

# The Balanced Hamiltonian Cycle on the Toroidal Mesh Graphs

Wen-Fang Peng, Justie Su-Tzu Juan

**Abstract**—The balanced Hamiltonian cycle problem is a quiet new topic of graph theorem. Given a graph  $G = (V, E)$ , whose edge set can be partitioned into  $k$  dimensions, for positive integer  $k$  and a Hamiltonian cycle  $C$  on  $G$ . The set of all  $i$ -dimensional edge of  $C$ , which is a subset by  $E(C)$ , is denoted as  $E_i(C)$ . If  $\|E_i(C) - E_j(C)\| \leq 1$  for  $1 \leq i < j \leq k$ ,  $C$  is called a balanced Hamiltonian cycle. In this paper, the proposed result shows that there exists a balanced Hamiltonian cycle for any Toroidal Mesh graph  $T_{m,n}$  if and only if  $m, n \geq 3$  and Toroidal Mesh graph  $nm \neq 2 \pmod{4}$ , and how to find a balanced Hamiltonian cycle on  $T_{m,n}$ , for  $n, m \geq 3$  and  $mn \neq 2 \pmod{4}$ .

**Keywords**—Hamiltonian cycle; balanced; Cartesian product

## I. INTRODUCTION

THE research of optimal encode uses gray-code encode to signify the information of  $n$ -bit about the application of 3D scanning, which has been mentioned in the references [1], [2], [3], [5] and [8]. The utility of gray-code will decrease the consumption of resource and increase the precision. Nevertheless, there would be some problem when deal with those information of transforming between 1 and 0, such as it will spend much more cost in identification. How to decrease the cost in dealing with such problems is important. Hence, in this paper, it discusses a method to decrease the number of transformation between 0 and 1 in the some dimension.

Balanced Hamiltonian cycle (BHC) problems are widely discussed in recent years. Several issues about BHC have been proposed by other researchers [2]. Wang et al proposed the BHC on  $C_n \times C_n$  for any positive integer  $n \geq 3$ . This paper proposes an extended research about the BHC on  $C_m \times C_n$ , also called  $T_{m,n}$ , for any positive integer  $m \geq 3, n \geq 3$ .

Next section introduces some background knowledge about the Hamiltonian cycle (HC) problem, Cartesian product, and some related definitions. Section 3 describes the main results, the research about the BHC problem on  $T_{m,n}$ , for  $m, n \geq 3$ , proposed by this paper. Finally, the last section makes a conclusion and lists the future work.

## II. DEFINITION AND NOTATION

This paper denotes the symbols below by referring to [4], [6], [7] and [9]. Define a walk  $W$ , which is in a graph  $G = (V, E)$ , is a sequence  $w = x_1e_1x_2e_2 \dots x_k e_k y$  for  $x_1, x_2, \dots, x_k, y \in V(G)$  and  $e_1, e_2, \dots, e_k \in E(G)$ . And let  $x$  be the *origin vertex* of  $W$ ,  $y$  be the *terminus vertex* of  $W$ . If all of vertices in this walk are different, a walk  $W$  is denoted a *path*. When the origin vertex and the terminus vertex are the same vertex, then this path is denoted a *cycle*.

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A *Hamiltonian path* of graph  $G = (V, E)$  is a path that contains all vertices. A *Hamiltonian cycle* of  $G$  is a cycle that contains all vertices.

Given a Hamiltonian cycle  $C$  on a graph  $G = (V, E)$ , whose edge set can be partitioned into  $k$  dimensions, for positive integer  $k$ . And let  $E_i(C)$  represents the set of all  $i$ -dimensional edge of  $C$  which is a subset by  $E(C)$ . If  $\|E_i(C) - E_j(C)\| \leq 1$  for  $1 \leq i \leq j \leq k$ ,  $C$  is called a balanced Hamiltonian cycle.

Let  $C_n$  denote a cycle with  $n$  vertices, given two graph  $G_1, G_2$ , the Cartesian product  $G_1 \times G_2$  of  $G_1$  and  $G_2$  is a graph with vertex set  $V(G_1 \times G_2) = \{(x, y) \mid x \in V(G_1), y \in V(G_2)\}$  and the edge set  $\{(u, v), (u', v') \mid u = u' \in V(G_1) \text{ and } (v, v') \in E(G_2) \text{ or } v = v' \in V(G_2) \text{ and } (u, u') \in E(G_1)\}$ . The *toroidal mesh graph*,  $T_{m,n}$  is the graph  $C_m \times C_n$ .

The dimension of  $T_{m,n}$  is 2. Given an Hamiltonian cycle  $C$  of  $T_{m,n}$ , let  $E_1(C) = \{(x_i, y_j)(x_{i+1}, y_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$  and  $E_2(C) = \{(x_i, y_j)(x_i, y_{j+1}) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$  mean the *1-dimension edge set* and *2-dimension edge set* of a Hamiltonian cycle  $C$ , respectively. Thus, the relation between vertex number and edge number of Hamiltonian cycle  $C$  is  $|V(C)| = |V(C_n \times C_m)| = |E_1(C)| + |E_2(C)| = mn$ . If  $C$  satisfied that  $\|E_1(C) - E_2(C)\| \leq 1$ , it presents that  $C$  is balanced.

In this paper, when we draw a figure of  $T_{m,n}$ ,  $m$  is denoted the number of vertices on  $x$ -axis, and  $n$  is the number of vertices on  $y$ -axis, respectively. Besides, for any vertices  $(x, y)$  of  $T_{m,n}$ ,  $x$  is called the *1<sup>st</sup>-dimension* and  $y$  is called the *2<sup>nd</sup>-dimension*. Furthermore, we define the lower-left vertex of  $T_{m,n}$  to be the *origin vertex* and set it as  $(1, 1)$ . Fig. 1 shows an example of  $T_{3,4}$ .

