

# Optimal Allocation Between Subprime Structured Mortgage Products and Treasuries

MP. Mulaudzi, MA. Petersen, J. Mukuddem-Petersen, IM. Schoeman, B. de Waal and JM. Manale

**Abstract**—This conference paper discusses a risk allocation problem for subprime investing banks involving investment in subprime structured mortgage products (SMPs) and Treasuries. In order to solve this problem, we develop a Lévy process-based model of jump diffusion-type for investment choice in subprime SMPs and Treasuries. This model incorporates subprime SMP losses for which credit default insurance in the form of credit default swaps (CDSs) can be purchased. In essence, we solve a mean swap-at-risk (SaR) optimization problem for investment which determines optimal allocation between SMPs and Treasuries subject to credit risk protection via CDSs. In this regard, SaR is indicative of how much protection investors must purchase from swap protection sellers in order to cover possible losses from SMP default. Here, SaR is defined in terms of value-at-risk (VaR). Finally, we provide an analysis of the aforementioned optimization problem and its connections with the subprime mortgage crisis (SMC).

**Keywords**—Investors; Jump Diffusion Process; Structured Mortgage Products; Treasuries; Credit Risk; Credit Default Swaps; Tranching Risk; Counterparty Risk; Value-at-Risk; Swaps-at-Risk; Subprime Mortgage Crisis.

## I. INTRODUCTION

The 2007-2010 subprime mortgage crisis (SMC) can be attributed to a confluence of factors such as lax screening by mortgage originators and a rise in the popularity of new structured financial products whose risks were difficult to evaluate. As far as the latter is concerned, subprime residential mortgage securitization involves the pooling of mortgages that are subsequently repackaged into interest-bearing securities. The interest and principal payments from mortgages are passed through to credit market investors such as subprime investing banks – herewith, simply known as investors. In so doing, the risks associated with mortgage securitization are transferred from originators to special purpose vehicles (SPVs) and structured mortgage product (SMP) bond holders such as investors.

Mortgage securitization thus represents an alternative and diversified source of housing finance based on the transfer of credit risk. Some of the other risks involved are tranching, counterparty and liquidity risks.

*Tranching risk* is the risk that arises from the intricacy associated with the slicing of SMPs into tranches in securitization deals. Another tranching risk that is of issue for SMPs

MP. Mulaudzi and JM Manale are with the University of South Africa, P O Box 392, UNISA, 0003, SA, e-mail: mulaump@unisa.ac.za, telephone: +27-12-429 4855, fax: +27-12-429 4898.

MA. Petersen, J. Mukuddem-Petersen and B. de Waal are with North-West University (Mafikeng Campus), Private Bag 2046, Mmabatho 2735, SA, e-mail: Mark.Petersen@nwu.ac.za, telephone: +27-18-389 2651, fax: +27-18-389 2090.

IM. Schoeman is with North-West University (Potchefstroom Campus), P.O. Bag x6001, Potchefstroom 2532, SA, e-mail: Ilse.Schoeman@nwu.ac.za, telephone: +27-18-299 2565, fax: +27-18-299 2570.

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is *maturity mismatch risk* that results from the discrepancy between the economic lifetimes of SMPs and the investment horizons of investors.

*Counterparty risk* that, in our case, is the risk that a banking agent does not pay out on a bond, credit derivative or credit insurance contract. It refers to the ability of banking agents – such as originators, mortgagors, servicers, investors, SPVs, trustees, underwriters and depositors – to fulfill their obligations towards each other. During the SMC, even banking agents who thought they had hedged their bets by buying insurance – via credit default swap contracts or monoline insurance – still faced the risk that the insurer will be unable to pay.

*Liquidity risk* arises from situations in which a banking agent interested in selling (buying) SMPs cannot do it because nobody in the market wants to buy (sell) those SMPs. Such risk includes funding and credit crunch risk. *Funding risk* refers to the lack of funds or deposits to finance mortgages and *credit crunch risk* refers to the risk of tightened mortgage supply and increased credit standards.

