned}&K_P(I-Q)N(x,\lambda)\\ &=\begin{pmatrix}\int_0^t F_1(s,\lambda)\,\mathrm{d}s-\frac{1}{\omega}\int_0^\omega\int_0^t F_1(s,\lambda)\,\mathrm{d}s\mathrm{d}t\\ \int_0^t F_2(s,\lambda)\,\mathrm{d}s-\frac{1}{\omega}\int_0^\omega\int_0^t F_2(s,\lambda)\,\mathrm{d}s\mathrm{d}t\\ \int_0^t F_3(s,\lambda)\,\mathrm{d}s-\frac{1}{\omega}\int_0^\omega\int_0^t F_3(s,\lambda)\,\mathrm{d}s\mathrm{d}t\\ +(\frac{1}{2}-\frac{t}{\omega})\int_0^\omega F_1(s,\lambda)\,\mathrm{d}s\\ +(\frac{1}{2}-\frac{t}{\omega})\int_0^\omega F_2(s,\lambda)\,\mathrm{d}s\\ +(\frac{1}{2}-\frac{t}{\omega})\int_0^\omega F_3(s,\lambda)\,\mathrm{d}s\end{pmatrix}\end{aligned}$$

where

$$F_1(s,\lambda)=b_1(s)-a_{11}(s)e^{x_1(s)}-\lambda a_{12}(s)e^{x_2(s)}-h_1(s)e^{-x_1(s)},$$

$$F_2(s,\lambda)=-b_2(s)+a_{21}(s)e^{x_1(s)}-a_{22}(s)e^{x_2(s)}-\lambda a_{23}(s)e^{x_3(s)}-h_2(s)e^{-x_2(s)},$$

$$F_3(s,\lambda)=-b_3(s)+a_{32}(s)e^{x_2(s)}-a_{33}(s)e^{x_3(s)}-h_3(\xi_3)e^{-x_3(s)}.$$

Obviously, QN and $K_P(I-Q)N$ are continuous. It is not difficult to show that $K_P(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega\subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega\subset\mathbf{X}$.

Now, we are in the position of searching for an appropriate open, bounded subset Ω for the application of the continuation theorem. Considering to the operator equation

$$Lx=\lambda N(x,\lambda)\quad \lambda\in(0,1),$$

we obtain

$$\begin{cases}\dot{x}_1(t)=\lambda(b_1(t)-a_{11}(t)e^{x_1(t)}-a_{12}(t)e^{x_2(t)}\\ \qquad -h_1(t)e^{-x_1(t)}),\\ \dot{x}_2(t)=\lambda(-b_2(t)+a_{21}(t)e^{x_1(t)}-a_{22}(t)e^{x_2(t)}\\ \qquad -a_{23}(t)e^{x_3(t)}-h_2(t)e^{-x_2(t)}),\\ \dot{x}_3(t)=\lambda(-b_3(t)+a_{32}(t)e^{x_2(t)}-a_{33}(t)e^{x_3(t)}\\ \qquad -h_3(t)e^{-x_3(t)}),\end{cases}$$

Assume that $x\in\mathbf{X}$ is a solution of Equation (2) for some $\lambda\in(0,1)$. Then there must exist ξ_i, $\eta_i(i=1,2,3)\in[0,\omega]$ such that

$$x_i(\xi_i)=\max_{t\in\mathbf{R}}x_i(t)\quad\text{and}\quad x_i(\eta_i)=\min_{t\in\mathbf{R}}x_i(t).$$