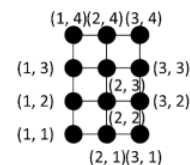


Fig. 1  $T_{3,4}$

The next section discusses the methods for getting the balanced Hamiltonian cycle on  $T_{m,n}$  for  $n, m \geq 3$ .

## III. MAIN RESULTS

This section gives theorem1 for prove  $\exists$ , and gives some cases to prove Theorem 3 that there exists a BHC on  $T_{m,n}$  for positive integers  $n, m \geq 3$ , except for the situation on  $mn \pmod{4} = 2$ .

*Theorem 1: For  $mn \pmod{4} = 2$ , there is no any balanced Hamiltonian cycle  $C$  exists on  $T_{m,n}$ .*

*Proof.*

When  $mn \pmod{4} = 2$ , one of the following case will hold. (i)  $n$

mod 4 = 2 and  $m$  is odd; (ii)  $m \bmod 4 = 2$  and  $n$  is odd. Without loss of generality, we say  $n \bmod 4 = 2$  and  $m$  is odd. Furthermore, let  $n = 4k_1 + 2$  and  $m = 2k_2 + 1$  for some positive  $k_1$  and  $k_2$ . Assume that there exists a balanced Hamiltonian cycle  $C^*$  on  $T_{m,n}$ . Since  $V(C^*) = mn = (4k_1 + 2)(2k_2 + 1) = 8k_1k_2 + 4k_2 + 4k_1 + 2 = 2(4k_1k_2 + 2k_1 + 2k_2 + 1)$ ,  $|E_1(C^*)| = |E_2(C^*)| = mn / 2 = 2(2k_1k_2 + k_1 + k_2) + 1$  is an odd integer.

We call a vertex  $u$  in  $V(C^*)$  is black if  $u \in \{(x, y) \mid 1 \leq x \leq m, 1 \leq y \leq n \text{ and } y \text{ is odd}\}$ ; white if  $u \in \{(x, y) \mid 1 \leq x \leq m, 1 \leq y \leq n \text{ and } y \text{ is even}\}$ . Hence the origin vertex is black. According to the definition of  $E_2(C^*)$ ,  $E_2(C^*)$  should trace black point to white point or white point to black point. After tracing all edges of  $C^*$ , find the terminate vertex of  $C^*$  is white due to  $|E_2(C^*)|$  is odd. Obviously, the origin vertex and the terminate vertex of  $C^*$  are different. That is a contradiction. So, there is no BHC on  $T_{m,n}$  when  $mn \bmod 4 = 2$ .

**Lemma 2:** For  $n = 3$ ,  $m \geq 3$  and  $m$  is odd, there is a balanced Hamiltonian cycle on  $T_{m,n}$ .

*Proof.*

The proof is divided into two cases. Case 1 discusses the condition on  $m \bmod 4 = 1$  and  $n = 3$ ; Case 2 discusses the state on  $m \bmod 4 = 3$  and  $n = 3$ .

**Case 1.  $m \bmod 4 = 1$  and  $n = 3$**

In this section,  $T_{m,3}$  consists of the BHC on  $T_{4,3}$  and the BHC on  $T_{5,3}$ , as shown in Fig. 2 and Fig. 3, respectively. Besides, Fig. 4 indicates how to connect all figures. First of all, let  $x = (m - 5) / 4$ , and inset Fig. 2 for  $x$  times on right side of Fig. 3 when  $m > 5$  and  $n = 3$ . Then, delete edge set  $E_1 = \{(6 + 4i, 3)(9 + 4i, 3) \mid 0 \leq i \leq (m - 9) / 4\} \cup (1, 3)(5, 3)$ , and add edge set  $E_2 = \{(5 + 4i, 3)(6 + 4i, 3) \mid 0 \leq i \leq (m - 9) / 4\} \cup (1, 3)(m, 3)$ . After these steps, a Hamiltonian cycle  $C$  on  $T_{m,3}$  is generated, whose  $|E_1(C)| = 7 + 6x$  and  $|E_2(C)| = 8 + 6x$ . Consequently,  $\|E_1(C) - E_2(C)\| = 1$ ,  $C$  satisfies the definition of BHC.

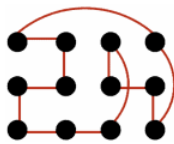


Fig. 2 The BHC on  $T_{4,3}$

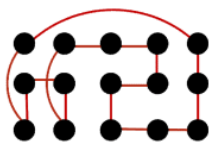


Fig. 3 The BHC on  $T_{5,3}$

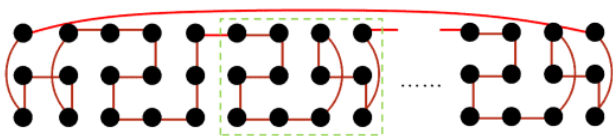


Fig. 4 The BHC on  $T_{m,3}$

**Case 2.  $m \bmod 4 = 3$  and  $n = 3$**

Compare Fig. 4 with Fig. 6, which indicates how to construct the BHC on  $T_{m,3}$ , there is only one difference at the beginning. As a result, refer to Case 3.1, replace Fig. 3 with Fig. 5, which is one of possible BHCs on  $T_{3,3}$ , and revise  $x = (m - 3) / 4$ . Then correct the edge set  $E_3 = \{(4 + 4i, 3)(7 + 4i, 3) \mid 1 \leq i \leq (m - 7) / 4\} \cup (1, 1)(3, 3)$  and  $E_4 = \{(3 + 4i, 3)(4 + 4i, 3) \mid 0 \leq i \leq (m - 7) / 4\} \cup (1, 3)(m, 3)$ , respectively. In the end, a Hamiltonian cycle  $C$  on  $T_{m,3}$  is built, which  $|E_1(C)| = 5 + 6x$  and  $|E_2(C)| = 4 + 6x$ . Obviously,  $C$  satisfies the definition of BHC as a result of  $\|E_1(C) - E_2(C)\| = 1$ .

