

# Deformation of Water Waves by Geometric Transitions with Power Law Function Distribution

E. G. Bautista, J. M. Reyes, O. Bautista, and J. C. Arcos

**Abstract**—In this work, we analyze the deformation of surface waves in shallow flows conditions, propagating in a channel of slowly varying cross-section. Based on a singular perturbation technique, the main purpose is to predict the motion of waves by using a dimensionless formulation of the governing equations, considering that the longitudinal variation of the transversal section obey a power-law distribution. We show that the spatial distribution of the waves in the varying cross-section is a function of a kinematic parameter,  $\kappa$ , and two geometrical parameters  $\varepsilon_h$  and  $\varepsilon_w$ . The above spatial behavior of the surface elevation is modeled by an ordinary differential equation. The use of single formulas to model the varying cross sections or transitions considered in this work can be a useful approximation to natural or artificial geometrical configurations.

**Keywords**—Surface waves, Asymptotic solution, Power law function, Non-dispersive waves.

## I. INTRODUCTION

AN essential aspect of the propagation of ocean waves described by the shallow water approximation is that they are strongly influenced by geometrical and physical parameters. Among others, we can include sharp variations in the depth and width that are characteristic parameters of the specific geometry of the open channel. The propagation of Tsunamis over the continental shelf and the propagation of tidal waves into estuaries are some examples of this typical phenomenon. On the other hand, in order to capture energy from waves, artificial channels can be used as efficient collector waves.

It is well known that one of the principal obstacles in obtaining adequate analytical solutions lies in the complicated geometry of natural estuaries; however, for some simplified geometries, it is possible to develop approximate analytical solutions. Today, computational methods offer a sophisticated tool for studying the dynamic propagation of shallow water waves. The modern use of digital computers can drastically reduce the numerical difficulties associated with complex geometries by enabling variations in depth and breadth to be incorporated in the calculations. This in turn increases the accuracy of the numerical solutions.

However, for some relevant limits the analytical solutions provide knowledge about the phenomenon of interest and can be very useful to simplify the numerical schemes. Reference [1] proposed a singular perturbation analysis to study the hydrodynamic performance of periodic ocean waves that are incident on an open parabolic channel of constant depth. Reference [2] developed a model for the monochromatic propagation on a smoothly varying bed profile divided into a series of shelves separated by abrupt steps. The submerged obstacles with different shapes, solids and porous solids have been studied extensively in order to determine the reflection and transmission of ocean waves and representative works can be found in [3]-[6]. In the specialized literature, approximated analytic solutions for the long wave equations with different propagation conditions based on Bessel functions can be found in [7]-[12]. The propagation of small amplitude water waves over variable bathymetry regions has been widely studied; [13] developed a consistent coupled-mode theory, which is derived from a variational formulation of the complete linear problem, representing the vertical distribution of the wave potential as a uniformly convergent series of local vertical models at each horizontal position. Previous investigations of tidal waves propagating in convergent channels with general shapes were investigated by [14]. In the previous works were studied separately the effect of the depth and width and analytical solutions were obtained by the well-known method of separation of variables, generating Bessel, Frobenius and other transcendental functions depending on the used geometry. On the other hand, other widely technique used is the well-known step method. However, the application of the above traditional technique turns out to be a complicated process for some geometry.

In this work, we develop a versatile analytical solution which allows us to study linear long water waves propagating along an open long channel of slowly varying cross-section, which obeys a power law function. The options of shapes in bottom and width can be so different ranging from rectangular, linear and parabolic transitions among others.

We consider the linear shallow water flow approximation by taking into account the influence of the width and bottom variables on the wave propagation. The governing equations for the ocean wave propagation are presented in dimensionless form as functions of the characteristic dimensionless parameters of the problem. The associate boundary-value problem is solved by using the well-known WKB (Wentzel-Kramers-Brillouin) singular perturbation technique [15]. In order to obtain the asymptotic perturbation solution for the motion of surface waves, we consider that our physical model

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is subdivided into three different regions. We anticipate that the continuity of kinematic and dynamic conditions will prevail and the solution of the surface elevation is function of a small kinematical parameter. The present mathematical formulation can be used in a first approximation to identify those geometries perturb significantly the amplitude of waves or which channels are less or more reflective than other geometries. This methodology can be widely used to select artificial tapered channels.

## II. FORMULATION

Let consider a linear long wave propagating in a channel of slowly varying cross-section. On Fig. 1 is depicted a schematic top view diagram of the physical model and Fig. 2 corresponds to the side view of the same physical model. The channel is divided into three different regions  $R_1$ ,  $R_2$  and  $R_3$ . In the selected Cartesian coordinate system, the  $x$  axis is positive to the right with origin in the intersection of regions  $R_2$  and  $R_3$ , meanwhile the  $z$  axis pointing outwards in a normal direction to the mean sea water level. The interval of the transition region  $R_2$  is  $0 \leq x \leq L$ , meanwhile the regions  $R_1$  and  $R_3$  are defined by the intervals  $L \leq x < \infty$  and  $-\infty < x \leq 0$ ; respectively. In addition, we assume that the corresponding widths and depths of the channel in these last two regions are constants. In the present analysis, we assume that the walls of the channel are impermeable which means no mass transfer in these boundaries. However, the vertical boundaries on the right and left sides of the system are completely open to the passage of the sum of incident and reflected waves on the right and only a transmitted wave on the left. Taking into account the above comments, the prescribed widths and depths are given by the following relationships:

$$b(x) = \begin{cases} b_1 & L \leq x < \infty \\ \beta x^m + b_3 & 0 \leq x \leq L \\ b_3 & -\infty < x \leq 0 \end{cases} \quad (1)$$

and

$$h(x) = \begin{cases} h_1 & L \leq x < \infty \\ \alpha x^n + h_3 & 0 \leq x \leq L \\ h_3 & -\infty < x \leq 0 \end{cases}$$