It is clear that $x_i'(\xi_i)=0$ and $x_i'(\eta_i)=0(i=1,2,3)$. From this and system (3.1), we have

$$\begin{cases}0=b_1(\xi_1)-a_{11}(\xi_1)e^{x_1(\xi_1)}-\lambda a_{12}(\xi_1)e^{x_2(\xi_1)}\\ \qquad -h_1(\xi_1)e^{-x_1(\xi_1)},\\ 0=-b_2(\xi_2)+a_{21}(\xi_2)e^{x_1(\xi_2)}-a_{22}(\xi_2)e^{x_2(\xi_2)}\\ \qquad -\lambda a_{23}(\xi_2)e^{x_3(\xi_2)}-h_2(\xi_2)e^{-x_2(\xi_2)},\\ 0=-b_3(\xi_3)+a_{32}(\xi_3)e^{x_2(\xi_3)}-a_{33}(\xi_3)e^{x_3(\xi_3)}\\ \qquad -h_3(\xi_3)e^{-x_3(\xi_3)},\end{cases}\qquad(3)$$

and

$$\begin{cases}0=b_1(\eta_1)-a_{11}(\eta_1)e^{x_1(\eta_1)}-\lambda a_{12}(\eta_1)e^{x_2(\eta_1)}\\ \qquad -h_1(\eta_1)e^{-x_1(\eta_1)},\\ 0=-b_2(\eta_2)+a_{21}(\eta_2)e^{x_1(\eta_2)}-a_{22}(\eta_2)e^{x_2(\eta_2)}\\ \qquad -\lambda a_{23}(\eta_2)e^{x_3(\eta_2)}-h_2(\eta_2)e^{-x_2(\eta_2)},\\ 0=-b_3(\eta_3)+a_{32}(\eta_3)e^{x_2(\eta_3)}-a_{33}(\eta_3)e^{x_3(\eta_3)}\\ \qquad -h_3(\eta_3)e^{-x_3(\eta_3)}.\end{cases}\qquad(4)$$

According to the first equation of (3), we have

$$\begin{aligned}b_1^M&\ge b_1(\xi_1)\\ &=a_{11}(\xi_1)e^{x_1(\xi_1)}+\lambda a_{12}(\xi_1)e^{x_2(\xi_1)}\\ &\quad +h_1(\xi_1)e^{-x_1(\xi_1)}\\ &> a_{11}^Le^{x_1(\xi_1)}+h_1^Le^{-x_1(\xi_1)},\end{aligned}$$

namely,

$$a_{11}^Le^{2x_1(\xi_1)}-b_1^Me^{x_1(\xi_1)}+h_1^L<0,$$

which implies that

$$\ln l_1^-<z_1(\xi_1)<\ln l_1^+.\qquad(5)$$

Similarly, by the first equation of (4),

$$\ln l_1^-<z_1(\eta_1)<\ln l_1^+.\qquad(6)$$

The second equation of (3) gives

$$\begin{aligned}a_{21}^Ml_1^+&\ge a_{21}^Me^{x_1(\xi_2)}>a_{21}(\xi_2)e^{x_1(\xi_2)}\\ &=b_2(\xi_2)+a_{22}(\xi_2)e^{x_2(\xi_2)}+\lambda a_{23}(\xi_2)e^{x_3(\xi_2)}\\ &\quad +h_2e^{-x_2(\xi_2)}\\ &> b_2^L+a_{22}^Le^{x_2(\xi_2)}+h_2^Le^{-x_2(\xi_2)},\end{aligned}$$

that is

$$a_{22}^Le^{2x_2(\xi_2)}-(a_{21}^Ml_1^+-b_2^L)e^{x_2(\xi_2)}+h_2^L<0,$$

which implies that

$$\ln l_2^-<x_2(\xi_2)<\ln l_2^+.\qquad(7)$$

Similarly, by the second equation of (4), we get

$$\ln l_2^-<x_2(\eta_2)<\ln l_2^+.\qquad(8)$$

The third equation of (3) gives

$$\begin{aligned}a_{32}^Ml_2^+&\ge a_{32}^Me^{x_x(\xi_3)}>a_{32}(\xi_3)e^{x_2(\xi_3)}\\ &=b_3(\xi_3)+a_{33}(\xi_3)e^{x_3(\xi_3)}+h_3e^{-x_3(\xi_3)}\\ &> b_3^L+a_{33}^Le^{x_3(\xi_3)}+h_3^Le^{-x_3(\xi_3)},\end{aligned}$$

