International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:8, 2011

A C^1-Conforming Finite Element Method for Nonlinear Fourth-Order hyperbolic Equation

Yang Liu, Hong Li, Siriguleng He, Wei Gao, Zhichao Fang

Abstract—In this paper, the C^1-conforming finite element method is analyzed for a class of nonlinear fourth-order hyperbolic partial differential equation. Some a priori bounds are derived using Lyapunov functional, and existence, uniqueness and regularity for the weak solutions are proved. Optimal error estimates are derived for both semidiscrete and fully discrete schemes.

Keywords—Nonlinear fourth-order hyperbolic equation; Lyapunov functional; Existence, uniqueness and regularity; Conforming finite element method; Optimal error estimates.

I. Introduction

In this paper, the C^1-conforming finite element method is analyzed for the following fourth-order hyperbolic equation

$$u_{tt}+\gamma\Delta^2 u-\Delta u-\Delta u_t+f(u)=0,(x,t)\in\Omega\times J, \quad (1)$$

with boundary conditions

$$u(x,t)=0,\frac{\partial u}{\partial \nu}=0,\ (x,t)\in\partial\Omega\times J, \quad (2)$$

or

$$u(x,t)=0,\Delta u=0,\ (x,t)\in\partial\Omega\times J, \quad (3)$$

and initial condition

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x)\ x\in\Omega, \quad (4)$$

where $f(u)=u^3-u$ and Ω is a bounded domain in $R^d, d\le 2$ with boundary $\partial\Omega$, $J=(0,T]$ with $0<T<\infty$.

In recent years, much attention has been given in the literature to the numerical solution of fourth-order partial differential equations. In [1], the finite element method was studied for the fourth-order Cahn-Hilliard equation. In [2], the stabilized finite element approximation for fourth order obstacle problem was discussed. In [3], the approximation for the fourth-order eigen-value problem with cubic Hermite elements on anisotropic meshes was investigated. Chen [4] proposed the expanded mixed finite element method for fourth-order elliptic equations. In [5], [6], the mixed time discontinuous finite element method was proposed for fourth-order parabolic partial differential equations. Li [7], [8], [9], [10], discussed some mixed finite element methods for fourth-order elliptic problems and parabolic problems. In [11], a new nonconforming element constructed by the Double Set Parameter method, is applied to the fourth order elliptic singular perturbation problem. In [12], the C^1-conforming finite element method is analyzed for the extended Fisher-Kolmogorov (EFK) equation, and some numerical results are given to illustrate the efficiency of the method.

In this paper, our purpose is to discuss and analyze the C^1-conforming finite element method[12] for the fourth-order hyperbolic equation. We derive some a priori bounds by using Lyapunov functional, and prove the existence, uniqueness and regularity for weak solutions, and obtain the optimal error estimates for both the semidiscrete and fully discrete schemes. Throughout this paper, C will denote a generic positive constant which does not depend on the spatial mesh parameter h and time discretization parameter Δt.

1.School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China. e-mail: mathliuyang@yahoo.cn(Y.Liu); smslh@imu.edu.cn(H.Li); cmml2005@163.com; mathgao@mail.ustc.edu.cn.
Manuscript received Aug 8, 2011.

II. Existence, uniqueness and regularity for weak solutions

In this section, we give existence uniqueness and regularity results for the fourth-order hyperbolic equation.

Take L^2-inner product of (1) with $v\in H_0^2$ and apply Green's formula to obtain the following weak formulation

$$\begin{aligned}(u_{tt},v)+\gamma(\Delta u,\Delta v)+(\nabla u,\nabla v)+(\nabla u_t,\nabla v)\\ +(f(u),v)=0,v\in H_0^2(\Omega),\end{aligned} \quad (5)$$

with $u(0)=u_0, u_t(0)=u_1$.
To prove existence and uniqueness results, we will derive a priori bound.