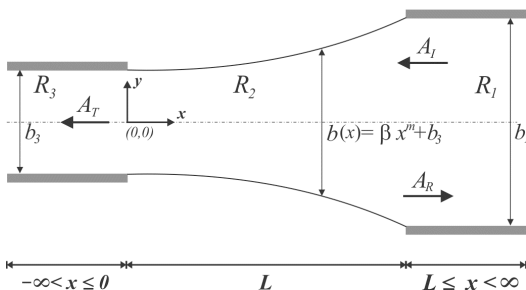


Fig. 1 Schematic top view diagram of the physical model studied

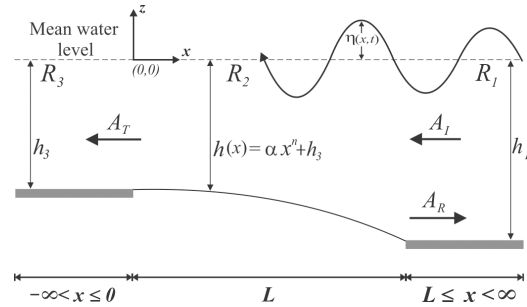


Fig. 2 Schematic side view diagram of the physical model studied

In the above equations, the slenderness ratios  $\alpha$  and  $\beta$  are given by;

$$\beta = \frac{b_1 - b_3}{L^m} \quad \text{and} \quad \alpha = \frac{h_1 - h_3}{L^n} \quad (2)$$

where  $b_1 = b(x = L)$  and  $h_1 = h(x = L)$ .

In the present work, we assume that the propagation of the water waves is described, in a first approximation, by utilizing one-dimensional linear long wave equations. Therefore, the governing equations can be written as follows (the details can be seen from [16]):

$$U \frac{\partial U(x,t)}{\partial t} + g \frac{\partial \eta(x,t)}{\partial x} = 0 \quad (3)$$

$$\frac{\partial \eta(x,t)}{\partial t} + \frac{1}{b(x)} \frac{\partial U(x,t) A(x)}{\partial x} = 0 \quad (4)$$

Equation (3) represents the conservation of momentum, while (4) is the conservation of mass. In the above equations,  $U$  is the average velocity defined as  $U = \int_{-h}^{\eta} u(x,z,t) dz / (\eta + h)$ , where  $u(x,z,t)$  is the horizontal instantaneous velocity;  $\eta$  is the level of the water surface measured from the mean sea level at rest,  $h(x)$ ;  $t$  is the physical time,  $g$  is gravitational acceleration and  $A(x) \approx b(x)h(x)$  is the area of the transversal section of the channel. The above set of equations can be written for each region by using appropriate dimensionless variables, and together with the boundary conditions, define the basic characteristics of the present model.

Firstly, we derive the governing equation and the corresponding solution of travelling waves which are propagating in the transition region  $R_2$ ; the analysis of the other two regions is trivial and is given at the end of this section.

The equations of continuity and momentum can be combined by differentiating (4) with respect to time, and then substituting (3) into the resulting equation can be readily obtained the following equation with the aid of

$$A(x) \approx b(x)h(x) = (\beta x^m + b_3)(\alpha x^n + h_3),$$

$$(\beta x^m + b_3) \frac{\partial^2 \eta}{\partial t^2} = g \left\{ \frac{(\beta x^m + b_3)(\alpha x^n + h_3) \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial \eta}{\partial x} \left[ (\beta x^m + b_3)(\alpha n x^{n-1}) + \frac{\partial}{\partial x} \left[ (\alpha x^n + h_3)(\beta m x^{m-1}) \right] \right]}{\alpha x^n + h_3} \right\}. \quad (5)$$

Equation (5) governs the unsteady oscillations of the water surface,  $\eta(x, t)$ . The equation is valid for long waves with small amplitude and controlled by the following relationships  $h(x)/\lambda \ll 1$  and  $\eta(x, t)/h(x) \ll 1$ , where  $\lambda$  is the characteristic wavelength.