that is

$$a_{33}^Le^{2x_3(\xi_3)}-(a_{32}^Ml_2^+-b_3^L)e^{x_3(\xi_3)}+h_3^L<0,$$

which implies that

$$\ln l_3^-<x_3(\xi_3)<\ln l_3^+.\qquad(9)$$

Similarly, by the third equation of (4), we get

$$\ln l_3^-<x_3(\eta_3)<\ln l_3^+.\qquad(10)$$

Moreover, from the first equation of (3), we have

$$b_1^M\ \ge\ b_1(\xi_1)=a_{11}(\xi_1)e^{x_1(\xi_1)}+\lambda a_{12}(\xi_1)e^{x_2(\xi_1)}$$

$$\begin{aligned} & +h_1(\xi_1)e^{-x_1(\xi_1)} \\ > & \ a_{11}(\xi_1)e^{x_1(\xi_1)} + \lambda a_{12}(\xi_1)e^{x_2(\xi_1)} \\ > & \ a_{11}^L e^{x_1(\xi_1)} + a_{12}^L l_2^- \\ > & \ a_{11}^L e^{x_1(\xi_1)}, \end{aligned}$$

which implies that

$$x_1(\xi_1) < \ln \frac{b_1^M}{a_{11}^L} \triangleq \ln H_1^-. \quad (11)$$

Similarly, from the first equation of (4), we obtain

$$\begin{aligned} h_1^L e^{-x_1(\eta_1)} \ \le & \ h_1(\eta_1)e^{-x_1(\eta_1)} + a_{11}(\eta_1)e^{x_1(\eta_1)} \\ < & \ a_{11}(\eta_1)e^{x_1(\eta_1)} \\ & +\lambda a_{12}(\eta_1)e^{x_2(\eta_1)} \\ & +h_1(\eta_1)e^{-x_1(\eta_1)} \\ = & \ b_1(\eta_1) \le b_1^M, \end{aligned}$$

which implies that

$$x_1(\eta_1) > \ln \frac{h_1^L}{b_1^M} \triangleq \ln H_1^+. \quad (12)$$

From the second equation of (3), we have

$$\begin{aligned} a_{21}^M l_1^+ \ \ge & \ a_{21}e^{x_1(\xi_2)} = b_2(\xi_2) + a_{22}(\xi_2)e^{x_2(\xi_2)} \\ & +\lambda a_{23}(\xi_2)e^{x_3(\xi_2)} + h_2(\xi_2)e^{-x_2(\xi_2)} \\ > & \ b_2^L + a_{22}^L e^{x_2(\xi_2)} > a_{22}^L e^{x_2(\xi_2)}, \end{aligned}$$

which implies that

$$x_2(\xi_2) < \ln \frac{a_{21}^M l_1^+}{a_{22}^L} \triangleq \ln H_2^-. \quad (13)$$

Similarly, from the second equation of (4), we obtain

$$\begin{aligned} a_{21}^M l_1^+ \ \ge & \ a_{21}(\eta_2)e^{-x_1(\eta_2)} = b_2(\eta_2) + a_{22}(\eta_2)e^{x_2(\eta_2)} \\ & +\lambda a_{23}(\eta_2)e^{x_3(\eta_2)} + h_2 e^{-x_2(\eta_2)} \\ > & \ b_2^L + h_2^L e^{-x_2(\eta_2)}, \end{aligned}$$

which implies that

$$x_2(\eta_2) > \ln \frac{h_2^L}{a_{21}^M l_1^+ - b_2^L} \triangleq \ln H_2^+. \quad (14)$$