Lemma 2.1: Assume that $u_0\in H_0^2$ and $u_1\in H^1$. Then there exists a positive constant C such that

$$||u(t)||_2+||u(t)||_\infty+||u_t||_1\le C(\gamma,||u_0||_2,||u_1||_1),\ t>0.$$

Proof. We consider the Lyapunov functional $\Xi(v)$ as

$$\Xi(v)=\int_\Omega\{\frac{\gamma}{2}|\Delta v|^2+\frac{1}{2}|\nabla v|^2+\frac{1}{2}|\nabla v_t|^2+F(v)\}dx, \quad (6)$$

where

$$F(v)=\frac{1}{4}(1-v^2)^2.$$

Note that $F'=f$. Differentiating (6) with respect to t and using (5), we obtain

$$\begin{aligned}\frac{d}{dt}\Xi(u)&=\gamma(\Delta u,\Delta u_t)+(\nabla u,\nabla u_t)+(\nabla u_t,\nabla u_{tt})\\ &+(f(u),u_t)=-\frac{1}{2}\frac{d}{dt}||u_t||^2,\end{aligned} \quad (7)$$

Integrating (7) from 0 to t, we obtain

$$\begin{aligned}\Xi(u)&=\Xi(u(0))+2||u_t(0)||^2-2||u_t||^2\\ &\le\Xi(u(0))+2||u_t(0)||^2,\\ ||u_t||^2&=\frac{1}{2}\Xi(u(0))+||u_t(0)||^2-\frac{1}{2}\Xi(u)\\ &\le\frac{1}{2}\Xi(u(0))+||u_t(0)||^2.\end{aligned} \quad (8)$$

Since $F(u) \geq 0$, using Poincaré inequality, we obtain

$$||u||_2 + ||u_t||_1 \leq C(\gamma, ||u_0||_2, ||u_1||_1). \quad (9)$$

Apply the Sobolev imbedding theorem to obtain

$$||u||_\infty \leq ||u||_{H^2} \leq C(\gamma, ||u_0||_2, ||u_1||_1). \quad (10)$$

In the following subsequent sections, we will discuss the global existence, uniqueness and regularity.

Theorem 2.2: Let $u_0 \in H_0^2(\Omega), u_1 \in H^1(\Omega)$. There exists a unique $u = u(x,t)$ in $\Omega \times (0,T]$ with

$$u \in L^\infty(J; H_0^2(\Omega)),\ u_t \in L^\infty(J; H^1(\Omega)).$$

such that u satisfies the initial condition $u(0) = u_0, u_t(0) = u_1$ and the equation (5) in the sense that

$$\begin{aligned}(u_{tt}, v) + \gamma(\Delta u, \Delta v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v)\\ +(f(u), v) = 0, v \in H_0^2(\Omega), t \in (0,T].\end{aligned} \quad (11)$$

Proof. Use the similar method to [12] to obtain the existence of equation (5). Below, we will prove the uniqueness of weak solutions.

Suppose u and w are two solutions of (5). Taking $\sigma = u - w$, we can obtain

$$\begin{aligned}(\sigma_{tt}, v) + \gamma(\Delta\sigma, \Delta v) + (\nabla\sigma, \nabla v) + (\nabla\sigma_t, \nabla v)\\ +(f(u) - f(w), v) = 0, v \in H_0^2(\Omega).\end{aligned} \quad (12)$$

Setting $v = \sigma_t$ and using the boundedness of $||u||_\infty$ and $||w||_\infty$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||\sigma_t||^2 + \gamma||\Delta\sigma||^2 + ||\nabla\sigma||^2) + ||\nabla\sigma_t||^2\\ &= -(f(u) - f(w), \sigma_t)\\ &= -(f'(\xi)\sigma, \sigma_t)\\ &= -(f'(\xi)\int_0^t \sigma_t(s)ds, \sigma_t)\\ &\leq C(\int_0^t ||\sigma_t(s)||^2 ds + ||\sigma_t||^2).\end{aligned} \quad (13)$$

Integrating (13) from 0 to t, we obtain

$$\begin{aligned}||\sigma_t||^2 + \gamma||\Delta\sigma||^2 + ||\nabla\sigma||^2 + \int_0^t ||\nabla\sigma_t||^2 ds\\ \leq C\int_0^t ||\sigma_t(s)||^2 ds.\end{aligned} \quad (14)$$

Using the Gronwall Lemma, we have $||\sigma_t||^2 = 0$ and $\int_0^t ||\sigma_t(s)||^2 ds = 0$, which imply $\sigma_t = 0$.
Note that $\sigma(0) = u(0) - w(0) = 0$ to have $\sigma = \int_0^t \sigma_t dt = 0$.

III. Error estimates for semidiscrete schemes

In this section, we apply Galerkin procedure for the fourth-order hyperbolic equation and obtain the semidiscrete scheme and a priori error estimates.