Further, (5) can be simplified in an ordinary differential equation by considering that the surface level  $\eta(x, t)$  is dictated by a simple harmonic motion with frequency  $\omega$ , given by:

$$\eta(x, t) = \text{Re}[\tilde{\eta}(x)e^{-i\omega t}] \quad (6)$$

where  $\tilde{\eta}(x)$  is the variable amplitude of the wave for any cross section along the coordinate  $x$  in the interval  $0 \leq x \leq L$ ,

Re "is the real part of" and  $i = \sqrt{-1}$ ,  $\omega = 2\pi/T$ , where  $T$  is the ocean wave period. Substituting (6) into (5), we can obtain the boundary-value problem for predicting the surface elevation  $\tilde{\eta}$ , which is given by:

$$\frac{g}{\omega^2} \frac{d^2 \tilde{\eta}}{dx^2} + \frac{g}{\omega^2} \left[ \frac{\alpha n x^{n-1}}{\alpha x^n + h_3} + \frac{\beta m x^{m-1}}{\beta x^m + b_3} \right] \frac{d\tilde{\eta}}{dx} + \frac{1}{\alpha x^n + h_3} \tilde{\eta} = 0 \quad (7)$$

Introducing the following dimensionless variables,

$$\tilde{\chi} = \frac{x}{L} \quad \text{and} \quad \tilde{\delta} = \frac{\tilde{\eta}}{h_1} \quad (8)$$

therefore, (7) can be rewritten as,

$$\kappa \frac{d^2 \tilde{\delta}}{d\tilde{\chi}^2} + \kappa \left[ \frac{n(\varepsilon_h - 1)\tilde{\chi}^{n-1}}{(\varepsilon_h - 1)\tilde{\chi}^n + 1} + \frac{d\tilde{\delta}}{d\tilde{\chi}} + \frac{\varepsilon_h}{(\varepsilon_h - 1)\tilde{\chi}^n + 1} \tilde{\delta} \right] = 0 \quad (9)$$

where the dimensionless parameters in the above equation are defined as

$$\kappa = \frac{gh_1}{(\omega L)^2}, \quad \varepsilon_h = \frac{h_1}{h_3}, \quad \varepsilon_w = \frac{b_1}{b_3} \quad (10)$$

### III. ASYMPTOTIC SOLUTION

Although (9) can be solved numerically for different values of the dimensionless parameter  $\kappa$ ; here, we prefer to develop an asymptotic approximation taking into account that the dimensionless parameter  $\kappa \ll 1$ . This procedure is capable of providing quickly insight into how the solution depends on the various parameters of the problem.

For small values of the dimensionless parameter  $\kappa$ , we can find an approximate solution by using perturbation techniques. This parameter represents the competition between the potential head  $gh_1$ , and the kinematical head,  $(\omega L)^2$ , which clearly is associated to the frequency  $\omega$  of the shallow-water wave. For linear non-dispersive long waves, the wave velocity is defined as  $c = \sqrt{gh(x)}$  or  $c = \omega/k = \lambda/T$ , where  $k = 2\pi/\lambda$  is the wave number. Therefore, the parameter  $\kappa$  can be rewritten as  $\kappa = [\lambda/(2\pi L)]^2$  with the aid of the above kinematical relationships. From a geometrical point of view, the limit of  $\kappa \ll 1$  corresponds to consider an horizontal projection  $L$  very large compared with  $\lambda$ , for finite values of the frequency  $\omega$  and the geometrical parameters  $\varepsilon_h$  and  $\varepsilon_w$ . In the following section and considering the relevance of the above limit in practical cases, we derive an asymptotic solution of (9).

The WKB method is a powerful tool for obtaining approximate solutions of linear differential equations with variable coefficients in which the higher order derivative is multiplied by a small parameter. We must anticipate that the WKB technique is an appropriate tool for obtaining global approximations to the solution of a linear differential equation whose dominant tendency has a dispersive or oscillating behavior. In order to apply the WKB perturbation technique, the solution can be proposed as a potential series given by the following:

$$\tilde{\delta}(\chi) \sim \exp \left[ \frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S_n(\chi) \right] \quad (11)$$

Substituting (11) into (9) and retaining terms of the same order, we obtain after some algebraic manipulations that the solution up to terms of order  $\kappa^{1/2}$  is given by (see Appendix A for more details):

$$\tilde{\delta}(\tilde{\chi}) = \frac{Ce^{i\left(\frac{\varepsilon_h}{\kappa}\right)^{1/2} F(\tilde{\chi}, n)} + De^{-i\left(\frac{\varepsilon_h}{\kappa}\right)^{1/2} F(\tilde{\chi}, n)}}{[(\varepsilon_h - 1)\tilde{\chi}^n + 1]^{1/4} [(\varepsilon_w - 1)\tilde{\chi}^m + 1]^{1/2}} \quad (12)$$

where  $C$  and  $D$  are constants which have to be determined with appropriated boundary conditions. In addition,  $F(\tilde{\chi}, n)$  is a specific integral given in Appendix (29).

For the region  $R_1$ , with constant values for the width and the depth and introducing the following dimensionless

variables for regions  $R_1$  and  $R_3$ ;

$$\bar{\chi} = k_1 x, \quad \bar{\delta} = \frac{\bar{\eta}(x)}{h_1} \quad \text{and} \quad \hat{\chi} = k_3 x, \quad \hat{\delta} = \frac{\hat{\eta}(x)}{h_3} \quad (13)$$

The surface elevation in dimensionless variables, in these regions can be expressed as:

$$\bar{\delta}(\bar{\chi}) = \beta_R e^{\bar{\chi}} + \beta_I e^{-\bar{\chi}} \quad (14)$$

and

$$\hat{\delta}(\hat{\chi}) = \beta_T e^{-\hat{\chi}} \quad (15)$$

where  $\beta_I = A_I / h_1$ ,  $\beta_R = A_R / h_1$  and  $\beta_T = A_T / h_3$ . In the following section, we present the patching conditions needed to obtain the previous constants  $C$ ,  $D$ ,  $\beta_R$  and  $\beta_T$ .

