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Contributions to Differential Geometry of Pseudo Null Curves in Semi-Euclidean Space

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Abstract — In this paper, first, a characterization of spherical Pseudo null curves in Semi-Euclidean space is given. Then, to investigate position vector of a pseudo null curve, a system of differential equation whose solution gives the components of the position vector of a pseudo null curve on the Frenet axis is established by means of Frenet equations. Additionally, in view of some special solutions of mentioned system, characterizations of some special pseudo null curves are presented.

Keywords **—** Semi-Euclidean Space, Pseudo Null Curves, Position Vectors.

I. INTRODUCTION

THE notion of null curves is due to E. Cartan (for more details see [1]). And, thereafter null curves are deeply studied by W.B. Bonnor [2] in Minkowski space-time. In the same space, Frenet equations for some special null; *Partially and Pseudo Null* curves are given in [6]. By means of Frenet equations, in [3] authors write characterizations of such kind null curves lying on the pseudohyperbolic space in E_1^4. Additionally, in [5] authors define Frenet equations of pseudo null and a partially null curves in Semi-Euclidean space E_2^4.

In this work, with an analogous way as in [3], first, we write a characterization of Lorentzian spherical curves in terms of Frenet equations defined in [5]. Thereafter, a system of differential equation whose solution gives the components of the position vector of a pseudo null curve on the Frenet axis is established. The general solution could not have obtained. Thus, by using special values, some important relations and characterizations of such kind curves are presented.

II. PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_2^4. are briefly presented.(A more complete elementary treatment can be found in [4].)

Semi-Euclidean space E_2^4 is a Euclidean space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 - dx_2^3 + dx_3^2 + dx_4^2, \tag{1}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_2^4. Since g is an indefinite metric, recall that a vector $\vec{v} \in E_2^4$ can have one of the three causal characters; it can be space-like if $g(\vec{v},\vec{v})\rangle 0$ or $\vec{v}=0$, time-like if $g(\vec{v},\vec{v})\langle 0$ and null (light-like) if $g(\vec{v},\vec{v})=0$ and $\vec{v}\neq 0$. Similarly, an arbitrary curve $\vec{\alpha}=\vec{\alpha}(s)$ in E_2^4 can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\vec{\alpha}'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{|g(\vec{v},\vec{v})|}$. Therefore, $\vec{v}$ is a unit vector if $g(\vec{v},\vec{v}) = \pm 1$. Next, vectors $\vec{v},\vec{w}$ in E_2^4 are said to be orthogonal if $g(v,w)=0$. The velocity of the curve $\vec{\alpha}$ is given by $\|\vec{\alpha}'\|$. Thus, a space-like or a time-like curve $\vec{\alpha}$ is said to be parameterized by arclength function s, if $g(\vec{\alpha}',\vec{\alpha}')=\pm 1$. The Lorentzian hypersphere of center $\vec{m}=(m_1,m_2,m_3,m_4)$ and radius $r \in R^+$ in the space E_2^4 defined by

$$S_2^3 = \left\{\vec{\alpha} = (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in E_2^4 : g(\vec{\alpha}-\vec{m},\vec{\alpha}-\vec{m}) = r^2\right\} \tag{2}$$

Denote by $\left\{\vec{T}(s),\vec{N}(s),\vec{B}_1(s),\vec{B}_2(s)\right\}$ the moving Frenet frame along the curve $\vec{\alpha}$ in the space E_2^4. Then $\vec{T},\vec{N},\vec{B}_1,\vec{B}_2$ are, the tangent, the principal normal respectively; the first binormal and the second binormal vector fields. For a pseudo null unit speed curve $\vec{\alpha}$ in E_2^4, the following Frenet equations are given in [5]

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}_1' \\ \vec{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & \sigma & 0 & -\varepsilon_2\tau \\ -\varepsilon_1\kappa & 0 & -\varepsilon_2\sigma & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \end{bmatrix}, \tag{3}$$

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where $\vec{T}, \vec{N}, \vec{B}_1$ and $\vec{B}_2$ are mutually orthogonal vectors satisfying equations

$$g(\vec{T},\vec{T}) = \varepsilon_1 = \pm 1, g(\vec{B}_1,\vec{B}_1) = \varepsilon_2 = \pm 1; \text{whereby } \varepsilon_1.\varepsilon_2 = -1,$$
$$g(\vec{N},\vec{N}) = g(\vec{B}_2,\vec{B}_2) = 0,\ g(\vec{N},\vec{B}_2) = 1,$$
$$g(\vec{T},\vec{N}) = g(\vec{T},\vec{B}_1) = g(\vec{T},\vec{B}_2) = g(\vec{N},\vec{B}_1) = g(\vec{B}_1,\vec{B}_2) = 0.$$

And here, $\kappa(s), \tau(s)$ and $\sigma(s)$ are first, second and third curvature of the curve $\vec{\alpha}$, respectively. And, a pseudo null curve's first curvature κ can take only two values: 0 when $\vec{\alpha}$ is a straight line or 1 in all other cases.