From the third equation of (3), we have

$$\begin{aligned} a_{32}^M l_2^+ \ \ge & \ a_{32}^M e^{x_x(\xi_3)} > a_{32}(\xi_3)e^{x_2(\xi_3)} \\ = & \ b_3(\xi_3) + a_{33}(\xi_3)e^{x_3(\xi_3)} + h_3 e^{-x_3(\xi_3)} \\ > & \ b_3^L + a_{33}^L e^{x_3(\xi_3)} + h_3^L e^{-x_3(\xi_3)} \\ > & \ b_3^L + a_{33}^L e^{x_3(\xi_3)}, \end{aligned}$$

which implies that

$$x_3(\xi_3) < \ln \frac{a_{32}^M l_2^+ - b_3^L}{a_{33}^L} \triangleq \ln H_3^-. \quad (15)$$

Similarly, from the third equation of (4), we obtain

$$\begin{aligned} a_{32}^M l_2^+ \ \ge & \ a_{32}^M e^{x_x(\eta_3)} > a_{32}(\eta_3)e^{x_2(\eta_3)} \\ = & \ b_3(\eta_3) + a_{33}(\eta_3)e^{x_3(\eta_3)} + h_3 e^{-x_3(\eta_3)} \\ > & \ b_3^L + a_{33}^L e^{x_3(\eta_3)} + h_3^L e^{-x_3(\eta_3)} \\ > & \ b_3^L + h_3^L e^{-x_3(\eta_3)}, \end{aligned}$$

which implies that

$$x_3(\eta_3) > \ln \frac{h_3^L}{a_{32}^M l_2^+ - b_3^L} \triangleq \ln H_3^+. \quad (16)$$

We claim that $\ln l_i^- < \ln H_i^-$, $\ln H_i^+ < \ln l_i^+ (i = 1, 2, 3)$. In fact, employing Lemma 2 and (T1)-(T3) , we have

$$\frac{b_1^M}{a_{11}^L} > \frac{b_1^M + \sqrt{(b_1^M)^2 - 4a_{11}^L h_1^L}}{2a_{11}^L} = l_1^+ > \frac{h_1^L}{b_1^M} = H_1^+,$$

$$l_1^- = \frac{b_1^M - \sqrt{(b_1^M)^2 - 4a_{11}^L h_1^L}}{2a_{11}^L} < \frac{b_1^M}{a_{11}^L} = H_1^-,$$

$$\frac{b_1^M}{a_{11}^L} > \frac{b_1^M + \sqrt{(b_1^M)^2 - 4a_{11}^L h_1^L}}{2a_{11}^L} = l_1^+ > \frac{h_1^L}{b_1^M} = H_1^+,$$

$$l_1^- = \frac{b_1^M - \sqrt{(b_1^M)^2 - 4a_{11}^L h_1^L}}{2a_{11}^L} < \frac{b_1^M}{a_{11}^L} = H_1^-,$$

$$\begin{aligned} \frac{a_{21}^M l_1^+ - b_2^L}{a_{22}^L} \ > & \ \frac{a_{21}^M l_1^+ - b_2^L + \sqrt{(a_{21}^M l_1^+ - b_2^L)^2 - 4a_{22}^L h_2^L}}{2a_{22}^L} \\ = & \ l_2^+ > \frac{h_2^L}{a_{21}^M l_1^+ - b_2^L} = H_2^+, \end{aligned}$$

$$\begin{aligned} l_2^- \ = & \ \frac{a_{21}^M l_1^+ - b_2^L - \sqrt{(a_{21}^M l_1^+ - b_2^L)^2 - 4a_{22}^L h_2^L}}{2a_{22}^L} \\ < & \ \frac{a_{21}^M l_1^+ - b_2^L}{2a_{22}^L} < \frac{a_{21}^M l_1^+}{a_{22}^L} = H_2^-, \end{aligned}$$

