

# Some New Subclasses of Nonsingular H-matrices

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**Abstract**—In this paper, we obtain some new subclasses of nonsingular H-matrices by using  $\alpha$  diagonally dominant matrix.

**Keywords**—H-matrix, diagonal dominance,  $\alpha$  diagonally dominant matrix.

## I. INTRODUCTION

LET  $A = (a_{ij})_{n \times n} \in C^{n \times n}$ ,  $M(A) = (m_{ij})$ , where

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases} \quad i = 1, 2, \dots, n.$$

Then we call  $M(A)$  is the comparison matrix of  $A$ . Suppose  $A$  is an  $n$  by  $n$  matrix over the field of real numbers. If  $A$  can be expressed in the form  $A = \sigma I - B$  where  $B$  is a nonnegative matrix and  $\sigma > \rho(B)$  the spectral radius of  $B$ , then  $A$  is called a nonsingular M-matrix. This class of matrices has been much studied [1].

If  $M(A)$  is nonsingular M-matrix, then  $A$  is called a nonsingular H-matrix. If

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, 2, \dots, n,$$

then we say  $A$  is strictly diagonally dominant. If there exist positive number  $x_1, x_2, \dots, x_n$  such that

$$x_i |a_{ii}| > \sum_{j \neq i} x_j |a_{ij}|, \quad i = 1, 2, \dots, n,$$

then we say  $A$  is generalized strictly diagonally dominant [2].

A matrix  $A$  be a nonsingular H-matrix is equivalent to that  $A$  be a generalized strictly diagonally dominant matrix [3].

H-matrices have important applications, for instance, in iterative methods of numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming. But how to determine whether an  $n$  by  $n$  complex matrix is a nonsingular H-matrix is not easy in practice. In this paper, we will give some new subclasses of nonsingular H-matrices.

## II. MAIN RESULTS

We will use the following notations:

$$R_i(\cdot) = \sum_{j \neq i} |a_{ij}|, \quad S_i(\cdot) = \sum_{j \neq i} |a_{ji}|, \\ i \in \langle n \rangle = \{1, 2, \dots, n\},$$

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$$I(A) = \left\{ \nu \in S(A) \mid \begin{array}{l} \prod_{i \in \nu} |a_{ij}| / \prod_{i \in \nu} R_i(A) \\ \text{or} \prod_{i \in \nu} |a_{ij}| / \prod_{i \in \nu} C_i(A) \end{array} \right\},$$

$$\frac{\alpha}{2} \{i \in \langle n \rangle \mid |a_{ij}| > \alpha R_i(A) + (1 - \alpha) S_i(A)\}, \\ \frac{\alpha}{1} \{i \in \langle n \rangle \mid \frac{\alpha}{2},$$

$$\beta_\alpha \{i \in \langle n \rangle \mid \alpha R_i(A) + (1 - \alpha) S_i(A) > 0\}.$$

**Definition** [4] Let  $A = (a_{ij}) \in C^{n \times n}$ . If there exists  $\alpha \in (0, 1)$ ,  $|a_{ii}| \geq \alpha R_i + (1 - \alpha) S_i$  ( $i \in \langle n \rangle$ ) holds, then we call  $A$  is  $\alpha$  diagonally dominant and denote  $A \in D_0(\alpha)$ . If all the inequations are strict, we denote  $A \in D(\alpha)$ .

**Lemma 1** [4] Let  $A = (a_{ij}) \in C^{n \times n}$ . If  $A \in D(\alpha)$ , then  $A$  is a nonsingular H-matrix.

**Lemma 2** [4] Let  $A = (a_{ij}) \in C^{n \times n}$ . If for  $\alpha \in (0, 1)$ ,  $|a_{ii}| \geq \alpha R_i + (1 - \alpha) S_i$  holds, and for  $i$  which satisfies  $|a_{ii}| < \alpha R_i + (1 - \alpha) S_i$  there exists a non-zero elements chain  $a_{i i_1}, a_{i_1 i_2}, \dots, a_{i_{p-1} i_p} \neq 0$  such that  $j \in J = \{j \in \langle n \rangle \mid |a_{ii}| > \alpha R_i + (1 - \alpha) S_i\} \cap \Phi$ , then  $A$  is a nonsingular H-matrix.