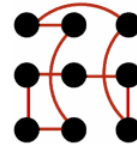


Fig. 5 The BHC on  $T_{3,3}$

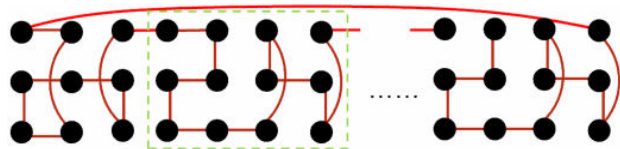


Fig. 6 The BHC  $T_{m,3}$

**Theorem 3:** For  $n, m \geq 3$ , there is a balanced Hamiltonian cycle on  $T_{m,n}$  except for the state on  $mn \bmod 4 = 2$ .

*Proof.*

According to the condition of even or odd on  $n, m$ , the proof is divided into three cases. Case 1 proposes the condition on  $n, m$  both are even; Case 2 proposes the condition one of  $m, n$  is even and the other is odd; Case 3 discusses the condition on  $n, m$  both are odd.

TABLE I  
THE RESULT OF THIS THEOREM

	$m \bmod 4 = 0$	$m \bmod 4 = 2$	$m \bmod 8 = 1$	$m \bmod 8 = 5$	$m \bmod 8 = 3$	$m \bmod 8 = 7$
$n \bmod 4 = 0$	Case 1.1		Case 2.1			
$n \bmod 4 = 2$	Case 1.2	Case 1.3	Case 2.2			
$n \bmod 4 = 1$	Case 2.1	Case 2.2	Case 3.1	Case 3.3	Case 3.2	Case 3.4
$n \bmod 4 = 3, n \geq 7$			Case 3.5	Case 3.7	Case 3.6	Case 3.8
$n = 3$	Lemma 2 Case 1			Lemma 2 Case 2		

**Case 1.  $n, m$  both are even**

This case is separated into three subcases for discussion. Case 1.1, Case 1.2 and Case 1.3 consider the states on  $m$  is even and  $n \bmod 4 = 0, m \bmod 4 = 0$  and  $n \bmod 4 = 2, m \bmod 4 = 2$  and  $n \bmod 4 = 2$ , respectively.

**Case 1.1.  $m$  is even and  $n \bmod 4 = 0$**

Fig. 7 and Fig. 8 show one of the possible HCs on  $T_{2,4}$  and one of the possible BHCs on  $T_{4,4}$ , respectively. When  $m > 4$  and

$n = 4$ , let  $x = (m - 4) / 2$ , and then duplicate Fig. 7 for  $x$  times. Next, inset them on the right side of Fig. 8 mentioned above. Then delete edge set  $E_5 = \{(i, 2)(i, 3) \mid 4 \leq i \leq m - 1\}$ , and insert edge set  $E_6 = \{(i, 2)(i + 1, 2) \cup (i, 3)(i + 1, 3) \mid 4 \leq i \leq m - 2 \text{ and } i \text{ is even}\}$ .

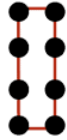


Fig. 7 The HC on  $T_{2,4}$

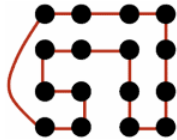


Fig. 8 The BHC on  $T_{4,4}$

After these steps, a Hamiltonian cycle  $C$  could be derived as shown in Fig. 9, whose  $|E_1(C)| = 8 + (2 + 2)x = 8 + 4x$ , and  $|E_2(C)| = 7 + 4x + 1 = 8 + 4x$ . Due to  $\|E_1(C) - E_2(C)\| = 0$ ,  $C$  is a BHC of  $T_{m,4}$ .

When  $n > 4$ , Fig. 10 illustrates the way of constructing a BHC on  $T_{m,n}$ . Let  $y = (n - 4) / 4$ , stack  $y + 1$  Cs on the top of each other. Next, delete edge set  $E_7 = \{(1, 1 + 4i)(1, 4 + 4i) \mid 0 \leq i \leq y\}$ , and add edge set  $E_8 = \{(1, 4i)(1, 4i + 1) \mid 1 \leq i \leq y\} \cup (1, 1)(1, n)$ . A BHC on  $T_{m,n}$  for  $m$  is even and  $n \bmod 4 = 0$  is generated.

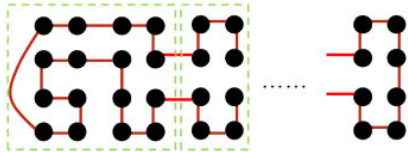


Fig. 9 The BHC on  $T_{m,4}$

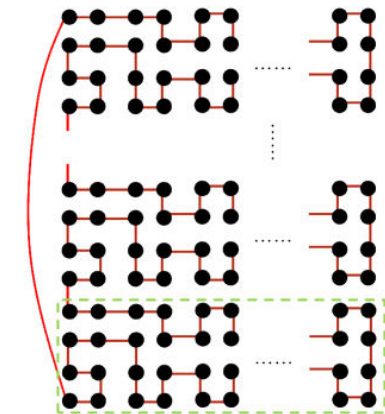


Fig. 10 The BHC on  $T_{m,n}$

Case 1.2  $m \bmod 4 = 0$  and  $n \bmod 4 = 2$

Fig. 11 and Fig. 12 are isomorphic BHC on  $T_{4,6}$ . The following steps indirect how to find a BHC on  $T_{m,6}$  when  $m > 4$ . First, inset Fig. 12 on the right side of Fig. 11 for  $x$  times, where  $x = (m - 4) / 4$ . Second, delete edge set  $E_9 = \{(1 + 4i, 6)(4 + 4i, 6) \mid 0 \leq i \leq x\}$ . Third, add edge set  $E_{10} = \{(4 + 4i, 6)(5 + 4i, 6) \mid 0 \leq i \leq (n - 8) / 4\}$ . By implementing these steps above, a

Hamiltonian cycle  $C$  is produced, whose  $|E_1(C)| = 12 + 12x$  and  $|E_2(C)| = 12 + 12x$ , as shown in Fig. 13. Because of  $\|E_1(C) - E_2(C)\| = 0$ ,  $C$  satisfies the definition of BHC.