$$\begin{aligned} \frac{a_{32}^M l_2^+ - b_3^L}{a_{33}^L} \ > & \ \frac{a_{32}^M l_2^+ - b_3^L + \sqrt{(a_{32}^M l_2^+ - b_3^L)^2 - 4a_{33}^L h_3^L}}{2a_{33}^L} \\ = & \ l_3^+ > \frac{h_3^L}{a_{32}^M l_2^+ - b_3^L} = H_3^+, \end{aligned}$$

$$\begin{aligned} l_3^- \ = & \ \frac{a_{32}^M l_2^+ - b_3^L - \sqrt{(a_{32}^M l_2^+ - b_3^L)^2 - 4a_{33}^L h_3^L}}{2a_{33}^L} \\ < & \ \frac{a_{32}^M l_2^+ - b_3^L}{2a_{33}^L} < \frac{a_{32}^M l_2^+}{a_{33}^L} = H_3^-. \end{aligned}$$

From (9)-(14), $\forall i = 1, 2, 3$, we obtain

$$\ln H_i^+ < x_i(\eta_i) < x_i(\xi_i) < \ln l_i^+ \quad (17)$$

or

$$\ln l_i^- < x_i(\eta_i) < x_i(\xi_i) < \ln H_i^-. \quad (18)$$

By (17) and (18), we have for all $t \in R$, $i \in \{1, 2, 3\}$,

$$\ln H_i^+ < x_i(t) < \ln l_i^+ \quad (19)$$

or

$$\ln l_i^- < x_i(t) < \ln H_i^-. \quad (20)$$

Clearly, $\ln l_i^\pm$ and $\ln H_i^\pm$ are independent of λ. Now let

$$\Omega_1 = \ \{ \ x = (x_1, x_2, x_3)^T \in X : \ln H_1^+ < x_1(t) < \ln l_1^+,$$

$$H_2^+ < x_2(t) < \ln l_2^+, H_3^+ < x_3(t) < \ln l_3^+\},$$

$$\Omega_2 = \{ \; x = (x_1, x_2, x_3)^T \in X : \ln H_1^+ < x_1(t) < \ln l_1^+, \; H_2^+ < x_2(t) < \ln l_2^+, l_3^- < x_3(t) < \ln H_3^-\},$$

$$\Omega_3 = \{ \; x = (x_1, x_2, x_3)^T \in X : \ln H_1^+ < x_1(t) < \ln l_1^+, \; l_2^- < x_2(t) < \ln H_2^-, H_3^+ < x_3(t) < \ln l_3^+\},$$

$$\Omega_4 = \{ \; x = (x_1, x_2, x_3)^T \in X : \ln H_1^+ < x_1(t) < \ln l_1^+, \; l_2^- < x_2(t) < \ln H_2^-, l_3^- < x_3(t) < \ln H_3^-\},$$

$$\Omega_5 = \{ \; x = (x_1, x_2, x_3)^T \in X : \ln l_1^- < x_1(t) < \ln H_1^-, \; H_2^+ < x_2(t) < \ln l_2^+, H_3^+ < x_3(t) < \ln l_3^+\},$$

$$\Omega_6 = \{ \; x = (x_1, x_2, x_3)^T \in X : \ln l_1^- < x_1(t) < \ln H_1^-, \; H_2^+ < x_2(t) < \ln l_2^+, l_3^- < x_3(t) < \ln H_3^-\},$$

$$\Omega_7 = \{ \; x = (x_1, x_2, x_3)^T \in X : \ln l_1^- < x_1(t) < \ln H_1^-, \; l_2^- < x_2(t) < \ln H_2^-, H_3^+ < x_3(t) < \ln l_3^+\}$$

and

$$\Omega_8 = \{ \; x = (x_1, x_2, x_3)^T \in X : \ln l_1^- < x_1(t) < \ln H_1^-, \; l_2^- < x_2(t) < \ln H_2^-, l_3^- < x_3(t) < \ln H_3^-\}.$$

Then $\Omega_i (i = 1, 2, 3, 4, 5, 6, 7, 8)$ are bounded open subsets of $X, \Omega_i \cap \Omega_j = \phi (i \neq j)$. Thus $\Omega_i (i = 1, 2, 3, 4, 5, 6, 7, 8)$ satisfies the requirement (a) in Lemma 1.