**Theorem 1.** Let  $A = (a_{ij})_{n \times n} \in C^{n \times n}$ , for  $\alpha \in (0, 1)$ , if

$$|a_{ii}| > \frac{\alpha}{x_i} \sum_{j \neq i} |a_{ij}| x_j + \frac{1 - \alpha}{y_i} \sum_{j \neq i} |a_{ji}| y_j, \quad i \in \langle n \rangle \quad (1)$$

$$|a_{ii}| \geq \frac{\alpha}{x_i} \left( \sum_{j \in N_1^\alpha} |a_{ij}| x_j + \sum_{j \in N_2^\alpha, j \neq i} |a_{ij}| \right) \\ + \frac{1 - \alpha}{y_i} \left( \sum_{j \in N_1^\alpha} |a_{ji}| y_j + \sum_{j \in N_2^\alpha, j \neq i} |a_{ji}| \right), \quad i \in \langle n \rangle \quad (2)$$

where  $0 < x_i < 1$ ,  $0 < y_i < 1$ ,  $i \in \langle n \rangle$ . Then  $A$  is a nonsingular H-matrix.

**Proof:** Let

$$b_i = \frac{x_i y_i |a_{ii}| - y_i \alpha \sum_{j \neq i} |a_{ij}| x_j - (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j x_i}{y_i \alpha \sum_{j \neq i} |a_{ij}| x_j + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j x_i}, \quad (3) \\ i \in \langle n \rangle.$$

From (1) we know that  $0 < b_i < +\infty$ . Let

$$c_i = \frac{\sum_{j \in N_1^\alpha} |a_{ij}| x_j}{\sum_{j \in N_2^\alpha} |a_{ij}|}, \quad f_i = \frac{\sum_{j \in N_1^\alpha} |a_{ji}| y_j}{\sum_{j \in N_2^\alpha} |a_{ji}|} \quad (4)$$

when  $\sum_{j \in N_2^\alpha} |a_{ij}| = 0$ ,  $\sum_{j \in N_2^\alpha} |a_{ji}| = 0$ , we denote  $c_i = \infty$ ,  $f_i = \infty$ , according to the hypothesis of this paper, we have  $c_i > 0$ ,  $f_i > 0$ .

We denote

$$\frac{\alpha}{x} = \{i \in \frac{\alpha}{2} | x_i = 1\}, \quad \frac{\alpha}{y} = \{i \in \frac{\alpha}{2} | y_i = 1\}.$$

There must exist a small enough positive number  $\varepsilon$ , such that

$$0 < \varepsilon < \min \left\{ \min_{i \in N_1^\alpha} \{c_i\}, \min_{i \in N_1^\alpha} \{f_i\}, \min_{i \in N_2^\alpha \setminus N_x^\alpha} \{1 - x_i\}, \min_{i \in N_2^\alpha \setminus N_y^\alpha} \{1 - y_i\} \right\}.$$

We choose positive diagonal matrix

$$\text{diag}(d_1, d_2, \dots, d_n)$$

and

$$\text{diag}(e_1, e_2, \dots, e_n),$$

where

$$d_i = \begin{cases} x_i & i \in \frac{\alpha}{1} \\ x_i & i \in \frac{\alpha}{x} \\ x_i + \varepsilon & i \in \frac{\alpha}{2} \setminus \frac{\alpha}{x} \end{cases}, \quad e_i = \begin{cases} y_i & i \in \frac{\alpha}{1} \\ y_i & i \in \frac{\alpha}{y} \\ y_i + \varepsilon & i \in \frac{\alpha}{2} \setminus \frac{\alpha}{y} \end{cases} \quad (b_{ij})$$

In the follows, we just need to prove that  $(b_{ij})$  is a strictly  $\alpha$  diagonally dominant matrix.

