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Heuristic method for judging the computational stability of the difference schemes of the Biharmonic equation

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Abstract—In this paper, we research the standard 13−point difference schemes for solving the biharmonic equation. Heuristic method is applied to judging the stability of multi-level difference schemes of the biharmonic equation. It is showed that the standard 13−point difference schemes are stable.

Keywords—Finite-difference equation; Computational stability; Hirt method.

I. Introduction

The computational stability of finite-difference equations has been studied for many years. It is so important in many subdisciplines, for example, computational mechanics, numerical weather prediction, computational physics, etc. To solve these problem, there are energy method, Von Neumann method, Fourier method and Heuristic[5], [6] method. The Heuristic method is our main concern, which is a method of approximate analysis. The biharmonic equation[1], [2], [3] which is one of the most important partial differential equations is applied in all subject areas of fundamental importance to the engineering sciences such as the theory of elasticity, mechanics of elastic plates and fracture mechanics. In general, the above problem can be solved by either a finite difference method[3], [4] or a finite element method. The finite difference method always seems to be the best choice not only for regular regions but also for more complex regions when appropriate extrapolation techniques are applied to the boundaries. In this paper, we research the standard 13−point difference schemes for solving the biharmonic equation and discuss how to use Heuristic method to analyze correctly the stability of multi-level difference schemes.

II. The standard 13-point difference schemes

In this section, we consider the biharmonic equation on the unit square $[0,1]^2$ with the mixed type of the clamped and the support boundary conditions,

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = f, in\ S, \tag{1}$$

$$u = g, on\ \Gamma, \tag{2}$$

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$$\frac{\partial u}{\partial n} = g^* on\ \Gamma_D, \frac{\partial^2 u}{\partial n^2} = g^{**} on\ \Gamma_N, \tag{3}$$

where $\Gamma = \partial S = AB \cup BC \cup CD \cup DA$, $\Gamma_D = AB \cup CD \cup DA$ and $\Gamma_N = BC$. The functions in (1)-(3) are supposed to be bounded

$$||f||_{0,S} \le C,\ ||g||_{0,\Gamma} \le C,\ ||g^*||_{0,\Gamma_D} \le C,\ ||g^{**}||_{0,\Gamma_N} \le C, \tag{4}$$

where C is a bounded constant, and

$$||f||_{0,S} = \sqrt{\int\int_s f^2 ds},\ ||g||_{0,\Gamma} = \sqrt{\int_\Gamma g^2 dl}. \tag{5}$$

Divide S by the uniform difference grids $x_i = ih, y_j = jh$, where $h = 1/N$. We use the standard 13-point finite difference equations (FDEs) to solve (1)-(3). The interior difference equations can be easily obtained as

$$\begin{aligned} &20u_{i,j} - 8[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}] \\ &+2[u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1}] \\ &+u_{i+2,j} + u_{i-2,j} + u_{i,j+2} + u_{i,j-2} = h^4 f_{i,j}, 3 \le i,j \le N-3. \end{aligned} \tag{6}$$

For the boundary of FDEs at $i = 2, N-2$ or $j = 2, N-2$, some values in (2) are known, to give a nontrivial contribution of the vector b. Take $(N-2, j)$ for example, we have the boundary of FDEs at $(N-2, j)$,

$$\begin{aligned} &2[u_{N-1,j+1} + u_{N-1,j-1} + u_{N-3,j+1} + u_{N-3,j-1}] \\ &+20u_{N-2,j} + u_{N-4,j} + u_{N-2,j+2} + u_{N-2,j-2} \\ &-8[u_{N-1,j} + u_{N-3,j} + u_{N-2,j+1} + u_{N-2,j-1}] \\ &= h^4 f_{N-2,j} - g_{N,j}, 3 \le j \le N-3, \end{aligned} \tag{7}$$

where $g_{i,j} = g(ih, jh)$. For the boundary of FDEs at $i = 1, N-1$ or $j = 1, N-1$, we have to use the boundary conditions in (3). Take $(N-1, j)$ for example. From (3), we have the approximation

$$u_{N+1,j} = u_{N-1,j} + 2hg^*_{N,j}. \tag{8}$$

Then for $(N-1, j)$ we obtain the boundary of FDEs from (6) and (8),

$$\begin{aligned} &2[u_{N-2,j+1} + u_{N-2,j-1}] + u_{N-3,j} + u_{N-1,j+2} + u_{N-1,j-2} \\ &+21u_{N-1,j} - 8[u_{N-2,j} + u_{N-1,j} + u_{N-1,j+1}] \\ &= h^4 f_{N-1,j} + 8g_{N,j} - 2[g_{N,j+1} + g_{N,j-1}] - 2hg^*_{N,j}, 3 \le j \le N-3. \end{aligned} \tag{9}$$

The FDEs at $(1,j)$ and $(i,1)$ can be obtained similarly.