In our contribution, we specifically investigate the securitization of mortgages which is undertaken as follows. The first step in the process involve originators that extend mortgages that are subsequently removed from their balance sheets and pooled into reference mortgage portfolios. Originators then sell these portfolios to SPVs – an entity set up by a financial institution, specifically to purchase mortgages and realize their off-balance-sheet treatment for legal and accounting purposes. Next, the SPV finances the acquisition of mortgage portfolios by issuing tradable, interest-bearing securities that are sold to investors. In addition, the mortgage portfolios are serviced by servicers who collect payments from the original mortgagors, and pass them on – less a servicing fee – directly to SPVs. Investors receive fixed or floating rate coupons from the SPV account funded by cash outflows generated by reference mortgage portfolios.

In the sequel, subprime mortgage securitization mainly refers to the securitization of such mortgages into SMPs such as residential mortgage-backed securities (RMBSs) and collateralized debt obligations (CDOs). The SMPs themselves are structured into *tranches*. In particular, this paper involves three such tranches: the *senior* (usually AAA rated and abbreviated as sen), *mezzanine* (usually AA, A, BBB rated and abbreviated as mezz) and *junior (equity) tranches* (usually BB, B rated and unrated and abbreviate as jun) in order of contractually specified claim priority. At this stage, the location and extent of subprime risk cannot be clearly described. This is due to the chain of interacting securities that cause the risk characteristics

to be opaque. Another contributing factor are the derivatives that resulted in negative basis trades moving CDO risk and credit derivatives that created additional long exposure to subprime mortgages. Determining the extent of the risk is also difficult because the effects on expected mortgage losses depend on house prices as the first order risk factor. Simulating the effects of this through the chain of interacting securities is very difficult.

SMP insurance is an important issue. Credit default swaps (CDSs) are financial instruments that are mainly used as a hedge and protection for debtholders, in particular subprime SMP investors, from the risk of default (see, for instance, [4]). Like all swaps and other credit derivatives, CDSs may also be used to profit from speculation. In the SMC, as subprime mortgage losses increased because of the high default rate, the probability increased that those issuing insurance (like CDS protection sellers) would have to compensate their counterparties (see [6] for further discussion). This created uncertainty across the system, as investors wondered which agents would be required to pay to cover such losses. Our work has a strong connection with this issue via an investor's optimal allocation problem subject swap-at-risk. CDSs are largely not regulated. As of 2008, there was no central clearinghouse to honor CDSs in the event a party to a CDS proved unable to perform its obligations under the CDS contract. Required disclosure of CDS-related obligations has been criticized as inadequate (compare with [2] and [4]). Next, a diagrammatic overview of SMP protection via CDSs is provided.

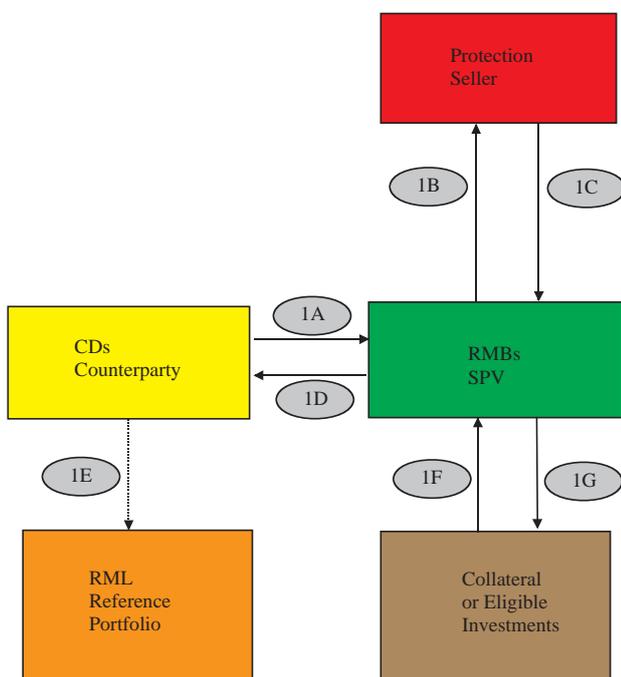


Fig. 1: Diagrammatic Overview of SMP Protection via CDSs

Our dynamic model allows for protection against SMP – in the form of SMPs – losses via CDS contracts. The CDS counterparty, OR, who is the protection buyer makes a regular