For  $\forall i \in \frac{\alpha}{1}$ , according to (3) we have

$$|a_{ij}| x_i y_i = (1 + b_i) \left( \alpha \sum_{j \neq i} |a_{ij}| x_j y_j + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j x_i \right). \quad (5)$$

There will be four cases:

Case one:  $\sum_{j \in N_2^\alpha} |a_{ij}| = 0$ ,  $\sum_{j \in N_2^\alpha} |a_{ji}| = 0$ , according to (1) we have:

$$b_{ii} = y_i |a_{ii}| x_i > y_i \alpha \sum_{j \neq i} |a_{ij}| x_j + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j \cdot x_i$$

$$\alpha \sum_{j \in N_1^\alpha} |b_{ij}| + (1 - \alpha) \sum_{j \in N_1^\alpha} |b_{ji}|$$

$$\alpha R_i(\cdot) + (1 - \alpha) S_i(\cdot).$$

Case two:  $\sum_{j \in N_2^\alpha} |a_{ij}| = 0$ ,  $\sum_{j \in N_2^\alpha} |a_{ji}| \neq 0$ , in this case,  $|a_{ij}| = 0$  for any  $j \in \frac{\alpha}{2}$ . according to (4) we have:

$$\varepsilon < f_i \Leftrightarrow \varepsilon \sum_{j \in N_2^\alpha} |a_{ji}| < b_i \sum_{j \neq i} |a_{ji}| y_j$$

$$\Leftrightarrow (1 + b_i) \sum_{j \neq i} |a_{ji}| y_j > \sum_{j \neq i} |a_{ji}| y_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ji}|. \quad (6)$$

With (5), (6) and the hypothesis of the paper, we have:

$$b_{ii} = y_i |a_{ii}| x_i = y_i \alpha \sum_{j \neq i} |a_{ij}| x_j + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j x_i$$

$$> y_i \alpha \sum_{\substack{j \in \frac{\alpha}{1} \\ j \neq i}} |a_{ij}| x_j$$

$$+ (1 - \alpha) \left( \sum_{j \neq i} |a_{ji}| y_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ji}| \right) x_i$$

$$> y_i \alpha \sum_{\substack{j \in \frac{\alpha}{1} \\ j \neq i}} |a_{ij}| x_j + x_i (1 - \alpha) \times$$

$$\left( \sum_{\substack{j \in \frac{\alpha}{1} \\ j \neq i}} |a_{ji}| y_j + \sum_{j \in N_y^\alpha} |a_{ji}| y_j + \sum_{j \in N_2^\alpha \setminus N_y^\alpha} |a_{ji}| (y_j + \varepsilon) \right)$$

$$\alpha R_i(\cdot) + (1 - \alpha) S_i(\cdot).$$

Case three:  $\sum_{j \in N_2^\alpha} |a_{ij}| \neq 0$ ,  $\sum_{j \in N_2^\alpha} |a_{ji}| = 0$ . As the same proof of case two, we can obtain

$$|b_{ii}| > R_i(\cdot)^\alpha C_i(\cdot)^{1-\alpha}.$$

Case four:  $\sum_{j \in N_2^\alpha} |a_{ij}| \neq 0$ ,  $\sum_{j \in N_2^\alpha} |a_{ji}| \neq 0$  according to (4) we have:

$$\varepsilon < c_i \Leftrightarrow \varepsilon \sum_{j \in N_2^\alpha} |a_{ij}| < b_i \sum_{j \neq i} |a_{ij}| x_j$$

$$\Leftrightarrow (1 + b_i) \sum_{j \neq i} |a_{ij}| x_j > \sum_{j \neq i} |a_{ij}| x_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ij}|.$$