Nextly, consider the boundary of FDEs at $\overline{BC}$. An approximation is obtained from (3)

$$u_{i,N+1} - 2u_{i,N} + u_{i,N-1} = h^2 g^{**}_{i,N}, \qquad (10)$$

to give

$$u_{i,N+1} = 2g_{i,N} - u_{i,N-1} + h^2 g^{**}_{i,N}. \qquad (11)$$

Hence the boundary of FDEs at $(i, N-1)$ are obtained from (6) and (11)

$$19u_{i,N-1} - 8[u_{i+1,N-1} + u_{i-1,N-1} + u_{i,N-2}]$$

$$+2[u_{i+1,N-2} + u_{i-1,N-2}] + u_{i+2,N-1} + u_{i-2,N-1} + u_{i,N-3}$$

$$= h^4 f_{i,N-1} + 6g_{i,N} - 2[g_{i+1,N} + g_{i-1,N}] - h^2 g^{*}_{i,N}, 3 \le i \le N-3. \qquad (12)$$

For the (i,j) near the corners, we may obtain difference equations similarly. Take the corner $(N-1, N-1)$ for example. We obtain the boundary of FDEs at $(N-1, N-1)$ from (6), (8) and (11),

$$20u_{N-1,N-1} - 8[u_{N-2,N-1} + u_{N-1,N-2}]$$

$$+2u_{N-2,N-2} + u_{N-3,N-1} + u_{N-1,N-3}$$

$$= h^4 f_{N-1,N-1} + 6g_{N-1,N} - 2hg^{*}_{N,N-1} - h^2 g^{**}_{N-1,N}$$

$$+8g_{N,N-1} - 2(g_{N,N} + g_{N,N-2} + g_{N-2,N}). \qquad (13)$$

For other corner nodes such as $(1,1)$, $(1, N-1)$ and $(N-1, 1)$, we can obtain the similar boundary of FDEs easily. From (6), (7), (9) and (12), we can also obtain all the FDEs for the other special nodes: $(2,2)$, $(2, N-2)$, $(N-2, 2)$, $(N-2, N-2)$, $(1,2)$, $(2,1)$, $(1, N-2)$, $(N-2, 1)$, $(2, N-1)$, $(N-1, 2)$, $(N-2, N-1)$ and $(N-1, N-2)$. In summary, we may write all those FDEs as the forms in matrix and vectors,

$$Ax = b, \qquad (14)$$

where

$$A = \begin{pmatrix} E_1 & F & I & & & & \\ F & E_2 & F & I & & & \\ I & F & E_2 & F & I & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & I & F & E_2 & F & I \\ & & & I & F & E_2 & F \\ & & & & I & F & E_1 \end{pmatrix},$$

with

$$E_1 = \begin{pmatrix} 20 & -8 & 1 & & & & \\ -8 & 21 & -8 & 1 & & & \\ 1 & -8 & 21 & -8 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -8 & 21 & -8 & 1 \\ & & & 1 & -8 & 21 & -8 \\ & & & & 1 & -8 & 20 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 19 & -8 & 1 & & & & \\ -8 & 20 & -8 & 1 & & & \\ 1 & -8 & 20 & -8 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -8 & 20 & -8 & 1 \\ & & & 1 & -8 & 20 & -8 \\ & & & & 1 & -8 & 19 \end{pmatrix},$$

$A \in R^{(N-1)^2 \times (N-1)^2}$, $E_1,\ E_2,\ I \in R^{(N-1)\times(N-1)}$ and I be the identity matrix, F is a tridiagonal $(N-1)\times(N-1)$ matrix with defined recursively(using Matlab notation) as

$$F = diag(a_1, 0) + diag(b_1, -1) + diag(c_1, 1),$$

where $a_1 = -8 * ones(N-1, 1)$, $b_1 = 2 * ones(N-2, 1)$ *and* $c_1 = 2 * ones(N-2, 1)$. The unknown vector $x = (u_{1,1}, \cdots, u_{1,N-1}, \cdots\cdots, u_{N-1,1}, \cdots, u_{N-1,N-1})^T$. The vector b has been known.

The biharmonic equation can be equivalent to the following difference operator equation

$$\Delta_h^2 u(x_i, y_j) = \frac{1}{h^4}(\delta_x^4 u_{i,j} + 2\delta_x^2\delta_y^2 u_{i,j} + \delta_y^4 u_{i,j}) \qquad (15)$$

where $\delta_x^4 = \delta_x^2(\delta_x^2)$, $\delta_y^4 = \delta_y^2(\delta_y^2)$, $\delta_x^2 = u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)$, $\delta_y^2 = u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})$.