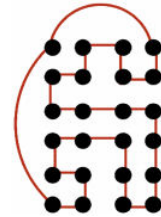


Fig. 11 The BHC on  $T_{4,6}$

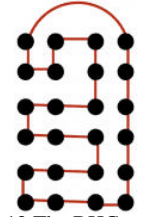


Fig. 12 The BHC on  $T_{4,6}$

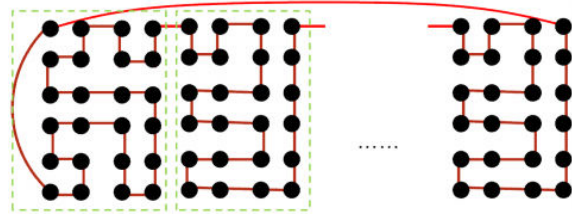


Fig. 13 The BHC on  $T_{m,6}$

When  $n > 6$ , let  $y = (n - 6) / 4$ . Stack Fig. 9 for  $y$  times on top of Fig. 13. Subsequently, delete edge set  $E_{11} = \{(1, 7 + 4i)(1, 10 + 4i) \mid 0 \leq i \leq (n - 10) / 4\} \cup (1, 1)(1, 6)$ , and put edge set  $E_{12} = \{(1, 6 + 4i)(1, 7 + 4i) \mid 0 \leq i \leq (n - 10) / 4\} \cup (1, 1)(1, n)$ . As a result, a BHC on  $T_{m,n}$  can be built, which connect the BHC on  $T_{m,6}$ , shown as Fig. 13, and the  $y$  BHCs on  $T_{m,4}$ , shown as Fig. 9.

Case 1.3.  $m \bmod 4 = 2$  and  $n \bmod 4 = 2$

Fig. 14 shows a BHC on  $T_{6,6}$  which can be used to build a BHC on  $T_{m,6}$ . Consider  $m > 6$ , make  $x = (m - 6) / 4$ . Use Fig. 14 as the beginning, and inset Fig. 12 on the right side for  $x$  times. After that, remove edge set  $E_{13} = \{(7 + 4i, 6)(10 + 4i, 6) \mid 0 \leq i \leq (m - 10) / 4\} \cup (1, 6)(6, 6)$ , and add edge set  $E_{14} = \{(6 + 4i, 6)(7 + 4i, 6) \mid 0 \leq i \leq (m - 10) / 4\} \cup (1, 6)(m, 6)$ . Finally, a Hamiltonian cycle  $C$  on  $T_{m,6}$  is built as shown in Fig. 15, whose  $|E_1(C)| = 18 + 12x$  and  $|E_2(C)| = 18 + 12x$ . Obviously,  $C$  satisfies the definition of BHC owing to  $\|E_1(C) - E_2(C)\| = 0$ .

When  $n > 6$ , the way of constructing the BHC on  $T_{m,n}$  is similar to Case 1.2. Only difference is to replace Fig. 13 with Fig. 15, else parts are the same.

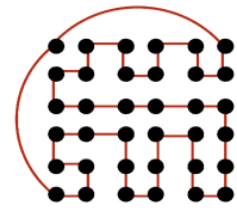


Fig. 14 The BHC on  $T_{6,6}$

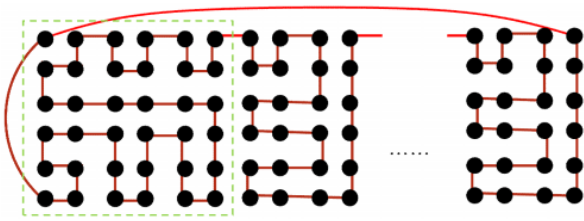


Fig. 15 The BHC on  $T_{m,6}$

Case 2. One of  $m, n$  is even, and the other is odd

For any positive integer  $n, m, T_{m,n}$  and  $T_{n,m}$  are isomorphic. Hence, if one of  $m, n$  is even, and the other is odd, without loss of generality, set that  $n$  is even and  $m$  is odd.

This case can be also divided into two subcases for discussion. Case 2.1 discusses the condition on  $m$  is odd and  $n \bmod 4 = 0$ ; Case 2.2 discusses the state on  $m$  is odd and  $n \bmod 4 = 2$ .

Case 2.1.  $m$  is odd and  $n \bmod 4 = 0$

Fig. 16 shows a possible BHC on  $T_{3,4}$ . When  $m > 3$ , the BHC on  $T_{m,4}$  consists of Fig. 7 and Fig. 16. Fig. 17 illustrates the way of connecting. First, for  $x = (m - 3) / 4$ , inset  $x$  duplicate BHC, which has been shown in Fig. 7, on the right side of Fig. 16. Second, eliminate edge set  $E_{15} = \{(3, 3)(i, 3)(i, 4) \mid 4 \leq i \leq m\} \cup (1, 4)(3, 4) \cup (1, 3)$ . Third, put edge set  $E_{16} = \{(3 + 2i, 3)(4 + 2i, 3) \cup (3 + 2i, 4)(4 + 2i, 4) \mid 0 \leq i \leq (m - 4) / 2\} \cup (1, 3)(m, 3) \cup (1, 4)(m, 4)$  on the graph produced by previous steps. After that, a Hamiltonian cycle  $C$  is established, whose  $|E_1(C)| = 6 + (2 + 2)x = 6 + 4x$  and  $|E_2(C)| = 6 + 4x$ . Due to  $\|E_1(C) - E_2(C)\| = 0$ ,  $C$  satisfies the definition of BHC.

