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On a New Nonlinear Sum-difference Inequality with Application

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Abstract—A new nonlinear sum-difference inequality in two variables which generalize some existing results and can be used as handy tools in the analysis of certain partial difference equation is discussed. An example to show boundedness of solutions of a difference value problem is also given.

Keywords—Sum-Difference inequality, Nonlinear, Boundedness.

I. Introduction

THE sum-difference inequalities which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of finite difference equations. During the past few years, motivated and inspired by their applications in various branches of difference equations, many such inequalities have been established. For example, see [1], [2], [7], [9] and the references given therein. In particular, Ma and Cheung [7] considered

$$\varphi(u(m,n)) \le a(m,n) + b(m,n)\sum_{s=m+1}^{\beta} c(s,n)\varphi(u(s,n)) + d(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\varphi'(u(s,t))[f(s,t)w(u(s,t)) + e(s,t)], \quad (1)$$

where $a(m,n)$, $b(m,n)$, $c(m,n)$ and $d(m,n)$ have certain monotonicity.

In this paper, we investigate more general nonlinear sum-difference inequality as follows,

$$u(m,n) \le a(m,n) + b(m,n)\sum_{s=m+1}^{\beta} c(s,n)u(s,n) + \sum_{k=1}^{2}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} f_k(m,n,s,t)w_k(u(s,t)), \quad (2)$$

where we do not require the monotonicity of $a(m,n)$, $b(m,n)$ and $f_k(m,n,s,t)$. Our result can generalize the results in [7] and be used more effectively to study the boundedness and uniqueness of the solutions of certain partial difference equation. Moreover, an example is presented to show the usefulness of our result.

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II. Main Result

Throughout this paper, we denote $\mathbf{N_0} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots\}$. As usual, the empty sum is taken to be 0. Assume that

(A_1) $a(m,n), b(m,n)$ and $c(m,n)$ are bounded and nonnegative for $m, n \in \mathbf{N_0}$ and $a(\infty,\infty) > 0$;

(A_2) $f_k(m,n,s,t)$ $(k = 1,2)$ is nonnegative for $m,n,s,t \in \mathbf{N_0}$;

(A_3) $w_k(u)$ $(k = 1,2)$ is continuous and nondecreasing function on $[0,\infty)$ and positive on $(0,\infty)$. They satisfy the relationship $w_1 \propto w_2$, that is, $\frac{w_2}{w_1}$ is nondecreasing on $(0,\infty)$.

and define that

(D_1) $\tilde{a}(m,n) = \sup_{m\le\tau,n\le\eta,\tau,\eta\in\mathbf{N_0}} a(\tau,\eta)$ and $\tilde{b}(m,n) = \sup_{m\le\tau,n\le\eta,\tau,\eta\in\mathbf{N_0}} b(\tau,\eta)$;

(D_2) $\tilde{f}_k(m,n,s,t) = \sup_{m\le\tau,n\le\eta,\tau,\eta\in\mathbf{N}_0} f_k(\tau,\eta,s,t)$;

(D_3) $\Delta_1 u(m,n) = u(m+1,n) - u(m,n)$ and $\Delta_3 g(m,n,s,t) = g(m,n,s+1,t) - g(m,n,s,t)$;

(D_4) For $u \ge u_k > 0$, $W_k(u) = \int_{u_k}^{u}\frac{dz}{w_k(z)}$ $(k = 1,2)$. From assumption (A_3), W_k is strictly increasing so its inverse W_k^{-1} is well defined, continuous and increasing in its corresponding domain.