III. A Characterization of Spherical Pseudo Null Curves

Theorem 1. Let $\vec{\alpha} = \vec{\alpha}(s)$ be a pseudo null unit speed curve in E_2^4 with curvatures $\kappa = 1, \tau \neq 0$ and $\sigma \neq 0$ for each $s \in I \subset R$. If $\vec{\alpha}$ lies on S_2^3, then;

$$\begin{cases} \dfrac{\sigma}{\tau} = \text{constant } \rangle 0 \ ; \text{ if } \varepsilon_1 = 1 \\ \dfrac{\sigma}{\tau} = \text{constant } \langle 0 \ ; \text{ if } \varepsilon_1 = -1. \end{cases} \tag{4}$$

Proof : Let us suppose that $\vec{\alpha} = \vec{\alpha}(s)$ lies on S_2^3 with center m. By the definition, we write that

$$g(\vec{\alpha} - \vec{m}, \vec{\alpha} - \vec{m}) = r^2. \tag{5}$$

Differentiating (5), four times with respect to s and using Frenet equations, we have, respectively,

$$\begin{cases} g(\vec{\alpha} - \vec{m}, \vec{T}) = 0, \\ g(\vec{\alpha} - \vec{m}, \vec{N}) = -\varepsilon_1, \\ g(\vec{\alpha} - \vec{m}, \vec{B}_1) = 0, \\ g(\vec{\alpha} - \vec{m}, \vec{B}_2) = -\dfrac{\sigma}{\tau}. \end{cases} \tag{6}$$

Let us decompose $\vec{\alpha} - \vec{m}$ by

$$\vec{\alpha} - \vec{m} = -\frac{\sigma}{\tau} N - \varepsilon_1 B_2. \tag{7}$$

If we calculate $g(\vec{\alpha} - \vec{m}, \vec{\alpha} - \vec{m}) = r^2$, we easily obtain

$$\frac{\sigma}{\tau} = \frac{r^2}{2\varepsilon_1}. \tag{8}$$

Finally, (8) yields (4) as desired.

IV. Position Vector of A Pseudo Null Curve in Semi-Euclidean Space

Let be $\vec{\alpha} = \vec{\alpha}(s)$ an unit speed pseudo null curve in E_2^4. Then, we can write position vector of $\vec{\alpha}$ with respect to frame $\{\vec{T}, \vec{N}, \vec{B}_1, \vec{B}_2\}$ as

$$\vec{\alpha} = \vec{\alpha}(s) = m_1 \vec{T} + m_2 \vec{N} + m_3 \vec{B}_1 + m_4 \vec{B}_2, \tag{9}$$

where m_i are arbitrary functions of s. Differentiating (9) with respect to s and using Frenet equations, we have a system of differential equation as follow:

$$\begin{cases} \dfrac{dm_1}{ds} - \varepsilon_1 m_4 - 1 = 0, \\ \dfrac{dm_2}{ds} + m_1 + m_3 \sigma = 0, \\ \dfrac{dm_3}{ds} + m_2 \tau - \varepsilon_2 m_4 \sigma = 0, \\ \dfrac{dm_4}{ds} - \varepsilon_2 m_3 \tau = 0. \end{cases} \tag{10}$$

Using (10), we have a fourth order differential equation with respect to m_1 such as

$$\frac{d}{ds}\left\{ \frac{1}{\tau}\left[\frac{d}{ds}\left(\frac{1}{\tau}\frac{d^2 m_1}{ds^2} \right) + \sigma\left(1 - \frac{dm_1}{ds} \right) \right] \right\} - \frac{\sigma}{\tau}\frac{d^2 m_1}{ds^2} + m_1 = 0. \tag{11}$$

Since, we give following result.
Corollary 1. Equation (11) is a characterization for $\vec{\alpha}$. Position vector of all pseudo null curves can be determined by means of solution of it.

V. Some Special Solutions of (11)

The general solution of the differential equation (11) could not have found. Due to this, in this section, we give some special values to the components.
Case 1. $m_1 = \text{constant} = c_1 \neq 0$. In this case, using $(10)_1$, $(10)_4$ and $(10)_3$ we have other components, respectively,

$$\begin{cases} m_2 = \dfrac{\sigma}{\tau}, \\ m_3 = 0, \\ m_4 = -\dfrac{1}{\varepsilon_1}. \end{cases} \tag{12}$$

Substituting (12) to $(10)_2$, we have

$$\frac{d}{ds}\left(\frac{\sigma}{\tau} \right) + c_1 = 0. \tag{13}$$

And, therefore (13) yields that

$$\frac{\sigma}{\tau}+c_1 s+c=0. \tag{14}$$

Considering obtained equations, we give the following theorem.