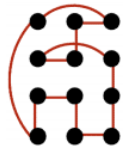


Fig. 16 The BHC on  $T_{3,4}$

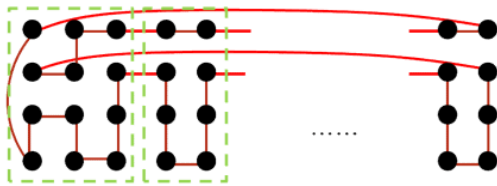


Fig. 17 The BHC on  $T_{m,3}$

When  $n > 4$ , for  $y = (n - 4) / 4$ , stack  $y + 1$  Cs. Next, remove edge set  $E_{17} = \{(1, 1 + 4i)(1, 4 + 4i) \mid 0 \leq i \leq y\}$ , and add edge set  $E_{18} = \{(1, 4i)(1, 4i + 1) \mid 1 \leq i \leq y\} \cup (1, 1)(1, n)$ . Therefore, a BHC on  $T_{m,n}$  is built, as shown in Fig. 18.

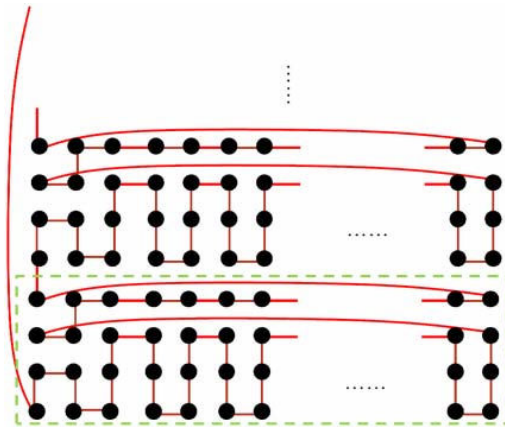


Fig. 18 The BHC on  $T_{m,n}$

Case 2.2:  $n \bmod 4 = 2$  and  $m$  is odd

According to theorem 1, there is no balanced Hamiltonian cycle on  $T_{m,n}$  for  $m \bmod 4$  is odd and  $n \bmod 4 = 2$ .

Case 3.  $m, n$  both are odd

This case is also separated into eight subcases for discussion. Case 3.1 discusses the state on  $n \bmod 4 = 1$  and  $m \bmod 8 = 1$ ; Case 3.2 proposes the state on  $n \bmod 4 = 3$  and  $m \bmod 8 = 3$ ; Case 3.3 consults the operations when  $n \bmod 4 = 1$  and  $m \bmod 8 = 5$ ; Case 3.4 concerns the details when  $n \bmod 4 = 1$  and  $m \bmod 8 = 7$ .

The other four remaining cases propose the method under the condition of  $n > 3$ . Case 3.5 discusses the state on  $n \bmod 4 = 3$  and  $m \bmod 8 = 1$ ; Case 3.6 considers the state on  $n \bmod 4 = 3$  and  $m \bmod 8 = 3$ ; Case 3.7 concerns the operations when  $n \bmod 4 = 3$  and  $m \bmod 8 = 5$ ; Case 3.8 consults the details when  $n \bmod 4 = 3$  and  $m \bmod 8 = 7$ .

Case 3.1.  $m \bmod 8 = 1$  and  $n \bmod 4 = 1$

Fig. 19 and Fig. 20 show one of the possible HCs on  $T_{8,5}$  and one of possible BHCs on  $T_{9,5}$ , respectively. When  $m > 9$  and  $n = 5$ , make  $x = (m - 9) / 8$ . First, use Fig. 20 as the beginning, and inset Fig. 19 for  $x$  times on its right side. Then eliminate edge set  $E_{19} = \{(10 + 8i, 4)(10 + 8i, 5) \cup (17 + 8i, 4)(17 + 8i, 5) \mid 0 \leq i \leq (m - 17) / 8\} \cup (1, 4)(9, 4) \cup (1, 5)(9, 5)$ , and add edge set  $E_{20} = \{(9 + 8i, 4)(10 + 8i, 4) \cup (9 + 8i, 5)(10 + 8i, 5) \mid 0 \leq i \leq (m - 17) / 8\} \cup (1, 4)(m, 4) \cup (1, 5)(1, m)$ . Finally, a Hamiltonian cycle  $C$  on  $T_{m,5}$  is produced, which is shown in Fig. 21. For  $|E_1(C)| = 22 + (18 + 2)x = 22 + 20x$  and  $|E_2(C)| = 23 + 20x$ ,  $C$  satisfies that  $\|E_1(C) - E_2(C)\| = 1$ . Undoubtedly,  $C$  is a BHC of  $T_{m,5}$ .

When  $n > 5$ , let  $y = (n - 5) / 4$ . Use Fig. 21 as base, then stack  $y$  BHCs, which is shown in Fig.12. Next, remove edge set  $E_{21} = \{(1, 6 + 4i)(1, 9 + 4i) \mid 0 \leq i \leq (n - 9) / 4\} \cup (1, 1)(1, 5)$ , and insert edge set  $E_{22} = \{(1, 1)(1, n) \mid 0 \leq i \leq (n - 9) / 4\} \cup (1, 5 + 4i)(1, 6 + 4i)$ . After complete all of the steps, a BHC on  $T_{m,n}$  for  $m \bmod 8 = 1$  and  $n \bmod 4 = 1$  is established.