#### A. Patching Conditions

In addition, we must provide kinematic and dynamic conditions at the common boundaries of these three regions in order to obtain the values of coefficients  $C$ ,  $D$ ,  $\beta_R$  and  $\beta_T$ . We apply, therefore, patching conditions across the common boundaries, thus we have

$$\bar{\delta}\Big|_{\bar{\chi}=\kappa^{-1/2}} = \tilde{\delta}\Big|_{\tilde{\chi}=1} \quad \text{and} \quad \kappa^{-1/2} \frac{d\bar{\delta}}{d\bar{\chi}}\Big|_{\bar{\chi}=\kappa^{-1/2}} = \frac{d\tilde{\delta}}{d\tilde{\chi}}\Big|_{\tilde{\chi}=1} \quad (16)$$

and

$$\varepsilon_h \tilde{\delta}\Big|_{\tilde{\chi}=0} = \hat{\delta}\Big|_{\hat{\chi}=0} \quad \text{and} \quad \varepsilon_h^{-1/2} \frac{d\tilde{\delta}}{d\tilde{\chi}}\Big|_{\tilde{\chi}=0} = \kappa^{-1/2} \frac{d\hat{\delta}}{d\hat{\chi}}\Big|_{\hat{\chi}=0}$$

From (12), (14), (15) and (16), we can easily derive the following system of equations,

$$\begin{aligned} C \frac{e^{i\bar{\theta}}}{\varepsilon_h^{1/4} \varepsilon_w^{1/2}} + D \frac{e^{-i\bar{\theta}}}{\varepsilon_h^{1/4} \varepsilon_w^{1/2}} - \beta_R e^{i\bar{\theta}} &= \beta_I \\ -C e^{i\bar{\theta}} \left[ \frac{1}{\varepsilon_h^{1/4} \varepsilon_w^{1/2}} + i \frac{\kappa^{1/2} \alpha_1}{2} \right] + \\ D e^{-i\bar{\theta}} \left[ \frac{1}{\varepsilon_h^{1/4} \varepsilon_w^{1/2}} - i \frac{\kappa^{1/2} \alpha_1}{2} \right] + \beta_R e^{i\bar{\theta}} &= \beta_I \\ C \varepsilon_h e^{i\bar{\theta}} + D \varepsilon_h e^{-i\bar{\theta}} - \beta_T &= 0 \\ C \left[ \varepsilon_h^{1/2} + i \frac{\kappa^{1/2} \alpha_2}{2} \right] \varepsilon_h^{1/2} e^{i\bar{\theta}} - \\ D \left[ \varepsilon_h^{1/2} - i \frac{\kappa^{1/2} \alpha_2}{2} \right] \varepsilon_h^{1/2} e^{-i\bar{\theta}} + \beta_T &= 0 \end{aligned} \quad (17)$$

with

$$\bar{\theta} = \frac{1}{\kappa^{1/2}} \left[ 1 + \varepsilon_h^{1/2} F(1, n) \right],$$

$$\hat{\theta} = \left( \frac{\varepsilon_h}{\kappa} \right)^{1/2} F(0, n), \quad \bar{\theta} = \frac{2}{\kappa^{1/2}}$$

$$\alpha_1 = \frac{m(\varepsilon_w - 1)}{\varepsilon_h^{1/4} \varepsilon_w^{3/2}} + \frac{n(\varepsilon_h - 1)}{2 \varepsilon_h^{5/4} \varepsilon_w^{1/2}} \quad (18)$$

and

$$\alpha_2 = m(\varepsilon_w - 1)q(0, m) + \frac{1}{2}n(\varepsilon_h - 1)g(0, n)$$

where  $F(0, n)$  and  $F(1, n)$  are the values of the integral (29), evaluated at  $\tilde{\chi} = 0$  and  $\tilde{\chi} = 1$ ; respectively, for different values of exponent  $n$ . It is necessary to precise that in the above integral, the exponent  $n$  takes different integer values, depending on the channel shape geometry, i.e.  $n(=1, 2, 3, \dots)$ .