Let $S_h^0, 0 < h < 1$ be a family of finite dimensional subspace of H_0^2 with the following approximation property[14]: For $v \in H^4(\Omega) \cap H_0^2(\Omega)$, there exists a constant C independent of h such that

$$\inf_{\chi \in S_h^0} ||v - \chi||_j \leq Ch^{4-j}||v||_4, j = 0, 1, 2. \quad (15)$$

The corresponding semidiscrete Galerkin approximation of (1)-(4) is defined to be a function $u_h : [0,T] \to S_h^0$ such that

$$\begin{aligned}(u_{htt}, v_h) + \gamma(\Delta u_h, \Delta v_h) + (\nabla u_h, \nabla v_h)\\ +(\nabla u_{ht}, \nabla v_h) + (f(u_h), v_h) = 0, v_h \in S_h^0,\end{aligned} \quad (16)$$

with $u_h(0) = u_{0,h}, u_{ht}(0) = u_{1,h}$.
To obtain optimal rate of convergence, following wheeler[13], we introduce $\tilde{u}$ be as an auxiliary projection of u defined by

$$A(u - \tilde{u}, v_h) = 0, v_h \in S_h^0, \quad (17)$$

where the bilinear form is introduced by

$$A(v,w) = \gamma(\Delta v, \Delta w) + (\nabla v, \nabla w) + (\nabla v_t, \nabla w), v, w \in H_0^2,$$

for our subsequent use note that $A(\cdot,\cdot)$ satisfies the following properties:
(i) Boundedness: There is a positive constant M such that

$$|A(v,w)| \leq M||v||_2||w||_2, v, w \in H_0^2.$$

(ii) Coercivity: There is a constant $\alpha_0 > 0$ such that

$$A(v,v) \geq ||v||_2, v \in H_0^2.$$

With $\eta = u - \tilde{u}$, the following estimates are well known [12], [13]: for $j = 0, 1, 2$

$$||\frac{\partial^l \eta}{\partial t^l}||_j \leq Ch^{4-j}\sum_{k=0}^{l}||\frac{\partial^k \eta}{\partial t^k}||_4. \quad (18)$$

Assuming quasi-uniformity condition on the triangulation, it is easy to check that

$$||\eta(t)||_{W^{j,\infty}} \leq Ch^{4-j}||u||_{4,\infty}, j = 0, 1. \quad (19)$$

For a priori error estimates, we decompose the errors as

$$u - u^h = u - \tilde{u} + \tilde{u} - u^h = \eta + \xi$$

Theorem 3.1: There exists a positive constant C independent of h such that

$$\begin{aligned}&||u_t - u_{ht}|| + ||\nabla(u_t - u_{ht})||_{L^2(J;L^2(\Omega))} + ||u - u_h||_{L^\infty(J;H^j(\Omega))}\\ &\leq Ch^{4-j}(||u||_{L^\infty(H^4)} + ||u_t||_{L^2(H^4)} + ||u_{tt}||_{L^2(H^4)}), 0 \leq j \leq 2\end{aligned}$$

Moreover, assuming quasi-uniformity condition on the triangulation $\mathcal{T}_h$, there exists a positive constant C independent of h such that

$$\begin{aligned}&||u - u_h||_{L^\infty(J;H^2(\Omega))}\\ &\leq Ch^4(||u||_{L^\infty(H^4)} + ||u_t||_{L^2(H^4)} + ||u_{tt}||_{L^2(H^4)}).\end{aligned}$$

Proof. Substracting (16) from (5) and using auxiliary projection, we obtain the following equation in ξ

$$\begin{aligned}(\xi_{tt}, v_h) + \gamma(\Delta\xi, \Delta v_h) + (\nabla\xi, \nabla v_h) + (\nabla\xi_t, \nabla v_h)\\ = -(\eta_{tt}, v_h) - (f(u) - f(u_h), v_h).\end{aligned} \quad (20)$$

Choosing $v_h = \xi_t$ in (20), and using the Cauchy-Schwarz's inequality and the Poincaré's inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||\xi_t||^2 + \gamma||\Delta\xi||^2 + ||\nabla\xi||^2) + ||\nabla\xi_t||^2\\ &=(\eta_{tt}, \xi_t) - (f'(\theta_u)(\eta + \xi), \xi_t)\\ &\leq C(||\eta_{tt}||^2 + ||\eta_t||^2 + ||\xi_t||^2 + ||\xi||^2)\\ &\leq C(||\eta_{tt}||^2 + ||\eta_t||^2 + ||\xi_t||^2 + ||\nabla\xi||^2).\end{aligned} \quad (21)$$