stream of payments constituting the *premium leg* (see **1A**) to the SMP SPV. This SPV, in turn, makes regular coupon payments to the protection seller (refer to **1B**). These payments are made until a credit event occurs or until maturity, whichever ever happens first. The size of premium payments is dependent on the quoted *default swap spread* which is paid on the face value of the protection and is directly related to credit ratings. If there is no credit event, the seller of protection receives the periodic fee from the buyer, and profits if the reference mortgage portfolio remains fully functional through the life of the contract and no payout takes place. However, the protection seller is taking the risk of big losses if a credit event occurs. Depending on the terms agreed upon at the onset of the contract when such an event takes place, the protection seller may deliver either the current cash value of the referenced bonds or the actual bonds to the protection buyer via the SMP SPV (refer to **1C** and **1D**). This payment to the protection buyer, is known as the *protection leg* (see **1D**). It equals the difference between par and the price of the cheapest to deliver (CTD) asset associated with the mortgage portfolio on the face value of the protection. The value of a CDS contract fluctuates based on the increasing or decreasing probability that a reference mortgage portfolio will have a credit event (compare with **1E**). Increased probability of such an event would make the contract worth more for the buyer of protection, and worth less for the seller. The opposite occurs if the probability of a credit event decreases. Collateral or eligible investments are highly rated, highly liquid financial instruments purchased from the sale proceeds of the initial SMP (represented by **1G**). These investments contribute the index portion (see **1F**) of the SMP coupon and provides protection payments or the return of principal to SMP bond holders.

The literature about SMPs and the SMC is growing and includes the following contributions. The article [5] (see, also, [10], [7] and [6]) shows that mortgage charge-offs are more pronounced among originators that are unable to sell their originate-to-distribute (OTD) mortgages to investors. This finding supports the view that the credit risk transfer through the OTD market resulted in the origination of inferior quality mortgages by originators. We believe that mortgage standards became slack because securitization gave rise to moral hazard, since each link in the mortgage chain made a profit while transferring associated credit risk to the next link (see, for instance, [6] and [5]). The increased distance between ORs and the ultimate bearers of risk potentially reduced ORs' incentives to screen and monitor mortgagors (see [6]). The increased complexity of SMPs and markets also reduces investor's ability to value them correctly (see, for instance, [6]). The main purpose of this paper is to address the problem of investment in subprime SMPs and Treasuries in a Lévy-process setting. To the best of our knowledge, Lévy process-driven models that deal with securitization and its relationship with the SMC were first introduced in [5]. However, the latter paper does not deal with optimization aspects of securitization. In this paper, there are several references to support the adoption of stochastic models for subprime SMP prices and investment as

well as SMP losses. For our study, the most relevant of these are [1] that discusses bank asset prices such as subprime SMPs prices that are driven by Brownian motion and, of course, [9] that is one of the standard references involving the stochastic dynamics of (bank) asset price processes.

In this conference paper, we solve a mean swaps-at-risk (SaR) optimization problem for investment which determines the optimal allocation of subprime SMPs and Treasuries subject to credit risk protection via CDSs. In this regard, SaR is indicative of how much protection investors must purchase from a swap protection seller in order to cover possible losses from credit events. The specific questions that we answer are as follows.

**Question 1.1: (Investment in Subprime SMPs and Treasuries Model):** Can we construct a stochastic dynamic model for investment in subprime SMPs and Treasuries which involves fund allocation as well as subprime SMP losses via a jump-diffusion process ? (see Subsection II-B).

**Question 1.2: (An Investor's Optimal Allocation Problem):** Can we solve an investor's optimization problem that determines the optimal proportion of funds invested in subprime SMPs and Treasuries subject to SaR ? (see Section III).