In the above system of equations, the variables  $g$  and  $q$  are defined as  $g(\tilde{\chi}, n) = \tilde{\chi}^{n-1}$  and  $q(\tilde{\chi}, n) = \tilde{\chi}^{m-1}$ , where  $g(0, n) = q(0, m) = 0$  for  $n, m > 1$  and  $g(0, n) = q(0, m) = 1$  for  $m, n = 1$ . Therefore, the solution of the system (17) provides us the constants  $C$  and  $D$ :

$$C = -2E \alpha_2 e^{i\left(\bar{\theta} - \frac{1}{\kappa^{1/2}}\right)} \quad (19)$$

$$D = 2E \left[ \alpha_2 - 4i \left( \frac{\varepsilon_h}{\kappa} \right)^{1/2} \right] e^{i\left(\bar{\theta} + \bar{\theta} - \frac{1}{\kappa^{1/2}}\right)} \quad (20)$$

with

$$\begin{aligned} E &= \frac{2\beta_I \varepsilon_h^{5/4} \varepsilon_w^{3/2} \kappa^{1/2} i}{\left[ \alpha_2 - 4i \left( \frac{\varepsilon_h}{\kappa} \right)^{1/2} \right] \left[ \alpha_3 + 4i \frac{\varepsilon_h \varepsilon_w}{\kappa^{1/2}} \right] \kappa e^{i2\bar{\theta}} - \alpha_2 \alpha_3 \kappa e^{i2\bar{\theta}}}, \\ \bar{\theta} &= \left( \frac{\varepsilon_h}{\kappa} \right)^{1/2} F(1, n) \end{aligned} \quad (21)$$

and

$$\alpha_3 = m \varepsilon_h (\varepsilon_w - 1) + \frac{1}{2} n \varepsilon_w (\varepsilon_h - 1).$$

#### IV. RESULTS AND DISCUSSION

We propose the following physical data in order to investigate the effect of the channel geometry on the wave propagation. Typical values for this kind of channels and sea water conditions are:  $L = 500$  m.,  $h_1 = 4.47$  m.,  $h_3 (= 4.47, 2.235, 1.11)$  m.,  $b_1 = 50$  m.,  $b_3 (= 50, 25, 12.5)$  m.; the wave length in region  $R_1$  is  $\lambda_1 = 99.34$ ,  $T = 15$  s.,  $A_I = 0.18$  m. With the above values, we obtain the next

dimensionless parameters  $\kappa = 0.001$ ,  $\varepsilon_h (= 1, 2, 4)$ ,  $\varepsilon_w (= 1, 2, 4)$  and  $\beta_l = 0.04$ . In this manner, the selected data satisfy the assumption of linear long wave theory, that is  $A_l / h_l = 0.04 \ll 1$  and  $kh_l = 0.282 < \pi / 10$ , [17]; therefore, the fundamental assumptions are fully satisfied.

As a particular case, considering  $\varepsilon_h = \varepsilon_w = 1$ , the region  $R_2$  degenerates into a channel with constant width and depth. Therefore, (9) is simplified as following

$$\kappa \frac{d^2 \tilde{\delta}}{d\tilde{\chi}^2} + \tilde{\delta} = 0 \quad (22)$$

From (19) and (20), we obtain that  $C = 0$  and  $D = \beta_l$ ; therefore, the analytical solution for this case is written as

$$\tilde{\delta}(\tilde{\chi}) = \beta_l e^{-i \frac{\tilde{\chi}}{\kappa^{1/2}}} \quad (23)$$

The distribution of the dimensionless surface elevation,  $\tilde{\delta}$ , as a function of the dimensionless coordinate  $\tilde{\chi}$  and calculated with (23) is shown in Fig. 3. From this figure is evident that the wave amplitude is not modified.

In Fig. 4 is plotted the dimensionless surface elevation for waves propagating in a channel with a parabolic depth transition  $n = 2$ ,  $\varepsilon_h = 4$  and constant width  $\varepsilon_w = 1$ . The results are compared with the Green's law for waves propagating over slowly depth transitions, which in terms of dimensionless proposed variables, is written as

$$\tilde{\delta}(\tilde{\chi}) = \beta_l \left[ \frac{1}{\varepsilon_h^{-1} + (1 - \varepsilon_h^{-1}) \tilde{\chi}^n} \right]^{1/4} \quad (24)$$

It should be noted that the surface elevation calculated with the present mathematical model is in good agreement with Green's law.

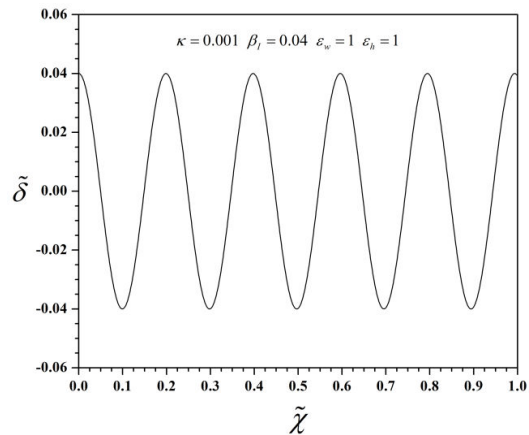


Fig. 3 Dimensionless surface elevation  $\tilde{\delta}$  as a function of dimensionless axis  $\tilde{\chi}$ , for constant values of the parameters  $\kappa = 0.001$ ,  $\beta_l = 0.04$ ,  $\varepsilon_h = 1$  and  $\varepsilon_w = 1$