Integrating (21) from 0 to t, we obtain

$$\begin{aligned}&||\xi_t||^2+\gamma||\Delta\xi||^2+||\nabla\xi||^2+\int_0^t||\nabla\xi_t||^2ds\\&\le C\int_0^t(||\eta_{tt}||^2+||\eta_t||^2+||\xi_t||^2+||\nabla\xi||^2)ds.\end{aligned}\tag{22}$$

Using the Gronwall's lemma, we have

$$\begin{aligned}&||\xi_t||^2+\gamma||\Delta\xi||^2+||\nabla\xi||^2+\int_0^t||\nabla\xi_t||^2ds\\&\le C\int_0^t(||\eta_{tt}||^2+||\eta_t||^2)ds.\end{aligned}\tag{23}$$

Substituting the estimates of $||\eta_{tt}||, ||\eta_t||$ and using the Poincaré' inequality, we obtain the following superconvergence result for $||\xi||_2$

$$\begin{aligned}&||\xi_t||+||\nabla\xi_t||_{L^2(J;L^2(\Omega))}+||\xi||_{L^\infty(J;H^2(\Omega))}\\&\le Ch^4(||u_t||_{L^2(H^4)}+||u_{tt}||_{L^2(H^4)}).\end{aligned}\tag{24}$$

From the Sobolev embedding theorem, we have

$$||\xi||_{L^\infty}\le C||\xi||_2.$$

and hence,

$$\begin{aligned}&||\xi||_{L^\infty(J;L^\infty(\Omega))}\\&\le Ch^4(||u_t||_{L^2(H^4)}+||u_{tt}||_{L^2(H^4)}).\end{aligned}\tag{25}$$

We apply the triangle inequality to get the the conclusion.

IV. Fully discrete schemes and error estimates

In this Section, we briefly describe a fully discrete scheme for approximating the solution u of (1) and discuss a priori error bounds.

Let $0=t_0<t_1<t_2<\cdots<t_M=T$ be a given partition of the time interval $[0,T]$ with step length $t_n=n\Delta t, \Delta t=T/M$, for some positive integer M. We use the following notation related to functions defined at discrete time levels. For a smooth function ϕ on $[0,T]$, define

$$\phi^n=\phi(t_n),\ \phi^{n+\frac12}=\frac12(\phi^{n+1}+\phi^n),\ \partial_t\phi^{n+\frac12}=\frac{\phi^{n+1}-\phi^n}{\Delta t},$$

$$\partial_t\phi^n=\frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t},\partial_t^2\phi^{n+\frac12}=\frac{\partial_t\phi^{n+\frac12}-\partial_t\phi^{n-\frac12}}{\Delta t},$$

$$\phi^{n;\frac14}=\frac14(\phi^{n+1}+2\phi^n+\phi^{n-1})=\frac12(\phi^{n+\frac12}+\phi^{n-\frac12}),$$

$$G(\chi,\psi)=\frac{F(\chi)-F(\psi)}{\chi-\psi}\to f(\chi),(\psi\to\chi).$$

Let U^n be the approximations of u at $t=t_n$ which we shall define through the following scheme. Given U^{n-1} in V_h, we now determine a pair U^n in V_h satisfying

$$\begin{aligned}&(\frac{2}{\Delta t}\partial_tU^{\frac12},v_h)+\gamma(\Delta U^{\frac12},\Delta v_h)\\&+(\nabla U^{\frac12},\nabla v_h)+(\nabla\partial_tU^{\frac12},\nabla v_h)\\&+(G(U^1,U^0),v_h)=(\frac{2}{\Delta t}u_t(0),v_h),\ \forall\ v_h\in S_h^0.\end{aligned}\tag{26}$$

$$\begin{aligned}&(\partial_t^2U^{n+\frac12},v_h)+\gamma(\Delta U^{n;\frac14},\Delta v_h)+(\nabla U^{n;\frac14},\nabla v_h)\\&+(\nabla\partial_tU^n,\nabla v_h)+(G(U^{n+1},U^{n-1}),v_h)=0,v_h\in S_h^0,n\ge1,\end{aligned}\tag{27}$$

where $u_t(0)=u_1$.