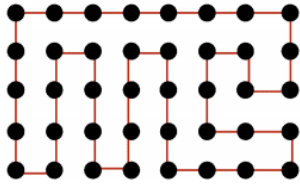


Fig. 19 The HC on  $T_{8,5}$

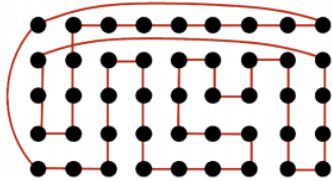


Fig. 20 The BHC on  $T_{9,5}$

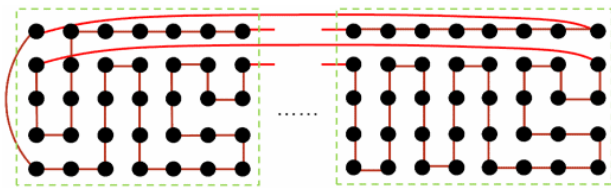


Fig. 21 The BHC on  $T_{m,5}$

Case 3.2.  $m \bmod 8 = 3$  and  $n \bmod 4 = 1$

Fig. 23 represents the way of constructing a BHC on  $T_{m,5}$ . When  $m > 3$ , let  $x = (m - 3) / 8$ , and then duplicate Fig. 19 for  $x$  times. Next, inset them on the right side of the BHC on  $T_{3,5}$ , which is shown in Fig. 22. In order to connect every figure, delete edge set  $E_{23} = \{(4 + 8i, 4)(4 + 8i, 5) \cup (11 + 8i, 4)(11 + 8i, 5) \mid 0 \leq i \leq (m - 11) / 8\} \cup (1, 4)(3, 4) \cup (1, 5)(3, 5)$ , and add edge set  $E_{24} = \{(3 + 8i, 4)(4 + 8i, 4) \cup (3 + 8i, 5)(4 + 8i, 5) \mid 0 \leq i \leq (m - 11) / 8\} \cup (1, 4)(m, 4) \cup (1, 5)(m, 5)$ . By implementing the steps above, a Hamiltonian cycle  $C$  is generated, whose  $|E_1(C)| = 8 + (18 + 2)x = 8 + 20x$  and  $|E_2(C)| = 7 + 20x$ . Due to  $\|E_1(C) - E_2(C)\| = 1$ ,  $C$  satisfies the definition of BHC.

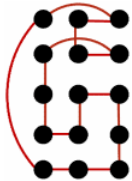


Fig. 22: The BHC on  $T_{3,5}$

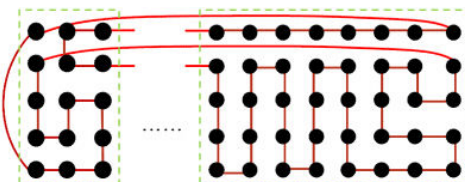


Fig. 23: The BHC on  $T_{m,5}$

When  $n > 5$ , the way of constructing the BHC on  $T_{m,n}$  is similar to Case 3.1. Only one difference is to replace Fig. 21 with Fig. 23.

Case 3.3.  $m \bmod 8 = 5$  and  $n \bmod 4 = 1$

Fig. 24 represents a BHC on  $T_{5,5}$ , which is used to construct the BHC on  $T_{m,5}$ . When  $m > 5$ , let  $x = (m - 5) / 8$ . To begin with, inset Fig. 19 for  $x$  times on the right side of Fig. 24. Next, delete edge set  $E_{25} = \{(6 + 8i, 4)(6 + 8i, 5) \cup (13 + 8i, 4)(13 + 8i, 5) \mid 0 \leq i \leq (m - 13) / 8\} \cup (1, 4)(5, 4) \cup (1, 5)(5, 5)$ , and put edge set  $E_{26} = \{(5 + 8i, 4)(6 + 8i, 4) \cup (5 + 8i, 5)(6 + 8i, 5) \mid 0 \leq i \leq (m - 13) / 8\} \cup (1, 4)(m, 4) \cup (1, 5)(m, 5)$ . Thus, a Hamiltonian cycle  $C$  is established, which is shown in Fig. 24. For  $|E_1(C)| = 12 + (18 + 2)x = 12 + 20x$  and  $|E_2(C)| = 13 + 20x$ ,  $C$  obviously satisfies  $\|E_1(C) - E_2(C)\| = 1$  that make it be a BHC of  $T_{m,5}$  for  $m \bmod 8 = 5$  and  $n \bmod 4 = 1$ .

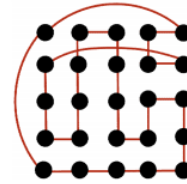


Fig. 24 The BHC on  $T_{5,5}$

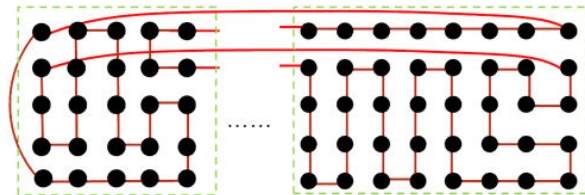


Fig. 24 The BHC on  $T_{m,5}$

Case 3.1 and Case 3.3 operate alike during  $n > 5$ . Only difference is to replace Fig. 21 with Fig. 24.

Case 3.4.  $m \bmod 8 = 7$  and  $n \bmod 4 = 1$

The following steps indirect how to construct a BHC on  $T_{m,5}$ . A BHC on  $T_{7,5}$  is shown in Fig. 25. When  $m > 7$ , make  $x = (m - 7) / 8$ . Then use Fig. 25 as the beginning, and inset Fig. 19 for  $x$  times on the right side. In order to connect all figures, eliminate edge set  $E_{27} = \{(8 + 8i, 4)(8 + 8i, 5) \cup (15 + 8i, 4)(15 + 8i, 5) \mid 0 \leq i \leq (m - 15) / 8\} \cup (1, 4)(7, 4) \cup (1, 5)(7, 5)$ , and insert edge set  $E_{28} = \{(7 + 8i, 4)(8 + 8i, 4) \cup (7 + 8i, 5)(8 + 8i, 5) \mid 0 \leq i \leq (m - 15) / 8\} \cup (1, 4)(m, 4) \cup (1, 5)(m, 5)$ . Therefore, a Hamiltonian cycle  $C$  is built, which as shown in Fig. 26. For  $|E_1(C)| = 18 + (18 + 2)x = 18 + 20x$  and  $|E_2(C)| = 17 + 20x$ ,  $C$  satisfies  $\|E_1(C) - E_2(C)\| = 1$ . Without a doubt,  $C$  is a BHC of  $T_{m,5}$  for  $m \bmod 8 = 7$  and  $n \bmod 4 = 1$ .