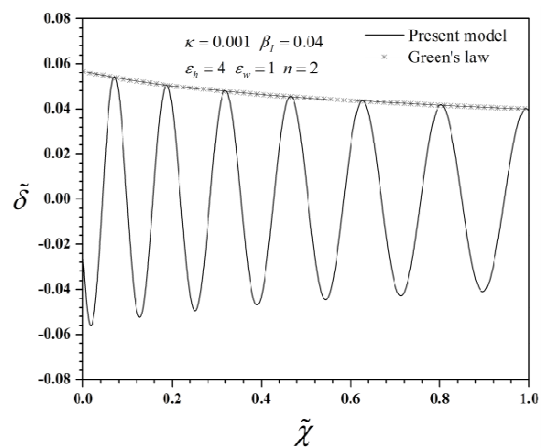


Fig. 4 Comparison of the dimensionless surface elevation  $\tilde{\delta}$  in a channel with parabolic depth transition with the wave amplitude calculated by Green's law

#### A. Particular Cases

##### 1. Channel with Constant Depth and Variable Width: $\varepsilon_h = 1$

In this part, we present the hydrodynamics of waves propagating in a channel with constant depth and variable width, considering that  $\varepsilon_h = 1$  and  $\varepsilon_w = 4$ , for different values of exponent  $m$  and constant values of the parameters  $\kappa = 0.001$ ,  $\beta_l = 0.04$ . Fig. 4 show the dimensionless surface elevation  $\tilde{\delta}$  as a function of the dimensionless coordinate  $\tilde{\chi}$  for three different horizontal shapes  $m (= 1, 2, 5)$ .

The analytical solution of waves propagating under these conditions reads now as

$$\tilde{\delta}(\tilde{\chi}) = \frac{1}{\sqrt{(\varepsilon_w - 1)\tilde{\chi}^m + 1}} \left[ C e^{i \frac{1}{\kappa^{1/2}} \tilde{\chi}} + D e^{-i \frac{1}{\kappa^{1/2}} \tilde{\chi}} \right] \quad (25)$$

The constants  $C$  and  $D$  are evaluated taking into account (19) and (20).

For large values of  $m$ , the dimensionless amplitude  $\tilde{\delta}$  is increased in the interval  $0 < \tilde{\chi} < 1$ . It should be noted, in addition, that the values of variable  $\tilde{\delta}$  at  $\tilde{\chi} = 0$  are twice greater than the corresponding values at  $\tilde{\chi} = 1$ . Fig. 5 shows that the wavelength remains constant, which satisfies the assumption that the ocean wave velocity for linear non-dispersive long waves propagating in a flat bottom, is constant, i. e.  $c = \lambda / T = \sqrt{gh}$ .

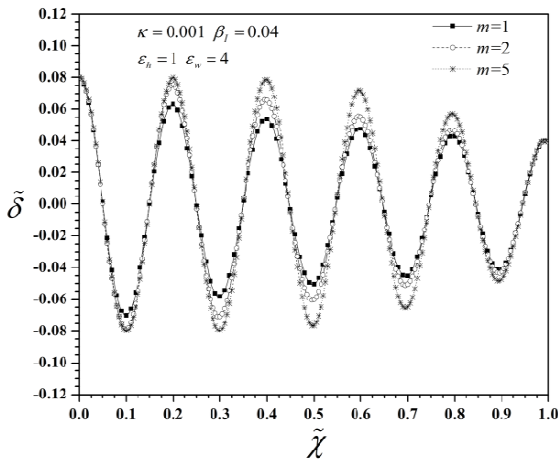


Fig. 5 Approximated analytical solution for the dimensionless surface elevation  $\tilde{\delta}$  as a function of the dimensionless coordinate  $\tilde{\chi}$ , for  $\kappa = 0.001$ ,  $\beta_l = 0.04$ ,  $\varepsilon_h = 1$ ,  $\varepsilon_w = 4$  and three different values of  $m (=1, 2, 5)$

## 2. Channel with Constant Width and Variable Depth:

$$\varepsilon_w = 1$$

In order to characterize in a first approximation the linear long waves propagating in depth with smooth transitions; here we show analytical results of the dimensionless water surface elevation and reflection and transmission coefficients for different bottom shapes.

The analytical solution of waves propagating under this condition reads now;

$$\tilde{\delta}(\tilde{\chi}) = \frac{Ce^{i\left(\frac{\varepsilon_h}{\kappa}\right)^{1/2} F(\tilde{\chi}, n)} + De^{-i\left(\frac{\varepsilon_h}{\kappa}\right)^{1/2} F(\tilde{\chi}, n)}}{[(\varepsilon_h - 1)\tilde{\chi}^n + 1]^{1/4}}. \quad (26)$$

In Fig. 6 is plotted the propagation of water waves in a channel with constant width and variable depth, calculated from (26). The parameters take values of  $\varepsilon_h = 4$ ,  $\kappa = 0.001$ ,  $\beta_l = 0.04$  and the bottom shapes are  $n (=1, 2, 5)$ . In this figure should be noted that the wavelength of the dimensionless water surface calculated for the three different channels, is varying along the channel, which is a consequence

of the variable depth. Additionally, in the channel with a shape  $n = 5$  generates amplitude greater than the other two cases. This is due to that for large values of  $n$ , the depth is practically flat along the channel as  $\tilde{\chi} \rightarrow 1$  and an abrupt transition is presented, limiting the values of  $n$ .