Theorem 2. Let $\vec{\alpha}=\vec{\alpha}(s)$ be a pseudo null unit speed curve in E_2^4 and first component of the position vector of $\vec{\alpha}$, c_1 be constant and nonzero. Then;

i) The position vector of $\vec{\alpha}$ can be written as follow

$$\vec{\alpha}(s)=c_1\vec{T}+\frac{\sigma}{\tau}\vec{N}-\frac{1}{\varepsilon_1}\vec{B}_2. \tag{15}$$

ii) There is a relation among curvatures of $\vec{\alpha}$ as (14).

iii) $\vec{\alpha}=\vec{\alpha}(s)$ never lies on S_2^3, thus $\vec{\alpha}$ cannot be a spherical curve.

Case 2. Let $m_2=0$ and $\vec{\alpha}$ lies on S_2^3. Thus, from $(10)_2$, we easily have $m_1+m_3\sigma=0$. Considering $(10)_3$ and $(10)_4$, we get second order differential equation respect to m_3

$$\frac{d}{ds}\left(\frac{1}{\sigma}\frac{dm_3}{ds}\right)-m_3\tau=0. \tag{16}$$

Substituting an exchange variable $t=\int_0^s \sigma\, ds$ to (16), we have

$$\frac{d^2m_3}{dt^2}-m_3\frac{\tau}{\sigma}=0. \tag{17}$$

And, here let us suppose $\frac{\tau}{\sigma}\rangle 0$ In this case, solution of (17) yields that

$$m_3(s)=A_1e^{\sqrt{\frac{\tau}{\sigma}}\int_0^s \sigma\, ds}+A_2e^{-\sqrt{\frac{\tau}{\sigma}}\int_0^s \sigma\, ds}, \tag{18}$$

where A_1 and A_2 real numbers. Thereby, we write other components

$$\begin{cases} m_1(s)=-\sigma\left(A_1e^{\sqrt{\frac{\tau}{\sigma}}\int_0^s \sigma\, ds}+A_2e^{-\sqrt{\frac{\tau}{\sigma}}\int_0^s \sigma\, ds}\right) \\ m_4(s)=\dfrac{1}{\varepsilon_2}\sqrt{\dfrac{\tau}{\sigma}}\left(A_1e^{\sqrt{\frac{\tau}{\sigma}}\int_0^s \sigma\, ds}-A_2e^{-\sqrt{\frac{\tau}{\sigma}}\int_0^s \sigma\, ds}\right) \end{cases}. \tag{19}$$

Remark 1. The case $\frac{\tau}{\sigma}\langle 0$ can be easily obtained in terms of (17).

Theorem 3. Let $\vec{\alpha}=\vec{\alpha}(s)$ be a pseudo null unit speed curve. And at the same time, suppose that second component of the position vector of $\vec{\alpha}$ is zero and $\vec{\alpha}$ lies on S_2^3. Then;

i) Position vector of $\vec{\alpha}$, which lies fully in TB_1B_2 subspace, can be composed by the equations (18), $(19)_1$ and $(19)_2$.

ii) There is a relation among curvatures of $\vec{\alpha}$,

$$\frac{d}{ds}\left(m_1\sigma\right)+\varepsilon_1 m_4+1=0. \tag{20}$$

Case 3. $m_3=0$. In this case, from $(10)_4$, we have $m_4=c_4=$constant. Solving equations $(10)_3$ and $(10)_1$, we write other components

$$\begin{cases} m_1=(1+\varepsilon_1c_4)s+c, \\ m_2=\varepsilon_2c_4\dfrac{\sigma}{\tau}. \end{cases} \tag{21}$$

Using $(10)_2$, we have

$$\frac{\sigma}{\tau}=-\frac{1}{c_4\varepsilon_2}\left(\frac{s^2}{2}+cs\right)+\frac{s^2}{2}. \tag{22}$$

Last, we give the following theorem.

Theorem 4. Let $\vec{\alpha}=\vec{\alpha}(s)$ be a pseudo null unit speed curve and third component of the position vector of $\vec{\alpha}$ be zero. Then;

i) Position vector of $\vec{\alpha}$, which lies fully in TNB_2, can be composed by previous equations.

ii) There are no spherical curves which lie fully in TNB_2 subspace.

iii) There is a relation among curvatures of $\vec{\alpha}$ as (22).

Remark 2. The case $m_4=c_4\neq 0$ is similar to Case 3.

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