From the above inequation and the inequation (6), we have

$$b_{ii} = y_i |a_{ii}| x_i = \alpha (1 + b_i) y_i \sum_{j \neq i} |a_{ij}| x_j + (1 - \alpha) (1 + b_i) x_i \sum_{j \neq i} |a_{ji}| y_j$$

$$> \alpha y_i \left( \sum_{j \neq i} |a_{ij}| x_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ij}| \right) + (1 - \alpha) x_i \left( \sum_{j \neq i} |a_{ji}| y_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ji}| \right)$$

$$\geq \alpha y_i \left( \sum_{j \in N_1^\alpha \cup N_y^\alpha} |a_{ij}| x_j + \sum_{j \in N_2^\alpha \setminus N_x^\alpha} (x_j + \varepsilon) |a_{ij}| \right)$$

$$+x_i(1-\alpha)\left(\sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{j \in N_2^\alpha \setminus N_y^\alpha} (y_j + \varepsilon)|a_{ji}|\right)$$

$$\alpha R_i(\cdot) + (1-\alpha)S_i(\cdot).$$

For any  $i \in \frac{\alpha}{2}$ , from the choice of  $\varepsilon$  and the positive diagonal matrices D and E, we know that  $0 < d_i, e_i \leq 1$ , for any  $i \in \frac{\alpha}{2}$ .

Case one:  $i \in \frac{\alpha}{x} \cap \frac{\alpha}{y}$

$$|b_{ii}| - |a_{ii}| \geq \alpha \left( \sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right)$$

$$+ (1-\alpha) \left( \sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right)$$

$$> \alpha \left( \sum_{j \in \langle n \rangle \setminus N_x^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{x} \\ j \neq i}} |a_{ij}| \right)$$

$$+ (1-\alpha) \left( \sum_{j \in \langle n \rangle \setminus N_y^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{y} \\ j \neq i}} |a_{ji}| \right)$$

$$\alpha R_i(\cdot) + (1-\alpha)S_i(\cdot).$$

Case two:  $i \in \frac{\alpha}{x}, i \notin \frac{\alpha}{y}$ , if  $\alpha \neq 1$ , from (2) we have

$$\begin{aligned} & (y_i + \varepsilon)|a_{ii}| \\ & \geq (y_i + \varepsilon)\alpha \left( \sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right) \\ & + (1-\alpha) \left( \sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right) \end{aligned} \quad (7)$$

Hence

$$|b_{ii}| - (y_i + \varepsilon)|a_{ii}|$$

$$\geq (y_i + \varepsilon)\alpha \left( \sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right)$$

$$+ (1-\alpha) \left( \sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right)$$

$$> (y_i + \varepsilon)\alpha \left( \sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{j \in N_2^\alpha \setminus N_x^\alpha} |a_{ij}|(x_j + \varepsilon) + \sum_{j \in N_x^\alpha} |a_{ij}| \right)$$

$$+ (1-\alpha) \left( \sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in N_2^\alpha \setminus N_y^\alpha \\ j \neq i}} |a_{ji}|(y_j + \varepsilon) + \sum_{j \in N_y^\alpha} |a_{ji}| \right)$$

$$\alpha R_i(\cdot) + (1-\alpha)S_i(\cdot).$$

Case three:  $i \notin \frac{\alpha}{x}, i \in \frac{\alpha}{y}$ , as the same proof of case two, we can obtain

$$|b_{ii}| > \alpha R_i(\cdot) + (1-\alpha)S_i(\cdot).$$

Case four:  $i \notin \frac{\alpha}{x}, i \notin \frac{\alpha}{y}$ , from (2) we have

$$(y_i + \varepsilon)|a_{ii}|(x_i + \varepsilon)$$

$$\geq (y_i + \varepsilon)\alpha \left( \sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right)$$

$$+ (1-\alpha) \left( \sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right) (x_i + \varepsilon)$$