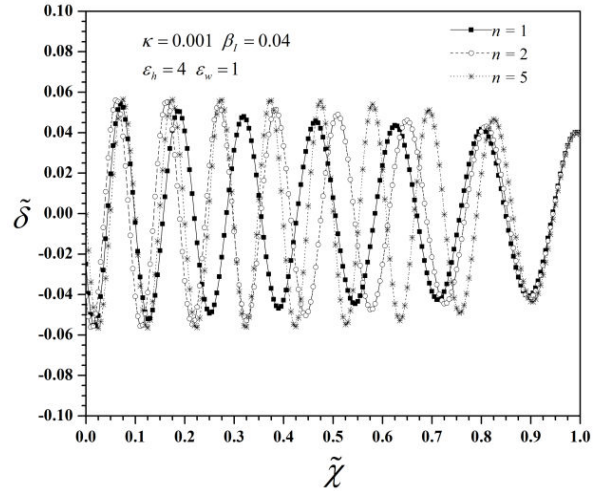


Fig. 6 Approximated analytical solution for the dimensionless surface elevation  $\tilde{\delta}$  as a function of the dimensionless coordinate  $\tilde{\chi}$ , for values of  $\kappa = 0.001$ ,  $\beta_l = 0.04$ ,  $\varepsilon_w = 1$ ,  $\varepsilon_h = 4$  and three different values of  $n (=1, 2, 5)$

## 3. Channel with Width and Depth Variables

The geometrical perturbation of the dimensionless water surface elevation by a channel with width and depth variables is shown in Fig. 7. For simplicity, we carry out the results, considering that the three cases have the same distribution geometry for the width and the depth; that is  $m = n (=1, 2, 5)$  for constant values of  $\varepsilon_h = \varepsilon_w = 2$ ,  $\kappa = 0.001$  and  $\beta_l = 0.04$ . Similar results as the previous cases are obtained and the variable  $\tilde{\delta}$  for a geometry with  $m = n = 5$ , shows larger values than the other two cases in the interval of  $0 < \tilde{\chi} < 1$ . At  $\tilde{\chi} \rightarrow 0$  and for the three different channels, the dimensionless surface elevation is almost 50% times greater than the corresponding amplitude at  $\tilde{\chi} \rightarrow 1$ .

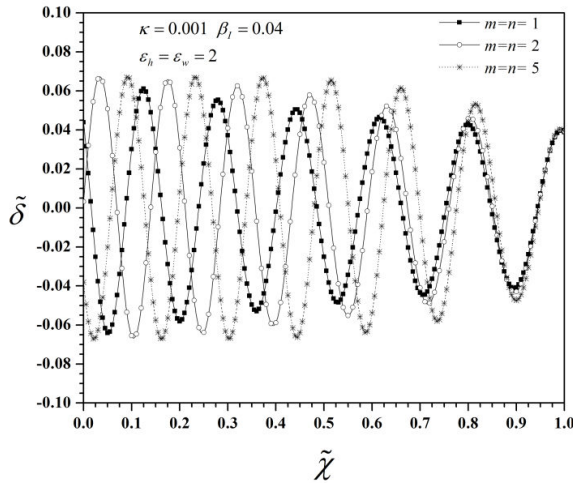


Fig. 7 Approximated analytical solution for the dimensionless surface elevation  $\tilde{\delta}$  as a function of the dimensionless coordinate  $\tilde{\chi}$ , for values of  $\kappa = 0.001$ ,  $\beta_I = 0.04$ ,  $\varepsilon_h = \varepsilon_w = 4$  and three different values of  $m=n(=1,2,5)$

#### V. CONCLUSIONS

An analytical model based on the WKB singular perturbation technique, is conducted to obtain a simple dimensionless equation for the deformation of ocean waves, propagating in a channel of slowly varying cross-section. The new solutions are general and easy to apply.

For a channel with parabolic bottom transition,  $n = 2$ , the analytical solution for the surface elevation reproduces properly the asymptotic solution Green's law for waves propagating on slowly depth transitions. We present three examples of the water surface oscillation; in the first case we analyze the effect of a channel with variable width transition and constant depth with the aim to identify the magnitude order of the amplification of the ocean waves. In this case, the dimensionless amplitude  $\tilde{\delta}$  is perturbed and the wavelength remains constant.

The second case depicts the effect that different bottom transition shapes with constant width have in the oscillation of progressive surface waves. It should be noted that the wave length is variable along the channel. Due to the versatility of the analytical solution, the hydrodynamic of progressive waves propagating in a channel of slowly varying cross-section, is also analyzed. The results show that the surface elevation at  $\tilde{\chi} \rightarrow 0$  is almost 50 % greater than the boundary condition at  $\tilde{\chi} = 1$ .

In the present model, the width and depth of the channel can take different values. In particular, we have recovered the wave propagating in a channel with flat bottom and constant width.

Furthermore, the present model can be used to study in a first approximation, the perturbation of surface waves in shallow flow conditions in order to amplify the heights of ocean waves.

The geometrical transitions studied in this work are useful approximations of estuarine configurations.