Theorem *1:* Under assumptions (A_1)-(A_3), if $\Delta_1 p(m,n)\tilde{a}(m,n)$ is nonpositive for m,n and nondecreasing in n and $u(m,n)$ is a nonnegative function for $m,n \in \mathbf{N}_0$ satisfying (2), then

$$u(m,n) \le W_2^{-1}\Big[W_2(\tilde{a}(\infty,n)) + p(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\tilde{f}_2(m,n,s,t) - \sum_{s=m}^{\infty}\frac{\Delta_3 g_2(m,n,s,n)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,s+1,\infty)))}\Big], \quad (3)$$

for $m \ge M_1, n \ge N_1$, where

$$\begin{aligned} &p(m,n) \\ &= 1 + \tilde{b}(m,n)\sum_{s=m+1}^{\beta} c(s,n)\prod_{i=n+1}^{s-1}(1 + c(i,n)\tilde{b}(i,n)), \\ &g_1(m,n,s,t) = p(s,t)\tilde{a}(s,t), \\ &g_2(m,n,s,t) = W_1(g_1(m,n,\infty,t)) + p(m,n)\sum_{\tau=m+1}^{\infty}\sum_{\eta=n+1}^{\infty}\tilde{f}_1(m,n,\tau,\eta) - \sum_{\tau=s}^{\infty}\frac{\Delta_3 g_1(m,n,\tau,t)}{w_1(g_1(\infty,\infty,\tau+1,\infty))}, \quad (4) \end{aligned}$$

$\phi_2(u) = \frac{w_2(u)}{w_1(u)}$, $\phi_1(u) = w_1(u)$, $W_0 = I$ (Identity), and M_1 and N_1 are positive integers satisfying

$$\begin{aligned} &W_k(\tilde{a}(\infty, N_1)) \\ &\quad +p(M_1, N_1) \sum_{s=M_1+1}^{\infty} \sum_{t=N_1+1}^{\infty} \tilde{f}_k(M_1, N_1, s, t) \\ &\quad + \sum_{s=M_1}^{\infty} \frac{\Delta_3 g_k(M_1, N_1, s, N_1)}{\phi_k(W_{k-1}^{-1}(g_k(\infty, \infty, s+1, \infty)))} \\ &\qquad \le \int_{u_k}^{\infty} \frac{dz}{w_k(z)}, \quad k = 1, 2. \end{aligned} \tag{5}$$

Remark *1:* If w_k $(k = 1, 2)$ satisfies $\int_{u_k}^{\infty} \frac{dz}{w_k(z)} = \infty$, then we may choose $N_1 = 0$ and $M_1 = 0$. As explained in [9], different choices of u_k in W_k do not affect our result.

Remark *2:* If $w_k(u)$ $(k = 1, 2)$ is continuous positive function on $(0, \infty)$ but the sequence of $\{w_k(u)\}$ does not satisfy $w_1 \propto w_2$, we can take a technique of monotonization of the sequence of functions $w_k(u)$ that calculated by

$$\tilde{w}_1(u) := \max_{\theta \in [0,u]} w_1(\theta),$$

$$\tilde{w}_2(u) := \max_{\theta \in [0,u]} \{\frac{w_2(\theta)}{\tilde{w}_1(\theta)}\} \tilde{w}_1(u).$$

Clearly, $\tilde{w}_k(u) \ge w_k(u)$ $(k = 1, 2)$ and the new function sequence $\{\tilde{w}_k(u)\}$ satisfies the assumption (A_3).

Remark *3:* We take $\varphi(u(m,n)) = v(m,n)$, then $u(m,n)$ = $\varphi^{-1}(v(m,n))$. Furthermore, if let $f_1(m,n,s,t) = d(m,n)$ $f(s,t)$, $f_2(m,n,s,t) = d(m,n)$ $e(s,t)$, $\tilde{w}_1(v) = \varphi'(u)w(u)$ = $\varphi'(\varphi^{-1}(v))w(\varphi^{-1}(v))$ and $\tilde{w}_2(v) = \varphi'(u)$=$\varphi'(\varphi^{-1}(v))$, (1) can be reduced to form of (2). So the result in [7] is the special case of our result.

Before we prove the theorem, the following lemma should be introduced [8].