III. The stability of the finite difference schemes

In this section we discuss how to use Heuristic stability analysis method to analyze correctly the stability of the finite difference schemes. This method can be used to judge the stabilities of difference schemes by deleting the high order error and lefting the lowest error item in the Taylor series expansion of the finite-difference schemes in some fixed point.

Theorem 3.1 *For the above finite difference schemes, they are of absolute stability.*

Proof. We consider the biharmonic equation (1), i.e.

$$\Delta^2 u(x,y) = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0. \qquad (16)$$

Its corresponding finite difference schemes is

$$\frac{1}{h^2}\{[\frac{\partial^2}{\partial x^2}u(x_{i+1}, y_j) - 2\frac{\partial^2}{\partial x^2}u(x_i, y_j) + \frac{\partial^2}{\partial x^2}u(x_{i-1}, y_j)]$$

$$+2 \times [\frac{\partial^2}{\partial y^2}u(x_{i+1}, y_j) - 2\frac{\partial^2}{\partial y^2}u(x_i, y_j) + \frac{\partial^2}{\partial y^2}u(x_{i-1}, y_j)]$$

$$+[\frac{\partial^2}{\partial y^2}u(x_i, y_{j+1}) - 2\frac{\partial^2}{\partial y^2}u(x_i, y_j) + \frac{\partial^2}{\partial y^2}u(x_i, y_{j-1})]\} = 0. \qquad (17)$$

Using Taylor series expansion at point $(x_i,\ y_j)$, we have

$$\frac{\frac{\partial^2}{\partial x^2}u(x_{i+1}, y_j) - 2\frac{\partial^2}{\partial x^2}u(x_i, y_j) + \frac{\partial^2}{\partial x^2}u(x_{i-1}, y_j)}{h^2}$$

$$= [\frac{\partial^4}{\partial x^4}u(x_i, y_j)]_i^j + \frac{h^2}{6}[\frac{\partial^6}{\partial x^6}u(x_i, y_j)]_i^j + O(h^2) \qquad (18)$$

$$\frac{\frac{\partial^2}{\partial y^2}u(x_i,y_{j+1})-2\frac{\partial^2}{\partial y^2}u(x_i,y_j)+\frac{\partial^2}{\partial y^2}u(x_i,y_{j-1})}{h^2}$$
$$=[\frac{\partial^4}{\partial y^4}u(x_i,y_j)]_i^j+\frac{h^2}{6}[\frac{\partial^6}{\partial y^6}u(x_i,y_j)]_i^j+O(h^2) \quad (19)$$
$$\frac{\frac{\partial^2}{\partial y^2}u(x_{i+1},y_j)-2\frac{\partial^2}{\partial y^2}u(x_i,y_j)+\frac{\partial^2}{\partial y^2}u(x_{i-1},y_j)}{h^2}$$
$$=[\frac{\partial^4 u(x_i,y_j)}{\partial x^2\partial y^2}]_i^j+\frac{h^2}{12}[\frac{\partial^6 u(x_i,y_j)}{\partial x^2\partial y^4}+\frac{\partial^6 u(x_i,y_j)}{\partial x^4\partial y^2}]_i^j+O(h^2). \quad (20)$$

Because of
$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}=-2\frac{\partial^4 u}{\partial x^2\partial y^2}$$
and (17)-(20), we obtain
$$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}=-\frac{h^2}{6}(\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial y^6}+\frac{\partial^6 u}{\partial x^2\partial y^4}+\frac{\partial^6 u}{\partial x^4\partial y^2})$$
$$=-\frac{h^2}{6}[\frac{\partial^2}{\partial x^2}(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4})+\frac{\partial^2}{\partial y^2}(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4})]$$
$$=\frac{h^2}{3}[\frac{\partial^6 u}{\partial x^2\partial y^4}+\frac{\partial^6 u}{\partial x^4\partial y^2}]+O(h^2). \quad (21)$$

After deleting the high order error item, we have
$$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}=\frac{h^2}{3}[\frac{\partial^6 u}{\partial x^2\partial y^4}+\frac{\partial^6 u}{\partial x^4\partial y^2}]. \quad (22)$$

Since $h^2/3>0$, the above finite difference schemes are of absolute stability. The proof of Theorem 3.1 is completed.

The heuristic analysis method is proved to be effective for the computational stability analysis of the difference schemes for the biharmonic equation.

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