Theorem 4.1: There exists a positive constant C such that

$$||U^{n+\frac12}||_\infty+||U^{n+\frac12}||_2\le C(||U^0||_2,||F(U^0)||,||u_1||,1)$$

Proof. Choosing $v_h=\partial_tU^n=\frac{U^{n+1}-U^{n-1}}{2\Delta t}=\frac{\partial_tU^{n+\frac12}+\partial_tU^{n-\frac12}}{2}=\frac{U^{n+\frac12}-U^{n-\frac12}}{\Delta t}$ in (27), we can get

$$\begin{aligned}&\frac{1}{2\Delta t}\Big(||\partial_tU^{n+\frac12}||^2-||\partial_tU^{n-\frac12}||^2+\gamma(||\Delta U^{n+\frac12}||^2\\&-||\Delta U^{n-\frac12}||^2)+||\nabla U^{n+\frac12}||^2-||\nabla U^{n-\frac12}||^2\\&+2\Delta t||\nabla\partial_tU^n||^2+(G(U^{n+1},U^{n-1}),U^{n+1}-U^{n-1})\Big)=0\end{aligned}\tag{28}$$

Using the definition $G(U^{n+1},U^{n-1})$, we obtain

$$\begin{aligned}&||\partial_tU^{n+\frac12}||^2+\gamma||\Delta U^{n+\frac12}||^2+||\nabla U^{n+\frac12}||^2\\&+2\Delta t||\nabla\partial_tU^n||^2+(F(U^{n+1}),1)\\=&||\partial_tU^{n-\frac12}||^2+\gamma||\Delta U^{n-\frac12}||^2+||\nabla U^{n-\frac12}||^2+(F(U^{n-1}),1)\end{aligned}\tag{29}$$

so, we can get

$$\begin{aligned}&||\partial_tU^{n+\frac12}||^2+\gamma||\Delta U^{n+\frac12}||^2+||\nabla U^{n+\frac12}||^2+(F(U^{n+1}),1)\\\le&||\partial_tU^{n-\frac12}||^2+\gamma||\Delta U^{n-\frac12}||^2+||\nabla U^{n-\frac12}||^2+(F(U^{n-1}),1)\\\le&||\partial_tU^{\frac12}||^2+\gamma||\Delta U^{\frac12}||^2+||\nabla U^{\frac12}||^2+(F(U^0),1)\end{aligned}\tag{30}$$

Choosing $v_h=\partial_tU^{\frac12}$ in (26), we can get

$$\begin{aligned}&4||\partial_tU^{\frac12}||^2+2\gamma(||\Delta U^1||^2-||\Delta U^0||^2)\\&+2(||\nabla U^1||^2-||\nabla U^0||^2)+2\Delta t||\nabla\partial_tU^{\frac12}||^2\\&+2(G(U^1,U^0),U^1-U^0)=4(u_t(0),\partial_tU^{\frac12})\end{aligned}\tag{31}$$

Using Cauchy-Schwartz inequality and Young inequality, we can get

$$\begin{aligned}&4||\partial_tU^{\frac12}||^2+2\gamma||\Delta U^1||^2+2||\nabla U^1||^2\\&+2\Delta t||\nabla\partial_tU^{\frac12}||^2+2(F(U^1),1)\\=&2\gamma||\Delta U^0||^2+2||\nabla U^0||^2+2(F(U^0),1)+4(u_t(0),\partial_tU^{\frac12})\\\le&2\gamma||U^0||_2^2+2(F(U^0),1)+4||u_1||^2+||\partial_tU^{\frac12}||^2.\end{aligned}\tag{32}$$

Noting that $F(U^1),F(U^0)\ge0$, we get

$$\begin{aligned}&3||\partial_tU^{\frac12}||^2+2\gamma||\Delta U^1||^2+2||\nabla U^1||^2\\&\le2\gamma||U^0||_2^2+2(F(U^0),1)+4||u_1||^2.\end{aligned}\tag{33}$$

Substitute (33) into (30) and use Poincaré inequality and $F(U^{n+1})\ge0$ to get

$$\begin{aligned}&||\partial_tU^{n+\frac12}||^2+||U^{n+\frac12}||_2^2\\&\le C(\gamma||U^0||_2^2+(F(U^0),1)+4||u_1||^2)<\infty.\end{aligned}\tag{34}$$

Use the Sobolev Imbedding theorem to get

$$\begin{aligned}&||\partial_tU^{n+\frac12}||^2+||U^{n+\frac12}||_\infty^2\\&\le C(\gamma||U^0||_2^2+(F(U^0),1)+4||u_1||^2)<\infty.\end{aligned}\tag{35}$$