Lemma *1:* (Pachpatte[8]) *Let $u_n, a_n, b_n \ge 0$, $q_n \ge 0$ be sequences defined for $n \in \mathbf{N_0}$ satisfying the inequality*

$$u_n \le a_n + q_n \sum_{s=n+1}^{m} b_s u_s, \qquad n \in \mathbf{N_0}. \tag{6}$$

Then

$$u_n \le a_n + q_n \sum_{s=n+1}^{m} b_s a_s \prod_{i=n+1}^{s-1} (1 + b_i q_i), \tag{7}$$

for $0 \le n \le m, m \in \mathbf{N_0}$.

III. Proof of Theorem

Proof. From the definitions (D_1) and D_2, $\tilde{a}(m,n)$, $\tilde{b}(m,n)$ and $\tilde{f}_k(m,n,s,t)$ are nonnegative and nonincreasing in m and n. Meanwhile, $\tilde{a}(m,n) \ge a(m,n)$, $\tilde{b}(m,n) \ge b(m,n)$ and $\tilde{f}_k(m,n,s,t) \ge f_k(m,n,s,t)$. $a(\infty,\infty) > 0$ in (A_1) implies that $\tilde{a}(m,n) > 0$ for all $m \ge M_1$, $n \ge N_1$. From (2), we have

$$\begin{aligned} u(m,n) &\le \tilde{a}(m,n) + \tilde{b}(m,n) \sum_{s=m+1}^{\beta} c(s,n) u(s,n) \\ &+ \sum_{k=1}^{2} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \tilde{f}_k(m,n,s,t) w_k(u(s,t)). \end{aligned} \tag{8}$$

Let

$$\begin{aligned} Y(m,n) &= \tilde{a}(m,n) + \\ &\sum_{k=1}^{2} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \tilde{f}_k(m,n,s,t) w_k(u(s,t)). \end{aligned} \tag{9}$$

Then

$$u(m,n) \le Y(m,n) + \tilde{b}(m,n) \sum_{s=m+1}^{\beta} c(s,n) u(s,n). \tag{10}$$

Keeping n fixed, from Lemma 1, we obtain

$$\begin{aligned} &u(m,n) \le Y(m,n) \\ &+\tilde{b}(m,n) \sum_{s=m+1}^{\beta} c(s,n) Y(s,n) \prod_{i=n+1}^{s-1} (1 + c(i,n)\tilde{b}(i,n)) \\ &\le Y(m,n) \\ &[1 + \tilde{b}(m,n) \sum_{s=m+1}^{\beta} c(s,n) \prod_{i=n+1}^{s-1} (1 + c(i,n)\tilde{b}(i,n))] \end{aligned} \tag{11}$$

where we apply the fact that $Y(m,n)$ is nonincreasing in m. For convenience, let

$$\begin{aligned} &p(m,n) = 1 + \\ &\tilde{b}(m,n) \sum_{s=m+1}^{\beta} c(s,n) \prod_{i=n+1}^{s-1} (1 + c(i,n)\tilde{b}(i,n)), \end{aligned} \tag{12}$$

where it can be seen easily that $p(m,n)$ is also increasing in m and n. So (11) can be rewritten as

$$\begin{aligned} &u(m,n) \le Y(m,n) p(m,n) \\ &= p(m,n)\tilde{a}(m,n) \\ &\quad +p(m,n) \sum_{k=1}^{2} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \tilde{f}_k(m,n,s,t) w_k(u(s,t)) \\ &= p(m,n)\tilde{a}(m,n) \\ &\quad + \sum_{k=1}^{2} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_k(m,n,s,t) w_k(u(s,t)), \end{aligned} \tag{13}$$

where

$$F_k(m,n,s,t) = p(m,n) \tilde{f}_k(m,n,s,t). \tag{14}$$

Obviously, $p(m,n)\tilde{a}(m,n)$ and $F_k(m,n,s,t)$ $(k = 1, 2)$ are nonincreasing in m and n.