#### APPENDIX

##### • Appendix A: WKB Perturbation Technique, for $\kappa \ll 1$

Differentiating once and twice (11), taking into account the solution given by (11) and applying a dominant balance, we obtain that,  $\delta = \kappa^{1/2}$ . Collecting terms of order  $\kappa^0$ , we obtain:

$$O(\kappa^0): (S'_0)^2 + \frac{\varepsilon_h}{(\varepsilon_h - 1)\tilde{\chi}^n + 1} = 0 \quad (27)$$

where

$$S_0(\tilde{\chi}) = \pm \varepsilon_h^{1/2} F(\tilde{\chi}, n) i \quad (28)$$

and

$$F(\tilde{\chi}, n) = \int \frac{1}{\sqrt{(\varepsilon_h - 1)\tilde{\chi}^n + 1}} d\tilde{\chi} \quad (29)$$

Similarly, we can collect terms of order  $\kappa^{1/2}$ , obtaining that,

$$O(\kappa^{1/2}): 2S'_0 S'_1 + S''_0 + \left[ \frac{(\varepsilon_h - 1)n\tilde{\chi}^{n-1}}{(\varepsilon_h - 1)\tilde{\chi}^n + 1} + \frac{(\varepsilon_w - 1)m\tilde{\chi}^{m-1}}{(\varepsilon_w - 1)\tilde{\chi}^m + 1} \right] S'_0 = 0 \quad (30)$$

with

$$S_1(\tilde{\chi}) = \ln[(\varepsilon_h - 1)\tilde{\chi}^n + 1]^{-\frac{1}{4}} + \ln[(\varepsilon_w - 1)\tilde{\chi}^m + 1]^{-\frac{1}{2}} + c_2 \quad (31)$$

and  $c_2$  is an integration constant. Therefore, the solution (12) is easily obtained by replacing (28) and (31) into (11).

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#### REFERENCES

- [1] E.G. Bautista, F. Méndez, O. Bautista, and A. Mora, "Propagation of shallow water waves in an open parabolic channel using the WKB perturbation technique," *Appl Ocean Res*, vol. 33, pp. 186-92, 2011.
- [2] T.J. O'Hare, and A.G. Davies, "A new model for surface wave propagation over undulating topography," *Coastal Eng*, vol. 18, pp. 251-66, 1992.
- [3] M.A. Losada, C. Vidal, and R. Medina, "Experimental study of the evolution of a solitary wave at an abrupt junction," *J Geophys Res*, vol. 94, pp. 14557-66, 1989.
- [4] L. Pengzhi, "A numerical study of solitary wave interaction with rectangular obstacles," *Coastal Eng*, vol. 5, pp. 35-51, 2004.
- [5] L. Pengzhi, and H.W. Liu, "Analytical study of linear long-wave reflection by a two dimensional obstacle of general trapezoidal shape," *J Eng Mech*, vol. 131, pp. 822-30, 2005.
- [6] S.K. Wang, T.W. Hsu, W.K. Weng, and S.H. Ou, "A three-point method for estimating wave reflection of obliquely incident waves over a

- sloping bottom," *Coastal Eng*, vol. 55, pp. 125-38, 2008.
- [7] C.J. Bender, and R.G. Dean, "Wave transformation by two-dimensional bathymetric anomalies with sloped transitions," *Coastal Eng*, vol. 50, pp. 61-84, 2003.
  - [8] R.G. Dean, and R.A. Dalrymple, (Editors), "Water wave mechanics for engineers and scientists," Ed. World Scientific Co-London, 1984.
  - [9] D.G. Goring, "Tsunamis-The propagation of long waves onto a shelf," Ed. Pasadena california: Institute of technology, 1979.
  - [10] C. Hsien-Kuo, and L. Jin-Cheng, "Long wave reflection from submerged trapezoidal breakwaters," *Ocean Eng*, vol. 34, pp.185-91, 2007.
  - [11] U. Kanoglu, and C.M. Synolakis, "Long wave runup on piecewise linear topographies," *J. Fluid Mech*, vol. 374, pp. 1-28, 1998.
  - [12] J. Tae-Hwa, S. Kyung-Duck, L. Seung, and C. Yong-Sik, "Linear wave reflection by trench with various shapes," *Ocean Eng*, vol. 35, pp. 1226-34, 2008.
  - [13] G.A. Athanassoulis, and K.A. Belibassakis, "Long wave runup on piecewise linear topographies," *J. Fluid Mech*, vol. 389, pp. 275-301, 1999.
  - [14] D. Prandle, and M. Rahman, "Tidal Response in Estuaries," *Journal of Physical Oceanography*, vol. 10(10), pp. 1552-1573, 1980.
  - [15] C.M. Bender, and S.A. Orzag, (Editors), "Advanced mathematical methods for scientists and engineers," Ed. New York: McGraw-Hill Company, 1978.
  - [16] C. Mei, M. Stiassnie, and D.P. Yue, (Editors), "Theory and applications of ocean surface waves. Part 1: Linear aspects. Advance on ocean engineering," vol. 23, Ed. World Scientific Publishing: New York (USA), 2005.
  - [17] M. Rahman, (Editor), "Water waves: relating modern theory to advanced engineering applications," Ed. Clarendon Press: Oxford (New York), 1995.

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