Since

$$(y_i + \varepsilon)\alpha \left( \sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right) > \alpha R_i(\cdot),$$

$$(1-\alpha) \left( \sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right) (x_i + \varepsilon)$$

$$> (1 - \alpha) S_i ( ),$$

we have

$$\begin{aligned} & |b_{ii}| - (y_i + \varepsilon) |a_{ii}| (x_i + \varepsilon) \\ & \geq (y_i + \varepsilon) \alpha \left( \sum_{j \in N_1^\alpha} |a_{ij}| x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right) \\ & + (1 - \alpha) \left( \sum_{j \in N_1^\alpha} |a_{ji}| y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right) (x_i + \varepsilon) \\ & > \alpha R_i ( ) + (1 - \alpha) S_i ( ). \end{aligned}$$

We see that for any  $i \in \langle n \rangle$ , we have  $|b_{ii}| > \alpha R_i ( ) + (1 - \alpha) S_i ( )$ . According to Lemma 1, we know that matrix  $B$  is a nonsingular H-matrix, so matrix  $A$  is a nonsingular H-matrix.

Let  $A = (a_{ij})_{n \times n} \in C^{n \times n}$ ,  $0 < x_i, y_i < 1, i \in \langle n \rangle$  satisfy the equation (2), we denote

$$K_\alpha = \left\{ i \in \langle n \rangle \mid |a_{ii}| > \frac{\alpha}{x_i} \sum_{j \neq i} |a_{ij}| x_j + \frac{1 - \alpha}{y_i} \sum_{j \neq i} |a_{ji}| y_j \right\}.$$

**Theorem 2** Let  $A = (a_{ij})_{n \times n} \in C^{n \times n}$ , for  $\alpha \in (0, 1)$ , if  $0 < x_i < 1, 0 < y_i < 1, i \in \langle n \rangle$  satisfy the inequations (2) and

$$|a_{ii}| \geq \frac{\alpha}{x_i} \sum_{j \neq i} |a_{ij}| x_j + \frac{1 - \alpha}{y_i} \sum_{j \neq i} |a_{ji}| y_j, \quad i \in \frac{\alpha}{1} \quad (8)$$

and  $K_\alpha \neq \emptyset$ , for any  $i_0 \in (\langle n \rangle \setminus K_\alpha)$ , there exists a nonzero elements chain  $a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_k} \neq 0$  such that  $i_k \in K_\alpha$ , then  $A$  is a nonsingular H-matrix.

**Proof:** we structure two positive diagonal matrices:  $D = \text{diag}(x_1, x_2, \dots, x_n)$  and  $E = \text{diag}(y_1, y_2, \dots, y_n)$ , and notes  $B = (b_{ij}) = EAD$ . So for any  $i \in \langle n \rangle$ , we have

$$|b_{ii}| \geq \alpha R_i (B) + (1 - \alpha) S_i (B).$$

Obviously,  $K_\alpha$  can be note

$$K_\alpha = \{ i \in \langle n \rangle \mid |b_{ii}| > \alpha R_i (B) + (1 - \alpha) S_i (B) \},$$

for any  $i_0 \in K_\alpha$ , we have  $b_{i_0 i_1} b_{i_1 i_2} \cdots b_{i_{k-1} i_k} \neq 0$  such that  $i_k \in K_\alpha$ . So according to Lemma 2, we know that matrix  $B$  is a nonsingular H-matrix, so matrix  $A$  is a nonsingular H-matrix.

From Theorem 2, we can get the following corollary.

**Corollary** Let  $A = (a_{ij})_{n \times n} \in C^{n \times n}$  be irreducible, for  $\alpha \in (0, 1)$ , if  $0 < x_i < 1, 0 < y_i < 1, i \in \langle n \rangle$  satisfy the inequations (2) and (8),  $\tilde{I} \neq \emptyset$ , where

$$\tilde{I} = \left\{ v \in S(A) \mid \begin{array}{l} y_i |a_{ii}| x_i / \tilde{R}_i(A) \\ \text{or } y_i |a_{ii}| x_i / \tilde{C}_i(A) \end{array} \right\},$$

$$\tilde{R}_i(A) = y_i \sum_{j \neq i} |a_{ij}| x_j, \quad \tilde{C}_i(A) = x_i \sum_{j \neq i} |a_{ji}| y_j,$$

then  $A$  is a nonsingular H-matrix.

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