**Question 1.3: (Connections with the SMC):** How does an investor's optimal allocation problem solved in Section III relate to the SMC ? (throughout the paper).

## II. SUBPRIME SECURITIZATION MODELS

In this section, we discuss subprime mortgage securitization and construct a stochastic model of investment in subprime SMPs and Treasuries under a jump-diffusion process. In this paper, we use the subprime SMPs to refer the subprime residential mortgage-backed securities (RMBSs). These securities are purchased by the investors from SPVs. In particular, the mortgage originators pass the subprime residential mortgage loans ( which we treat them as adjustable-rate mortgages) to the SPVs that are subsequently purchased by the investors in the form of subprime RMBSs. In order to model the uncertainty associated with these issues, we consider the filtered probability space,  $(\Omega, \mathbf{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ , throughout.

### A. Subprime mortgage securitization

In this subsection, we consider the modeling of subprime SMPs price, Treasuries and SMPs losses.

1) *Subprime SMP price process with jumps:* In the sequel, an investor invests a proportion of its funds in subprime SMPs issued by SPV with an interest rate,  $r^M$ . Generally, SMP bonds deals pay a floating coupon,  $r^M$ , while reference mortgage portfolios (collateral) typically pay a fixed rate,  $r^A$ , until the reset date on hybrid adjustable rate mortgages (ARMs). In this case, the risk that interest paid into the deal from the reference portfolio,  $r^A$ , is not sufficient to make coupon payments,  $r^M$ , to SMP bond holders may arise. To mitigate this situation, the deal may be subject to an *available funds cap* (AFC). Here, investors receive interest as the minimum of the sum of the index rate,  $r^L$ , (i.e., 6-month LIBOR) and margin,  $\varrho$ , or the weighted average AFC,  $r^a$ . Symbolically, this means that

$$r^M = \min[r^L + \varrho, r^a]. \quad (1)$$

In our contribution, the mathematical expectation of the rate of return on subprime SMPs is denoted by  $\mu = \mathbf{E}[r^M]$ . Moreover, assume that the dynamics of the subprime SMP price process,  $M_t$ , is given by

$$dM_t = M_{t-} \left\{ \left( \mu - \sum_{i=1}^n \alpha_i \lambda_i \right) dt + \sigma dZ_t + \sum_{i=1}^n \alpha_i dN_i(t) \right\}, \\ t \geq 0, \quad \mu \in \mathbb{R}, \quad M_0 = p \in \mathbb{R}^+, \quad (2)$$

where  $\sigma > 0$  is a constant volatility of  $M_t$ ,  $n \in \mathbb{N}$ ,  $Z_t$  is a Brownian motion and for  $i = 1, \dots, n$  the process  $N_i$  is a homogeneous Poisson process with intensity  $\lambda_i$ ,  $\alpha_i$  is the jump size of the process  $M_t$ . In order to avoid negative subprime SMP prices, we make the assumption that

$$-1 < \alpha_1 < \dots < \alpha_n < \infty.$$

2) *Treasuries:* *Treasuries* are bonds issued by national Treasuries and are the debt financing instruments of the federal government. There are four types of Treasuries: Treasury bills, Treasury notes, Treasury bonds and savings bonds. All Treasuries besides savings bonds are very liquid and are heavily traded on the secondary market. During the SMC, investors held their investments in safe assets such as Treasuries. However, such investments contribute to the prolonging of the crisis because it resulted in a credit crunch. In the sequel, we denote the *interest rate on Treasuries* or *Treasuries* rate by  $r^T$ . Suppose that the value<sup>1</sup> process of the *Treasuries*,  $T$ , is given by

$$dT(t) = r^T T(t) dt, \quad t > 0, \quad T(0) = 1. \quad (3)$$

This form is indicative of the fact that the value process for the Treasuries is riskless because it does not contain a noise term.