Take any arbitrary positive integers M and N with $M \ge M_1, N \ge N_1$. From (13) we have an auxiliary inequality as follows

$$\begin{aligned} u(m,n) &\le g_1(M,N,m,n) \\ &+ \sum_{k=1}^{2} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_k(M,N,s,t) w_k(u(s,t)) \end{aligned} \tag{15}$$

for $m \ge M$, $n \ge N$, where $g_1(M,N,m,n)$ = $p(m,n)\tilde{a}(m,n)$. Let

$$z(m,n) = \sum_{k=1}^{2} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_k(M,N,s,t) w_k(u(s,t)). \tag{16}$$

Then $z(\infty,n)=0$. $z(m,n)$ is nonnegative and nonincreasing in m and n and satisfies

$$u(m,n)\le g_1(M,N,m,n)+z(m,n). \quad (17)$$

Moreover,

$$\begin{aligned}&-\Delta_1 z(m,n)\\&=\sum_{k=1}^{2}\sum_{t=n+1}^{\infty}F_k(M,N,m+1,t)w_k(u(m+1,t))\\&\le\sum_{k=1}^{2}\sum_{t=n+1}^{\infty}F_k(M,N,m+1,t)\\&\qquad w_k(g_1(M,N,m+1,t)+z(m+1,t)). \quad (18)\end{aligned}$$

Since $\Delta_3 g_1(M,N,m,n)=\Delta_1 p(m,n)\tilde{a}(m,n)$ is nonpositive and nondecreasing in m and n, so

$$\begin{aligned}&-\frac{\Delta_1 z(m,n)+\Delta_3 g_1(M,N,m,n)}{w_1(z(m+1,n)+g_1(M,N,m+1,n))}\\&\le\sum_{t=n+1}^{\infty}F_1(M,N,m+1,t)\\&\quad+\sum_{t=n+1}^{\infty}[F_2(M,N,m+1,t)\\&\quad\phi_2(g_1(M,N,m+1,t)+z(m+1,t))]\\&\qquad-\frac{\Delta_3 g_1(M,N,m,n)}{w_1(z(m+1,n)+g_1(M,N,m+1,n))}\\&\le\sum_{t=n+1}^{\infty}F_1(M,N,m+1,t)\\&\quad+\sum_{t=n+1}^{\infty}[F_2(M,N,m+1,t)\\&\quad\phi_2(g_1(M,N,m+1,t)+z(m+1,t))]\\&\quad-\frac{\Delta_3 g_1(M,N,m,n)}{w_1(z(m+1,n)+g_1(\infty,\infty,m+1,\infty))} \quad (19)\end{aligned}$$

where $\phi_2(u)=\frac{w_2(u)}{w_1(u)}$. Noticing that

$$\begin{aligned}&-\int_{z(m,n)+g_1(M,N,m,n))}^{z(m+1,n)+g_1(M,N,m+1,n)}\frac{d\tau}{w_1(\tau)}\\&\le-\int_{z(m,n)+g_1(M,N,m,n)}^{z(m+1,n)+g_1(M,N,m+1,n)}\\&\frac{d\tau}{w_1(z(m+1,n)+g_1(M,N,m+1,n))}\\&=-\frac{\Delta_1 z(m,n)+\Delta_3 g_1(M,N,m,n)}{w_1(z(m+1,n)+g_1(M,N,m+1,n))}\\&\le\sum_{t=n+1}^{\infty}F_1(M,N,m+1,t)\\&+\sum_{t=n+1}^{\infty}[F_2(M,N,m+1,t)\\&\phi_2(g_1(M,N,m+1,t)+z(m+1,t))]\\&\qquad-\frac{\Delta_3 g_1(M,N,m,n)}{w_1(g_1(\infty,\infty,m+1,\infty))},\end{aligned}$$

we have

$$\begin{aligned}&-\int_{z(m,n)+g_1(M,N,m,n)}^{z(\infty,n)+g_1(M,N,\infty,n)}\frac{d\tau}{w_1(\tau)}\\&=-\sum_{s=m}^{\infty}\int_{z(s,n)+g_1(M,N,s,n)}^{z(s+1,n)+g_1(M,N,s+1,n)}\frac{d\tau}{w_1(\tau)}\\&\le\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}F_1(M,N,s,t)\\&+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[F_2(M,N,s,t)\\&\quad\phi_2(g_1(M,N,s,t)+z(s,t))]\\&\qquad-\sum_{s=m}^{\infty}\frac{\Delta_3 g_1(M,N,s,n)}{w_1(g_1(\infty,\infty,s+1,\infty))}.\end{aligned}$$