For fully discrete error estimates, we now split the errors

$$u(t_n)-U^n=(u(t_n)-\tilde{u}(t_n))+(\tilde{u}(t_n)-U^n)=\eta^n+\xi^n$$

Use (5), (17), and (27), we then obtain

$$\begin{aligned}&(\frac{2}{\Delta t}\partial_t\xi^{\frac{1}{2}},v_h)+\gamma(\Delta\xi^{\frac{1}{2}},\Delta v_h)+(\nabla\xi^{\frac{1}{2}},\nabla v_h)\\&+(\nabla\partial_t\xi^{\frac{1}{2}},\nabla v_h)+(f(u(t_{\frac{1}{2}}))-G(U^1,U^0),v_h)\\=&-(\frac{2}{\Delta t}\partial_t\eta^{\frac{1}{2}}+2\tau^0,v_h)+(\nabla\varepsilon^0,\nabla v_h),\ \forall\ v_h\in S_h^0.\end{aligned}\quad(36)$$

$$\begin{aligned}&(\partial_t^2\xi^{n+\frac{1}{2}},v_h)+\gamma(\Delta\xi^{n;\frac{1}{4}},\Delta v_h)+(\nabla\xi^{n;\frac{1}{4}},\nabla v_h)\\&+(\nabla\partial_t\xi^n,\nabla v_h)+(f(u(t_n))-G(U^{n+1},U^{n-1}),v_h)\\=&-(\partial_t^2\eta^{n+\frac{1}{2}}+\tau^n,v_h)+(\nabla\varepsilon^n,\nabla v_h),v_h\in S_h^0,n\geq 1,\end{aligned}\quad(37)$$

where

$$\tau^0=\frac{1}{2}u_{tt}^{\frac{1}{2}}+\frac{1}{\Delta t}(u_t(0)-\partial_t u^{\frac{1}{2}})=O(\Delta t),$$

$$\tau^n=(u_{tt})^{n;\frac{1}{4}}-\partial_t^2u(t_n)=O(\Delta t^2),$$

and

$$\varepsilon^0=u_t^{\frac{1}{2}}-\partial_t u^{\frac{1}{2}}=O(\Delta t^2),\varepsilon^n=(u_t)^{n;\frac{1}{4}}-\partial_t u^n=O(\Delta t^2).$$

Theorem 4.2: Assume that $U^0=\tilde{u}_h(0)$. Then there exists a positive constant C independent of h and Δt such that for $j=0,1,2$ and $J=0,1,\cdots,M$

$$||u(t_{J+\frac{1}{2}})-U^{J+\frac{1}{2}}||_j\leq C(h^{4-j}+\Delta t^2).\quad(38)$$

Moreover, assume the quasi-uniformity condition on the triangulation T_h to obtain the following estimate:

$$||u(t_{J+\frac{1}{2}})-U^{J+\frac{1}{2}}||_\infty\leq C(h^4+\Delta t^2).\quad(39)$$

Proof. Set $v_h=\partial_t\xi^n=\frac{\partial_t\xi^{n+\frac{1}{2}}+\partial_t\xi^{n-\frac{1}{2}}}{2}$ in (37) and have used integration by parts on spaces to obtain

$$\begin{aligned}&(\partial_t^2\xi^{n+\frac{1}{2}},\partial_t\xi^n)+\gamma(\Delta\xi^{n;\frac{1}{4}},\Delta\partial_t\xi^n)+(\nabla\xi^{n;\frac{1}{4}},\nabla\partial_t\xi^n)\\&+(\nabla\partial_t\xi^n,\nabla\partial_t\xi^n)+(f(u(t_n))-G(U^{n+1},U^{n-1}),\partial_t\xi^n)\\&=-(\partial_t^2\eta^{n+\frac{1}{2}}+\tau^n,\partial_t\xi^n)+(\nabla\varepsilon^n,\nabla\partial_t\xi^n).\end{aligned}\quad(40)$$

Note that

$$(\partial_t^2\xi^{n+\frac{1}{2}},\partial_t\xi^n)=\frac{1}{2\Delta t}(||\partial_t\xi^{n+\frac{1}{2}}||^2-||\partial_t\xi^{n-\frac{1}{2}}||^2),$$

$$(\Delta\xi^{n;\frac{1}{4}},\Delta\partial_t\xi^n)=\frac{1}{2\Delta t}(||\Delta\xi^{n+\frac{1}{2}}||^2-||\Delta\xi^{n-\frac{1}{2}}||^2),$$

$$(\nabla\xi^{n;\frac{1}{4}},\nabla\partial_t\xi^n)=\frac{1}{2\Delta t}(||\nabla\xi^{n+\frac{1}{2}}||^2-||\nabla\xi^{n-\frac{1}{2}}||^2).$$

$$\begin{aligned}&||f(u(t_n))-G(U^{n+1},U^{n-1})||\\\leq&||f(u(t_n))-f(u_{n;\frac{1}{4}})||+||f(u_{n;\frac{1}{4}})-G(u(t_{n+1}),u(t_{n-1}))||\\&+||G(u(t_{n+1}),u(t_{n-1}))-G(U^{n+1},u(t_{n-1}))||\\&+||G(U^{n+1},u(t_{n-1}))-G(U^{n+1},U^{n-1})||\leq C\Delta t^2.\end{aligned}\quad(41)$$