3) *Investor subprime SMP losses:* We suppose that losses suffered by investors due to SMP defaults is a random variable,  $S$ , with the distribution function,  $F$ . In the sequel, we define this loss as

$$S : \Omega \rightarrow \mathbb{R}^+, \quad (4)$$

where  $\Omega$  takes on nonnegative real values that may not necessarily be measurable. Moreover, let  $\theta \geq 0$  be a nonnegative real number which is an upper bound of  $S(w)$ , for all  $w \in \Omega$ , where  $w$  is defined as the state of the mortgage market. Therefore,  $\{w \in \Omega : S(w) > \theta\}$  is empty. This enables us to define the smallest essential upper bound for the aggregate securitization losses,  $S$ , as

$$\text{ess sup } S(w) = \inf \{ \theta \in \mathbb{R}^+ : \mathbf{P}(\{w : S(w) > \theta\}) = 0 \}.$$

<sup>1</sup>There is a difference between price and value: the amount of cash that one pays is a price and value is what it worth. However, in our contribution we are concerned with the value of the security. Therefore, when we use term price, what we really mean is value.

Furthermore, we assume that  $S$  is modeled as a compound Poisson process, for which  $N$  is a Poisson process with a deterministic frequency parameter,  $\phi(t)$ . In this case,  $N$  is stochastically independent of the Brownian motion,  $Z$ .

*B. A model for investment in subprime SMPs and Treasuries*

In this subsection, we construct a stochastic model of an investor's investment in subprime SMPs and Treasuries. Such an investor starts with initial funds,  $B_0^\pi$ , and the expected rate of return on subprime SMPs,  $\mu$ . The investment in SMPs may yield substantial returns but may also result in losses. Moreover, for a fixed term  $[0, T]$ , we consider a characterization of the investor's net investment in subprime SMPs and Treasuries,  $B^{\prime\pi}$ , of the form

$$\begin{aligned} & \text{Nett Investment in SMPs \& Treasuries } (B^{\prime\pi}) & (5) \\ & = \text{Total Investment in SMPs \& Treasuries } (B^\pi) \\ & \quad - \text{Subprime SMP Losses } (S), \end{aligned}$$

where the investor's total investment in subprime SMPs and Treasuries,  $B^\pi$ , is the stochastic process  $(B_t^\pi)_{t \geq 0}$ , defined on the filtered probability space,  $(\Omega, \mathbf{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ . Let  $\pi \in [0, 1]$  be the proportion of the investor's funds invested in subprime SMPs, and  $1 - \pi$  is the proportion of funds invested in Treasuries. For  $t > 0$  and  $M_t$ ,  $T(t)$  and  $S$ , given by (2), (3) and (4), respectively, we use (5) to represent the dynamics of the investor's total and nett investment in subprime SMPs & Treasuries by

$$\begin{aligned} dB_t^\pi &= B_t^\pi \left\{ \left( (1 - \pi)r^T + \pi\mu - \pi \sum_{i=1}^n \alpha_i \lambda_i \right) dt + \pi \sigma dZ_t \right. \\ & \quad \left. + \pi \sum_{i=1}^n \alpha_i dN_i(t) \right\} & (6) \end{aligned}$$

and

$$dB_t^{\prime\pi} = dB_t^\pi - dS_t, \quad t \geq 0, \quad B_0^\pi = \ell \in \mathbb{R}^+, \quad (7)$$

respectively. The solution of (6) is obtained via Itô's formula and for  $B_0^\pi = \ell$  and  $t = T$  is found to be

$$\begin{aligned} B_T^\pi &= \ell \exp \left\{ \left( r^T + (\mu - r^T)\pi - \pi \sum_{i=1}^n \alpha_i \lambda_i - \frac{1}{2} \pi^2 \sigma^2 \right) T \right. \\ & \quad \left. + \pi \sigma Z_T + \sum_{i=1}^n (N_i(T)) \ln(1 + \pi \alpha_i) \right\}. & (8) \end{aligned}$$

In the sequel, to avoid the possibility that the total investment value process,  $B^\pi$ , is negative, by restricting the investment strategy,  $\pi$ , as follows.

$$\begin{cases} -\frac{1}{\alpha_n} \leq \pi < -\frac{1}{\alpha_1} & \text{if } \alpha_n > 0 > \alpha_1; \\ -\infty < \pi \leq -\frac{1}{\alpha_1} & \text{if } \alpha_n < 0; \\ -\frac{1}{\alpha_n} \leq \pi < \infty & \text{if } \alpha_1 > 0. \end{cases} \quad (9)$$

The next result provides an explicit formula for the expectation of  $B_T^\pi$ . In this regard, we assume that the moments of  $B_T^\pi$  exist.