Now we show that (b) of Lemma 1 holds, i.e., we prove when $x \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap R^3, QNx \neq (0, 0, 0)^T, i = 1, 2, 3, 4, 5, 6, 7, 8$. If it is not true, then when $x \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap R^3, i = 1, 2, 3, 4, 5, 6, 7, 8$ constant vector $x = (x_1, x_2, x_3)^T$ with $x \in \partial\Omega_i, i = 1, 2, 3, 4, 5, 6, 7, 8$ satisfies

$$\begin{cases} 0 = \int_0^\omega b_1(t)\mathrm{d}t - \int_0^\omega a_{11}(t)e^{x_1}\mathrm{d}t - \int_0^\omega a_{12}(t)e^{x_2}\mathrm{d}t \\ \quad - \int_0^\omega h_1(t)e^{-x_1}\mathrm{d}t, \\ 0 = -\int_0^\omega b_2(t)\mathrm{d}t + \int_0^\omega a_{21}(t)e^{x_1}\mathrm{d}t - \int_0^\omega a_{22}(t)e^{x_2}\mathrm{d}t \\ \quad - \int_0^\omega a_{23}(t)e^{x_3}\mathrm{d}t - \int_0^\omega h_2(t)e^{-x_2}\mathrm{d}t. \\ 0 = -\int_0^\omega b_3(t)\mathrm{d}t + \int_0^\omega a_{32}(t)e^{x_2}\mathrm{d}t - \int_0^\omega a_{33}(t)e^{x_3}\mathrm{d}t \\ \quad - \int_0^\omega h_3(t)e^{-x_3}\mathrm{d}t = 0. \end{cases}$$

In terms of differential mean value theorem, there exist three points $t_i (i = 1, 2, 3)$ such that when $x \in \partial\Omega_i \cap \operatorname{Ker}L = \partial\Omega_i \cap R^3, i = 1, 2, 3, 4, 5, 6, 7, 8,$

$$QN(x, 0) = \begin{pmatrix} F_1(0, t_1) \\ F_2(0, t_2) \\ F_3(0, t_3) \end{pmatrix}$$

If for $x \in \partial\Omega_i \cap \operatorname{Ker}L = \partial\Omega_i \cap R^3$, $QN(x, 0) = (0, 0, 0)^T$ holds, then the constant vector $x \in \partial\Omega_i (i = 1, 2, 3, 4, 5, 6, 7, 8)$ satisfies

$$F(0, t_1) = 0,$$

$$F(0, t_2) = 0$$

and

$$F(0, t_3) = 0,$$

Then $x \in \Omega_1 \cap R^3$ or $x \in \Omega_2 \cap R^3$ or $x \in \Omega_3 \cap R^2$ or $x \in \Omega_4 \cap R^3$, or $x \in \Omega_5 \cap R^3$, or $x \in \Omega_6 \cap R^3$, or $x \in \Omega_7 \cap R^3$, or $x \in \Omega_8 \cap R^3$. This contradicts the fact that $x \in \partial\Omega_i \cap R^3, i = 1, 2, 3, 4, 5, 6, 7, 8$. This proves (b) in Lemma 1 holds. Finally, we show that (c) in Lemma 1 holds. Note that the system of algebraic equations:

$$\begin{cases} b_1(t_1) - a_{11}(t_1)e^{x_1} - h_1(t_1)e^{-x_1} = 0, \\ -b_2(t_2) + a_{21}(t_2)e^{x_1} - a_{22}(t_2)e^{x_2} - h_2(t_2)e^{-x_2} = 0, \\ -b_3(t_3) + a_{32}(t_3)e^{x_2} - a_{33}(t_3)e^{x_3} - h_3(t_3)e^{-x_3} = 0. \end{cases}$$