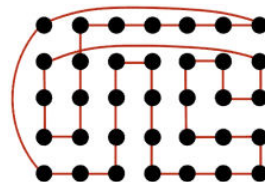


Fig. 25 The BHC on  $T_{7,5}$

Refer to Case 3.1 when  $n > 5$ . Replace Fig. 21 with Fig. 26, else parts are similar to Case 3.1. Finally, a BHC on  $T_{m,n}$  is produced.

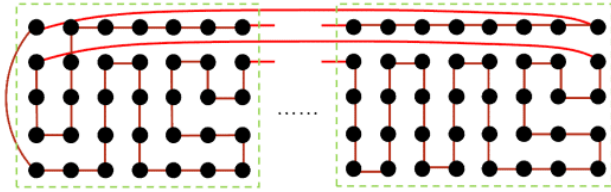


Fig. 27 The BHC on  $T_{m,5}$

Case 3.5.  $m \bmod 8 = 1$  and  $n \bmod 4 = 3$

Fig. 27 and Fig. 28 show one of the possible HCs on  $T_{8,7}$  and one of possible BHCs on  $T_{9,7}$ , respectively. Furthermore, Fig. 29 illustrates the way of constructing a BHC on  $T_{m,7}$ , which is described in the content below. When  $m > 9$ , let  $x = (m - 9) / 8$ . First, inset  $x$  HCs, which has been mentioned above, on the right side of Fig. 28. Second, remove edge set  $E_{29} = \{(10 + 8i, 6)(10 + 8i, 7) \cup (17 + 8i, 6)(17 + 8i, 7) \mid 0 \leq i \leq (m - 17) / 8\} \cup (1, 6)(9, 6) \cup (1, 7)(9, 7)$ . Third, add edge set  $E_{30} = \{(9 + 8i, 6)(10 + 8i, 6) \cup (9 + 8i, 7)(10 + 8i, 7) \mid 0 \leq i \leq (m - 17) / 8\} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$ . Finally, a Hamiltonian cycle  $C$  is yielded, whose  $|E_1(C)| = 32 + (26 + 2)x = 32 + 28x$  and  $|E_2(C)| = 31 + 28x$ . As a result of  $\|E_1(C) - E_2(C)\| = 1$ , it verify that  $C$  is a BHC.

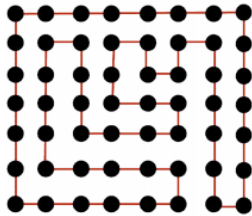


Fig. 27 The HC on  $T_{8,7}$ .

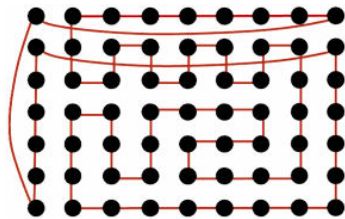


Fig. 28 The BHC on  $T_{9,7}$

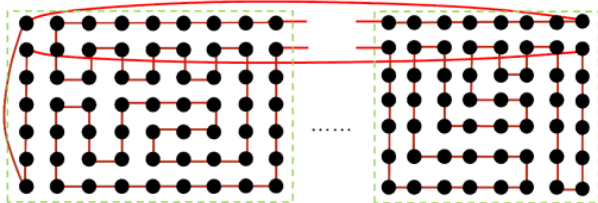


Fig. 29 The BHC on  $T_{m,7}$

When  $n > 7$ , let  $y = (n - 7) / 4$ . Stack  $y$  BHCs, which is shown in Fig. 12, above Fig. 28. Then delete edge set  $E_{31} = \{(1, 8 + 4i)(1, 11 + 4i) \mid 0 \leq i \leq (n - 11) / 4\} \cup (1, 1)(1, 7)$ , and add edge set  $E_{32} = \{(1, 7 + 4i)(1, 8 + 4i) \mid 0 \leq i \leq (n - 11) / 4\} \cup (1, 1)(1, n)$ . After that, a BHC on  $T_{m,n}$  is generated.

Case 3.6.  $m \bmod 8 = 3$  and  $n \bmod 4 = 3$

Fig. 30 represents a BHC on  $T_{3,7}$ , which is used to construct a BHC on  $T_{m,7}$ . Besides, Fig. 31 illustrates how to connect Fig. 27 and Fig. 30. Let  $x = (m - 3) / 8$  for  $m > 3$ . Then, copy Fig. 27 for  $x$  times, and inset them on the right side of Fig. 30. Next, eliminate edge set  $E_{33} = \{(4 + 8i, 6)(4 + 8i, 7) \cup (11 + 8i, 6)(11 + 8i, 7) \mid 0 \leq i \leq (m - 11) / 8\} \cup (1, 6)(3, 6) \cup (1, 7)(3, 7)$ , and put edge set  $E_{34} = \{(3 + 8i, 6)(4 + 8i, 6) \cup (3 + 8i, 7)(4 + 8i, 7) \mid 0 \leq i \leq (m - 11) / 8\} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$ . Therefore, a Hamiltonian cycle  $C$  is established, whose  $|E_1(C)| = 10 + (26 + 2)x = 10 + 28x$  and  $|E_2(C)| = 11 + 28x$ . Because of  $\|E_1(C) - E_2(C)\| = 1$ ,  $C$  satisfies the definition of BHC.