**Proposition 2.1: (Explicit Formula for  $E[B_T^\pi]$ ):** Suppose that (8) holds, and the moment generating function of homogeneous Poisson process  $N_i(T)$  exists, that is,

$$E[\exp\{uN_i(T)\}] = \varphi_{N_i(T)}(u) = \exp[\lambda_i T(e^u - 1)], \quad u > 0.$$

Then we have the expectation of  $B_T^\pi$  given by

$$E[B_T^\pi] = \ell \exp \left\{ (r^T + (\mu - r^T)\pi)T \right\}. \quad (10)$$

In Subsection II-B, we have seen that the dynamic model of investment given by (7) involves subprime SMP losses,  $S$ , Treasuries rate,  $r^T$ , expected return on subprime SMPs,  $\mu$ , jump heights,  $\alpha_i$ , proportions of investor funds invested in subprime SMP,  $\pi$ , and Treasuries,  $1 - \pi$ . Before the SMC, we believe that  $0 < \alpha_i < \infty$ . This was mainly due to the house price appreciation which caused the value of subprime SMP to increase. In this regard, investment could have performed well as a result of positive returns,  $\mu > 0$ , due to an increase in subprime SMP prices as in (2) in Subsection II-A1. In particular, investors experienced lower default rates because of increases in house prices. This trend attracted investment in the subprime mortgage market. However, as the SMC unravelled, house prices depreciated dramatically. Accordingly, the value of subprime SMP declined and  $-1 < \alpha_i < 0$ . As a consequence, the default rate increased considerably with investors incurring large losses from their investments. In addition, some investors started to allocate away from risky assets towards safe assets such as Treasuries. Although this strategy was considered to be safe, it was another factor that prolonged the SMC.

III. AN INVESTOR'S OPTIMAL ALLOCATION PROBLEM

In this section, we state and solve a constrained optimization problem for investment in subprime SMPs and Treasuries with SaR. In this regard, we assume that the dynamics of investment, subprime SMPs and Treasuries are given by (8), (2) and (3), respectively.

A. Definition of SaR

In this subsection, we provide a definition for SaR. In this regard, suppose that  $\ell$  is the initial investor funds and  $[0, T]$  is a fixed term. Moreover, let  $q_\gamma$  be the  $\gamma$ -quantile of the distribution of  $\pi \sigma Z_T + \sum_{i=1}^n (N_i(T)) \ln(1 + \pi \alpha_i)$  for the proportion of funds  $\pi \in [0, 1]$ , invested in subprime SMPs, and  $B_T^\pi$  the corresponding terminal value of an investor's investment. This means that  $q_\gamma$  is a real number such that

$$\begin{aligned} & \mathbf{P} \left( \pi \sigma Z_T + \sum_{i=1}^n (N_i(T)) \ln(1 + \pi \alpha_i) \leq q_\gamma \right) \\ &= \sum_{n_1, \dots, n_n=0}^{\infty} \left\{ \chi \left( \frac{1}{|\pi \sigma| \sqrt{T}} \left( q_\gamma - \sum_{i=1}^n (n_i \ln(1 + \pi \alpha_i)) \right) \right) \right. \\ & \quad \left. \times \exp \left( -T \sum_{i=1}^n \lambda_i \right) \prod_{i=1}^n \frac{(T \lambda_i)^{n_i}}{n_i!} \right\} = \gamma \end{aligned} \quad (11)$$

In this case, we define the value-at-risk (VaR) associated with investment as

$$\text{VaR}(\ell, \pi, T) = \ell \exp \left\{ \left( r^T + (\mu - r^T)\pi - \pi \sum_{i=1}^n \alpha_i \lambda_i - \frac{1}{2} \pi^2 \sigma^2 \right) T + q_\gamma \right\}. \quad (12)$$

We are now able to define SaR.