has eight distinct solutions since $(T1) - (T3)$ hold,

$$\begin{aligned} (x_1^*, y_1^*, z_1^*) &= (\ln x_{1+}, \ln x_{2+}, \ln x_{3+}), \\ (x_2^*, y_2^*, z_2^*) &= (\ln x_{1+}, \ln x_{2+}, \ln x_{3-}), \\ (x_3^*, y_3^*, z_3^*) &= (\ln x_{1+}, \ln x_{2-}, \ln x_{3+}), \\ (x_4^*, y_4^*, z_4^*) &= (\ln x_{1+}, \ln x_{2-}, \ln x_{3-}), \\ (x_5^*, y_5^*, z_5^*) &= (\ln x_{1-}, \ln x_{2+}, \ln x_{3+}), \\ (x_6^*, y_6^*, z_6^*) &= (\ln x_{1-}, \ln x_{2+}, \ln x_{3-}), \\ (x_7^*, y_7^*, z_7^*) &= (\ln x_{1-}, \ln x_{2-}, \ln x_{3+}), \\ (x_8^*, y_8^*, z_8^*) &= (\ln x_{1-}, \ln x_{2-}, \ln x_{3-}), \end{aligned}$$

where

$$x_{1\pm} = \frac{b_1(t_1) \pm \sqrt{(b_1(t_1))^2 - 4a_{11}(t_1)h_1(t_1)}}{2a_{11}(t_1)},$$

$$x_{2\pm} = \frac{a_{21}(t_2)x_{1\pm} - b_2(t_2)}{2a_{22}(t_2)} \pm \frac{\sqrt{(a_{21}(t_2)x_{1\pm} - b_2(t_2))^2 - 4a_{22}(t_2)h_2(t_2)}}{2a_{22}(t_2)},$$

$$x_{3\pm} = \frac{a_{32}(t_3)x_{2\pm} - b_3(t_2)}{2a_{33}(t_3)} \pm \frac{\sqrt{(a_{32}(t_3)x_{2\pm} - b_3(t_3))^2 - 4a_{33}(t_3)h_3(t_3)}}{2a_{33}(t_3)}.$$

It is easy to verify that

$$\ln l_i^- < \ln x_{i-} < \ln H_i^- < \ln H_i^+ < \ln x_{i+} < \ln l_i^+.$$

Therefore,

$$\begin{aligned} &(x_1^*, y_1^*, z_1^*) \in \Omega_1, (x_2^*, y_2^*, z_2^*) \in \Omega_2, (x_3^*, y_3^*, z_3^*) \in \Omega_3, \\ &(x_4^*, y_4^*, z_4^*) \in \Omega_4, (x_5^*, y_5^*, z_5^*) \in \Omega_5, (x_6^*, y_6^*, z_6^*) \in \Omega_6, \\ &(x_7^*, y_7^*, z_7^*) \in \Omega_7, (x_8^*, y_8^*, z_8^*) \in \Omega_8. \end{aligned}$$

Since $\operatorname{Ker} L = \operatorname{Im} Q$, we can take $J = I$. In the light of the definition of the Leray-Schauder degree, a direct computation gives for $i = 1, 2, 3, 4, 5, 6, 7, 8,$

$$\deg\{JQN, \Omega_i \cap \operatorname{Ker}L, (0, 0, 0)^T\} = -1 \text{ or } 1 \neq 0,$$

here, J is taken as the identity mapping. So far we have proved that $\Omega_i (i = 1, 2, 3, 4, 5, 6, 7, 8)$ satisfy all the conditions in Lemma 1. Thus, system (2) has at least eight positive periodic solutions in $\bar{\Omega}$, that is system (1) has at least eight positive periodic solutions. This completes the proof. ∎

Remark 1. *From the proof of Theorem 1, we can see that if the harvesting terms* $h1(t) = h2(t) = h3(t) = 0,$

system (1) has at least one positive periodic solution, but we could not conclude that system (1) has at least eight positive periodic solutions because we could not construct $\Omega_i, i = 1,2,3,4,5,6,7,8$ satisfying $\Omega_i \cap \Omega_j = \phi$. Therefore, adding the harvesting terms to population models can make biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena.