Multiplying by $2\Delta t$ and summing from $n=1$ to J, we find that

$$\begin{aligned}&||\partial_t\xi^{J+\frac{1}{2}}||^2+||\Delta\xi^{J+\frac{1}{2}}||^2\\&+||\nabla\xi^{J+\frac{1}{2}}||^2+\Delta t\sum_{n=1}^{J}||\nabla\partial_t\xi^n||^2\\\leq\ &C\Big[||\partial_t\xi^{\frac{1}{2}}||+||\Delta\xi^{\frac{1}{2}}||^2+||\nabla\xi^{\frac{1}{2}}||^2+\Delta t\sum_{n=1}^{J}||\partial_t\xi^{n+\frac{1}{2}}||^2\\&+\Delta t\sum_{n=1}^{J}(||f(u(t_n))-G(U^{n+1},U^{n-1})||^2\\&+||\partial_t^2\eta^{n+\frac{1}{2}}+\tau^n||^2+||\nabla\varepsilon^n||^2)\Big].\end{aligned}\quad(42)$$

Use the Gronwall lemma and (41) to obtain

$$\begin{aligned}&||\partial_t\xi^{J+\frac{1}{2}}||^2+||\Delta\xi^{J+\frac{1}{2}}||^2+||\nabla\xi^{J+\frac{1}{2}}||^2+\Delta t\sum_{n=1}^{J}||\nabla\partial_t\xi^n||^2\\\leq&C\Big[||\partial_t\xi^{\frac{1}{2}}||+||\Delta\xi^{\frac{1}{2}}||^2+||\nabla\xi^{\frac{1}{2}}||^2+||\partial_t^2\eta^{n+\frac{1}{2}}||^2\\&+||\tau^n||^2+||\nabla\varepsilon^n||^2\Big]+C\Delta t^4\\\leq&C\Big[||\partial_t\xi^{\frac{1}{2}}||+||\Delta\xi^{\frac{1}{2}}||^2+||\nabla\xi^{\frac{1}{2}}||^2+||\partial_t^2\eta^{n+\frac{1}{2}}||^2\Big]+C\Delta t^4.\end{aligned}\quad(43)$$

For the estimation of the first three terms on the right-hand side of the above inequality, we now choose $w_h=\partial_t\xi^{\frac{1}{2}}$ in (36) to obtain

$$\begin{aligned}&(\frac{2}{\Delta t}\partial_t\xi^{\frac{1}{2}},\partial_t\xi^{\frac{1}{2}})+\gamma(\Delta\xi^{\frac{1}{2}},\Delta\partial_t\xi^{\frac{1}{2}})+(\nabla\xi^{\frac{1}{2}},\nabla\partial_t\xi^{\frac{1}{2}})\\&+||\nabla\partial_t\xi^{\frac{1}{2}}||^2+(f(u(t_{\frac{1}{2}}))-G(U^1,U^0),\partial_t\xi^{\frac{1}{2}})\\=\ &-(\frac{2}{\Delta t}\partial_t\eta^{\frac{1}{2}}+2\tau^0,\partial_t\xi^{\frac{1}{2}})+(\nabla\varepsilon^0,\nabla\partial_t\xi^{\frac{1}{2}}).\end{aligned}\quad(44)$$

Therefor, multiply by Δt and use the Cauchy-Schwartz inequality and the Young inequality to obtain

$$\begin{aligned}&2||\partial_t\xi^{\frac{1}{2}}||^2+\gamma||\Delta\xi^1||^2+||\nabla\xi^1||^2+\Delta t||\nabla\partial_t\xi^{\frac{1}{2}}||^2\\\leq\ &\Delta t^2||f(u(t_{\frac{1}{2}}))-G(U^1,U^0)||^2+||\partial_t\xi^{\frac{1}{2}}||^2+||\partial_t\eta^{\frac{1}{2}}||^2\\&+\frac{\Delta t}{2}||\nabla\partial_t\xi^{\frac{1}{2}}||^2+C(\Delta t)^2||\tau^0||^2+C\Delta t||\nabla\varepsilon_0||^2.\end{aligned}\quad(45)$$