**Definition 3.1: (Definition of SaR):** Suppose that the  $\mathbf{P}$  and  $\text{VaR}(\ell, \pi, T)$  are as in the above. In this case,  $\text{SaR}$  is defined as

$$\text{SaR}(\ell, \pi, T) = \ell \exp(r^T T) - \text{VaR}(\ell, \pi, T), \quad (13)$$

where  $\pi$  is the proportion of investor funds invested in SMP with  $\ell$  and  $[0, T]$  being an investor's initial funds and planning term, respectively. In particular, (13) reveals that the  $\text{SaR}$  is the difference between the expected value of the profit-loss distribution and  $\text{VaR}$  of the investment in subprime SMPs and Treasuries.

#### B. Statement of an investor's optimal allocation problem

In this subsection, we formulate a constrained optimization problem for investment in subprime SMPs and Treasuries subject to SaR, where SaR is defined as a measure of the protection required against possible subprime SMP losses. For a constant bound, A, the mathematical formulation of this problem is given below.

**Problem 3.2: (Statement of an investor's optimal allocation problem):**

Assume that  $\mathbf{B}_T^\pi$  and  $\mathbf{E}[\mathbf{B}_T^\pi]$  are given by (8) and (10), respectively. For  $t \geq 0$ , we give a mathematical statement of an investor's investment problem with SaR as follows

$$\max_{\pi \in [0,1]} \mathbf{E}[\mathbf{B}_T^\pi] \quad \text{subject to (8) and } \text{SaR}(\ell, \pi, T) \leq A, \quad (14)$$

where  $[0, T]$  is a fixed term and A is the maximum protection that can be acquired.

In Subsection III-B, we consider a mean-SaR optimization problem (see Problem 3.2). This problem maximizes the expected terminal value of total investment subject to SaR – defined as a measure of the protection required against possible losses from subprime SMP investments. Moreover, this problem enables us to analyze the features of CDSs which was another derivative that fueled the SMC.

The solution of Problem 3.2 can only be obtained through numerical methods. However, this is beyond the scope of this paper. By way of partially addressing this problem, Subsection III-C only contains an illustration of the behavior of the solution. In particular, through differential calculus, we found that the solution to our problem has to be the largest investment strategy,  $\pi$ , that satisfies both the SaR-constraint in (14) and conditions in (9). Furthermore, Subsection III-D discussed a numerical algorithm that can be followed in order to approximate a solution. Also, Problem 3.2 reveals that investors will

not only be exposed to credit risk but counterparty risk as well. The latter occurs when investors experience SMP losses and the swap protection seller fails to honor its obligation. In this case will have to use its capital to absorb all these losses. In the situation where investors do not have enough capital, they will go bankrupt. This situation can cause systemic risk in the case where investors engaged in interbank lending. On the other hand, if investments perform well, minimal mortgage losses will ably be compensated for by credit default insurance via CDS contracts.

#### C. Solution of an investor's optimal allocation problem

The problem we face is that  $q_\gamma$  cannot be represented explicitly. This brings us to the conclusion that the analytical solution of Problem 3.2 is not possible. However, using (10) we are able to tell the behavior of the solution. In this regard, we note that

$$\frac{\partial \mathbf{E}[\mathbf{B}_T^\pi]}{\partial \pi} > 0$$

provided  $\mu > r^T$ , i.e.,  $\mathbf{E}[\mathbf{B}_T^\pi]$  is increasing function over the interval  $0 \leq \pi \leq 1$ . It then follows that the optimal solution of Problem 3.2 is the largest proportion of the investor's initial funds,  $\ell$ , i.e.  $\pi \in [0, 1]$  that satisfies the SaR constraint and condition (9). For optimization problems of this kind, the solution(s) can only be found through numerical methods.