IV. An example

Example 1. *Consider the following periodic predator-prey-chain system with harvesting:*

$$\begin{cases} \dot{N}_1(t) = N_1(t)\Big]3 + \sin t - \dfrac{4+\sin t}{10}N_1(t) \\ \qquad -\dfrac{9+\sin t}{10}N_2(t)\Big] - \dfrac{9+\cos t}{20}, \\ \dot{N}_2(t) = N_2(t)\Big[-2 - \cos t + \dfrac{5+\cos t}{10}N_1(t) \\ \qquad -\dfrac{5+\cos t}{10}N_2(t) - \dfrac{5+\cos t}{10}N_3(t)\Big] \\ \qquad -\dfrac{2+\cos t}{100}, \\ \dot{N}_3(t) = N_3(t)\Big[-3 - \sin t + \dfrac{8+\sin t}{10}N_2(t) \\ \qquad -\dfrac{4+\sin t}{8}N_3(t)\Big] - \dfrac{8+\cos t}{10}. \end{cases} \tag{21}$$

In this case, $b_1(t) = 3 + \cos t, b_2(t) = 2 + \cos t, b_3(t) = 3 + \sin t, a_{11}(t) = \frac{4+\sin t}{10}, a_{12}(t) = \frac{9+\sin t}{10}, a_{21}(t) = \frac{5+\cos t}{10}, a_{22}(t) = \frac{6+\cos tt}{10}, a_{23}(t) = \frac{6+\cos tt}{10}, a_{32}(t) = \frac{8+\sin t}{10}, a_{33}(t) = \frac{4+\sin t}{8}, h_1(t) = \frac{9+\cos t}{20}, h_2(t) = \frac{2+\cos t}{20}$ and $h_3(t) = \frac{8+\cos t}{10}$. Since

$$\begin{aligned} & b_1^L = 2, 2\sqrt{a_{11}^M h_1^M} = 2\sqrt{\frac{5}{10} \times \frac{10}{10}} = 1, \\ & l_1^- = \frac{b_1^M - \sqrt{(b_1^M)^2 - 4a_{11}^L h_1^L}}{2a_{11}^L} = 0.05, \\ & l_1^+ = \frac{b_1^M + \sqrt{(b_1^M)^2 - 4a_{11}^L h_1^L}}{2a_{11}^L} = 13.28, \\ & l_2^+ = \frac{a_{21}^M l_1^+ - b_2^L + \sqrt{(a_{21}^M l_1^+ - b_2^L)^2 - 4a_{22}^L h_2^L}}{2a_{22}^L} = 13.9, \\ & a_{21}^M l_1^+ - b_2^L = 6.97, \ \ 2\sqrt{a_{22}^L h_2^L} = 0.32, \\ & a_{32}^M l_2^+ - b_3^L = 10.51, \ \ 2\sqrt{a_{33}^L h_3^L} = 1, \end{aligned}$$

then

$$\begin{aligned} & 2 = b_1^L > 2\sqrt{a_{11}^M h_1^M} = 1, \\ & a_{21}^M l_1^+ - b_2^L = 6.97 > 0.32 = 2\sqrt{a_{22}^L h_2^L}, \\ & a_{32}^M l_2^+ - b_3^L = 10.51 > 1 = 2\sqrt{a_{33}^L h_3^L}. \end{aligned}$$

Hence, all conditions of Theorem 1 are satisfied. By Theorem 1, system (21) has at least eight positive 2π-periodic solutions.

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