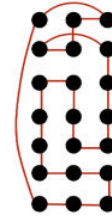


Fig. 30 The BHC on  $T_{3,7}$

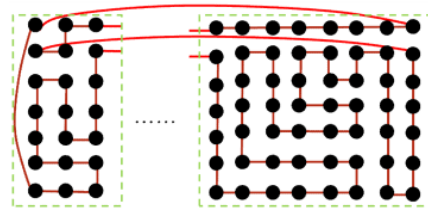


Fig. 31 The BHC on  $T_{m,7}$

Refer to Case 3.5 when  $n > 7$ . Fig. 31 substitutes for Fig. 29, and the else parts are same as Case 3.5. Eventually, a BHC on  $T_{m,n}$  is built.

Case 3.7.  $m \bmod 8 = 5$  and  $n \bmod 4 = 3$

In this section,  $T_{m,7}$  consists of the HC on  $T_{8,7}$ , as shown in Fig. 27, and the BHC on  $T_{5,7}$ , as shown in Fig. 32. When  $m > 5$ , make  $x = (m - 5) / 8$ . First of all, inset Fig. 27 for  $x$  times on the right side of Fig. 32. So as to connect all figures, delete edge set  $E_{35} = \{(6 + 8i, 6)(6 + 8i, 7) \cup (13 + 8i, 6)(13 + 8i, 7) \mid 0 \leq i \leq (m - 13) / 8\} \cup (1, 6)(5, 6) \cup (1, 7)(5, 7)$ , and add edge set  $E_{36} = \{(5 + 8i, 6)(6 + 8i, 6) \cup (5 + 8i, 7)(6 + 8i, 7) \mid 0 \leq i \leq (m - 13) / 8\} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$ . As a result, a Hamiltonian cycle  $C$  on  $T_{m,7}$  is yielded, whose  $|E_1(C)| = 18 + (26 + 2)x = 18 + 28x$  and  $|E_2(C)| = 17 + 28x$ , as shown in Fig. 33. Without a doubt,  $C$  is a BHC due to  $\|E_1(C) - E_2(C)\| = 1$ .

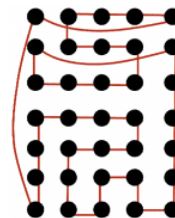


Fig. 32 The BHC on  $T_{5,7}$

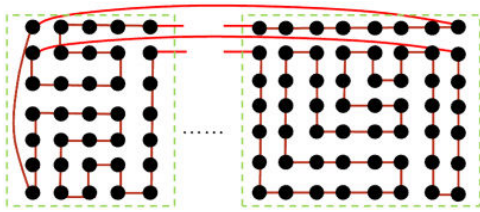


Fig. 33 The BHC on  $T_{m,7}$

When  $n > 7$ , the way of constructing the BHC on  $T_{m,n}$  is similar to Case 3.5. Only one difference is to replace Fig. 29 with Fig. 33.

Case 3.10.  $m \bmod 8 = 7$  and  $n \bmod 4 = 3$

There is a BHC on  $T_{7,7}$  as shown in Fig. 34. When  $m > 7$ , use the BHC on  $T_{7,7}$  mentioned above as the beginning. Make  $x = (m - 7) / 8$ , then inset  $x$  HCs, which is shown in Fig. 27, on the right side of Fig. 34. Then, remove edge set  $E_{37} = \{(15 + 8i, 6)(15 + 8i, 7) \mid 0 \leq i \leq (m - 15) / 8\} \cup (1, 6)(7, 6) \cup (1, 7)(7, 7) \cup (8 + 8i, 6)(8 + 8i, 7)$ , and add edge set  $E_{38} = \{(7 + 8i, 6)(8 + 8i, 6) \cup (7 + 8i, 7)(8 + 8i, 7) \mid 0 \leq i \leq (m - 15) / 8\} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$ . Thus, a Hamiltonian cycle  $C$  on  $T_{m,7}$  is built, which is shown in Fig. 35.

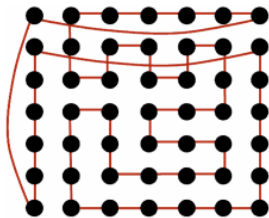


Fig. 34 The BHC on  $T_{7,7}$

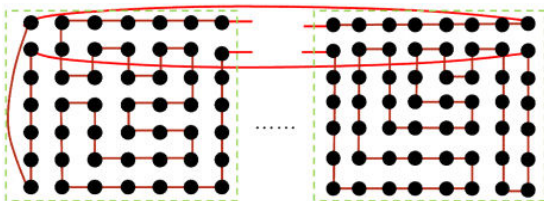


Fig. 35 The BHC on  $T_{m,7}$

For  $|E_1(C)| = 24 + (26 + 2)x = 24 + 28x$  and  $|E_2(C)| = 25 + 28x$ ,  $C$  satisfies that  $||E_1(C)| - |E_2(C)|| = 1$ . Undoubtedly,  $C$  is a BHC of  $T_{m,7}$ .

Compare Case 3.5 with Case 3.8 for  $n > 7$ , the only difference is that Fig. 29 is replaced with Fig. 35, and the else parts of operating are all the same. Then a BHC on  $T_{m,n}$  is constructed.

#### IV. CONCLUSION

By giving Theorem 1 and 2 in this paper, the main result below can be verified. In general cases, there exists a BHC on  $T_{m,n}$  for positive integers  $n, m$ , except for the situation of  $mn \equiv 2 \pmod{4}$ . How to find a balanced Hamiltonian cycle in the graph of  $k$ -dimension Cartesian product for any positive integer  $k$ , will be the future work.

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