The definition of W_1 and $z(\infty,n)=0$ show

$$\begin{aligned}&W_1(z(m,n)+g_1(M,N,m,n))\\&\le W_1(g_1(M,N,\infty,n))+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}F_1(M,N,s,t)\\&\quad+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[F_2(M,N,s,t)\\&\quad\phi_2(g_1(M,N,s,t))+z(s,t))]\\&\qquad-\sum_{s=m}^{\infty}\frac{\Delta_3 g_1(M,N,s,n)}{w_1(g_1(\infty,\infty,s+1,\infty))} \quad (20)\end{aligned}$$

or equivalently

$$\begin{aligned}&\Xi(m,n)\le g_2(M,N,m,n)\\&+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}F_2(M,N,s,t)\phi_2(W_1^{-1}(\Xi(s,t))) \quad (21)\end{aligned}$$

where

$$\begin{aligned}&\Xi(m,n)=W_1(z(m,n)+g_1(M,N,m,n)),\\&g_2(M,N,m,n)=W_1(g_1(M,N,\infty,n))\\&\qquad+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}F_1(M,N,s,t)\\&\qquad-\sum_{s=m}^{\infty}\frac{\Delta_3 g_1(M,N,s,n)}{w_1(g_1(\infty,\infty,s+1,\infty))}. \quad (22)\end{aligned}$$

Letting

$$\begin{aligned}&\tilde{z}(m,n)\\&=\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}F_2(M,N,s,t)\phi_2(W_1^{-1}(\Xi(s,t))), \quad (23)\end{aligned}$$

from (21), we get

$$\Xi(m,n)\le g_2(M,N,m,n)+\tilde{z}(m,n). \quad (24)$$

On one hand,

$$-\int_{\tilde{z}(m,n)+g_2(M,N,m,n)}^{\tilde{z}(m+1,n)+g_2(M,N,m+1,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))}$$
$$\leq -\int_{\tilde{z}(m,n)+g_2(M,N,m,n)}^{\tilde{z}(m+1,n)+g_2(M,N,m+1,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tilde{z}(m+1,n)+g_2(M,N,m+1,n)))}$$
$$= -\frac{\Delta_1\tilde{z}(m,n)+\Delta_3 g_2(M,N,m,n)}{\phi_2(W_1^{-1}(\tilde{z}(m+1,n)+g_2(M,N,m+1,n)))}$$
$$\leq \sum_{t=n+1}^{\infty} F_2(M,N,m+1,t) - \frac{\Delta_3 g_2(M,N,m,n)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,m+1,\infty)))},$$

on the other hand,

$$\int_{h(m,n)}^{h(m+1,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))} = \int_{h(m,n)}^{h(m+1,n)} \frac{w_1(W_1^{-1}(\tau))d\tau}{w_2(W_1^{-1}(\tau))}$$
$$= \int_{W_1^{-1}(h(m,n))}^{W_1^{-1}(h(m+1,n))} \frac{d\tau}{w_2(\tau)}$$
$$= W_2(W_1^{-1}(h(m+1,n))) - W_2(W_1^{-1}(h(m,n))).$$