So, we can get

$$\begin{aligned}&||\partial_t\xi^{\frac{1}{2}}||^2+\gamma||\Delta\xi^1||^2+||\nabla\xi^1||^2+\frac{\Delta t}{2}||\nabla\partial_t\xi^{\frac{1}{2}}||^2\\&\leq C\Delta t^4+||\partial_t\eta^{\frac{1}{2}}||^2.\end{aligned}\quad(46)$$

Noting that

$$\begin{aligned}||\partial_t\eta^{n+\frac{1}{2}}||^2&\leq\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}||\eta_t(s)||^2ds\\&\leq Ch^8||u_t||^2_{L^\infty(0,\Delta t,H^{r+1}(\Omega))}\end{aligned}$$

and, using (46) and (43), we have

$$\begin{aligned}&||\partial_t\xi^{J+\frac{1}{2}}||^2+||\Delta\xi^{J+\frac{1}{2}}||^2+||\nabla\xi^{J+\frac{1}{2}}||^2\\&+\Delta t\sum_{n=1}^{J}||\nabla\partial_t\xi^n||^2\leq C(h^8+\Delta t^4).\end{aligned}\quad(47)$$

Using (47), the Poincaré inequality and the triangle inequality completes the proof of the theorem 4.2.

Acknowledgment

This work is supported by National Natural Science Fund (No. 11061021), Program of Higher-level talents of Inner Mongolia University (No. Z200901004), the Scientific Research Projection of Higher Schools of Inner Mongolia (No. NJ10006, No. NJ10016) and YSF of Inner Mongolia University (No. ND0702)

References

[1] T. ZHANG, Finite element analysis for Cahn-Hilliard equation, Mathematica Numerica Sinica, 2006, 28(3): 281-292.
[2] R. AN, K.T LI, Stabilized finite element approximation for fourth order obstacle problem[J]. Acta Mathematicae Applicatae Sinica, 2009, 32(6): 1068-1078.
[3] D.Y. SHI, Y.C. PENG. The finite element methods for fourth order eigenvalue problems on anisotropic meshes, Chinese Journal of Engineering Mathematics, 2008, 25(6): 1029-1034.
[4] Z.X. CHEN, Analysis of expanded mixed methods for fourth-order elliptic problems, Numer. Methods Partial Differential Equations, 1997, 13: 483-503.
[5] H. LI, Y. LIU, Mixed discontinuous space-time finite element method for the fourth-order parabolic integro-differential equations, Mathematica Numerica Sinica, 2007, 29(4): 413-420.
[6] S. He, H. Li, The mixed discontinuous space-time finite element method for the fourth order linear parabolic equation with generalized boundary condition, Mathematica Numerica Sinica, 2009, 31(2): 167-178.
[7] J.C. LI, Mixed methods for fourth-order elliptic and parabolic problems using radial basis functions, Adv. Comput. Math., 2005, 23: 21-30.
[8] J.C. LI, Full-order convergence of a mixed finite element method for fourth-order elliptic equations, J. Math. Anal. Appl., 1999, 230: 329-349.
[9] J.C. LI, Optimal convergence analysis of mixed finite element methods for fourth-order elliptic and parabolic problems, Numer. Methods Partial Differential Equations, 2006, 22: 884-896.
[10] J.C. LI, Optimal error estimates of mixed finite element methods for a fourth-order nonlinear elliptic problem, J. Math. Anal. Appl., 2007, 334: 183-195.
[11] S.C. CHEN, M.F. LIU, Z.H. QIAO, An anisotropic nonconforming element for fourth order elliptic singular perturbation problem, International Journal of Numerical Analysis and Modeling, 2010, 7(4): 766-784.
[12] P. DANUMJAYA, A.K. PANI, Numerical methods for the extended fisher-kolmogorov (EFK) equation, International Journal of Numerical Analysis and Modeling, 2006, 3(2): 186-210.
[13] M.F. WHEELER, A priori L^2-error estimates for Galerkin approximations to parabolic differential equation, SIAM J. Numer. Anal., 1973, 10: 723-749.
[14] P.G. CIARLET, The Finite Element Method for Elliptic Problems, Amsterdam: North-Holland, 1978.