Therefore,

$$-\int_{\tilde{z}(m,n)+g_2(M,N,m,n)}^{\tilde{z}(\infty,n)+g_2(M,N,\infty,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))}$$
$$\leq -\sum_{s=m}^{\infty}\int_{\tilde{z}(s,n)+g_2(M,N,s,n))}^{\tilde{z}(s+1,n)+g_2(M,N,s+1,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))}$$
$$\leq \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} F_2(M,N,s,t) - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(M,N,s,n)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,s+1,\infty)))}, \quad (25)$$

that is,

$$W_2(W_1^{-1}(\tilde{z}(m,n)+g_2(M,N,m,n)))$$
$$\leq W_2(W_1^{-1}(\tilde{z}(\infty,n)+g_2(M,N,\infty,n))) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} F_2(M,N,s,t) - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(M,N,s,n)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,s+1,\infty)))}$$
$$= W_2(g_1(M,N,\infty,n)) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} F_2(M,N,s,t) - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(M,N,s,n)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,s+1,\infty)))}, \quad (26)$$

where we apply the fact that $\tilde{z}(\infty,n) = 0$ and $g_2(M,N,\infty,n) = W_1(g_1(M,N,\infty,n))$. From (17) and (22), we get

$$u(m,n) \leq g_1(M,N,m,n) + z(m,n) = W_1^{-1}(\Xi(m,n))$$
$$\leq W_1^{-1}(g_2(M,N,m,n)+\tilde{z}(m,n))$$
$$\leq W_2^{-1}[W_2(g_1(M,N,\infty,n)) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} F_2(M,N,s,t) - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(M,N,s,n)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,s+1,\infty)))}] \quad (27)$$

for $m \geq M$ and $n \geq N$. Replacing m and n by M and N in (27) respectively, we have

$$u(M,N) \leq W_2^{-1}[W_2(g_1(M,N,\infty,n)) + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty} F_2(M,N,s,t) - \sum_{s=M}^{\infty} \frac{\Delta_3 g_2(M,N,s,N)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,s+1,\infty)))}].$$

Since M and N are arbitrary, we replace M and N by m and n and get

$$u(m,n) \leq W_2^{-1}[W_2(g_1(m,n,\infty,n)) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} F_2(m,n,s,t) - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(m,n,s,n)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,s+1,\infty)))}] \quad (28)$$

for $m \geq M_1$ and $n \geq N_1$. From (12) and (14), $g_1(M,N,\infty,n)$=$p(\infty,n)\tilde{a}(\infty,n) = \tilde{a}(\infty,n)$. Hence, it follows from (28) that

$$u(m,n) \leq W_2^{-1}[W_2(\tilde{a}(\infty,n)) + p(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} \tilde{f}_2(m,n,s,t) - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(m,n,s,n)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,s+1,\infty)))}],$$

This completes the proof. □

IV. Application to a difference equation

Consider a nonlinear difference equation

$$v(m,n) = \alpha(m,n) + \sum_{s=m+1}^{\beta} G(s,n,v(s,n)) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} H(m,n,s,t,v(s,t)) \quad (29)$$

for $m,n \in \mathbf{N_0}$, where $v(m,n)$ is an unknown function for $m,n \in \mathbf{N_0}$. Suppose that

$$|G(m,n,v(m,n))| \leq c(m,n)|v(m,n)|,$$
$$|H(m,n,s,t,v(s,t))| \leq f_1(m,n,s,t)w_1(|v(s,t)|) + f_2(m,n,s,t)w_2(|v(s,t)|),$$

where $w_1 \propto w_2$ holds. If $v(m,n)$ is a solution of (29), then

$$u(m,n) \le a(m,n) + \sum_{s=m+1}^{\beta} c(s,n)u(s,n) + \sum_{k=1}^{2}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} f_k(m,n,s,t)w_k(u(s,t)), \quad (30)$$

where $u(m,n) = |v(m,n)|$ and $a(m,n) = |\alpha(m,n)|$. Applying Theorem 2.1, we have

$$u(m,n) \le W_2^{-1}[W_2(\tilde{a}(\infty,n)) + p(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} \tilde{f}_2(m,n,s,t) + \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(m,n,s,n)}{\phi_2(W_1^{-1}(g_2(\infty,\infty,s+1,\infty)))}] \quad (31)$$

where $p(m,n) = 1 + \sum_{s=m+1}^{\beta} c(s,n)\prod_{i=n+1}^{s-1}(1+c(i,n))$ and other functions are defined in Theorem 2.1. Clearly, (31) implies the boundedness of solutions of equation (29).

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