#### D. Numerical procedure

In this subsection, we focus on the numerical analysis of subprime RMBSs models developed in our paper. However, the numerical aspects of CDOs are left out for future research. In order to achieve this goal, we employ techniques from Gaussian stochastic processes (see, for instance, [8]) to construct a numerical algorithm that can be used to approximate the solution of Problem 3.2.

**1) Gaussian diffusion model for subprime mortgage securitization:** In this subsection, we replace the aforementioned model in Subsection II-A1 with the generalized inverse Gaussian (GIG) diffusion model. In particular, we present the stochastic analysis related to the GIG model and apply it to Problem 3.2. Suppose that the dynamics of Treasuries,  $T$ , is given by (3) and the subprime SMP price process from (2) is given by

$$dM_t = M_t \{ \mu dt + dL_t \}, t \geq 0, M_0 = p, \quad (15)$$

$$\text{where } L_t = W(t) - \frac{1}{2} \sigma^2 \int_0^t W^{2\beta}(s) ds - w, \quad L_0 = 0.$$

Using (3) and (15), we write the value process of the investor's total investment as

$$\mathbf{B}_t^\pi = \ell \exp \left\{ \left( (1 - \pi)r^T + \pi\mu \right) t + \pi \left( W(t) - \frac{1}{2} \sigma^2 \int_0^t W^{2\beta}(s) ds - w \right) \right\}, \quad (16)$$

$$\mathbf{B}_0^\pi = \ell.$$

Moreover,  $W(t)$  is defined as the GIG diffusion process which satisfies the stochastic differential equation (SDE) given by

$$dW(t) = \frac{1}{4}\sigma^2 W^{2\beta-2}(t) \left( \varpi + 2(2\beta + \lambda - 1)W(t) - vW^2(t) \right) dt + \sigma W^\beta(t) dZ_t, \quad W(0) = w, \quad (17)$$

where  $\sigma > 0$ ,  $\beta \geq \frac{1}{2}$ ,  $v, \varpi \geq 0$ ,  $\max(v, \varpi) > 0$ , and

$$\begin{aligned} \lambda \in \mathbb{R} & \quad \text{if } v, \varpi > 0; \\ \lambda \leq \min(0, 2(1 - \beta)) & \quad \text{if } v = 0, \varpi > 0; \\ \lambda \geq \min(0, 2(1 - \beta)) & \quad \text{if } v > 0, \varpi = 0. \end{aligned} \quad (18)$$

The Gaussian diffusion model is a generalization of the Black-Scholes model, which correspond to  $\beta = \varpi = 0$ ,  $\lambda = 1$ ,  $v = -2$ .

In the following lemma we decompose the subprime SMP price process presented in (15). The idea of the lemma is to demonstrate the useful features of the GIG diffusion model construction.

**Lemma 3.3: (Decomposition of the Subprime SMP Price):** Suppose that  $M_t$  and  $W(t)$  satisfy (15) and (17), respectively. Then we can decompose  $M_t$  into a drift term multiplied by a local martingale, i.e.,

$$M_t = p \exp \left\{ \mu t + \frac{1}{4}\sigma^2 \int_0^t W^{2\beta-2}(s) \left( \varpi + 2(2\beta + \lambda - 1)W(s) - vW^2(s) \right) ds \right\} \times \exp \left\{ \sigma \int_0^t W^\beta(s) dZ_s - \frac{1}{2}\sigma^2 \int_0^t W^{2\beta}(s) ds \right\}, \quad t \geq 0.$$

The next lemma contains useful results concerning (17), which will be required in order to describe the investor's total investment value process.

**Lemma 3.4: (Scalar Multiple of GIG Diffusion is also GIG Diffusion):** Suppose that  $W(t)$  is the GIG diffusion given by (17) and  $\pi > 0$  is the proportion of investor funds invested in subprime SMP. Then the process  $\tilde{W}(t) = \pi W(t)$  is also a GIG diffusion with  $\tilde{W}(0) = \pi W(0)$  and parameters  $\tilde{\sigma} = \sigma\pi^{1-\beta}$ ,  $\tilde{\varpi} = \varpi\pi$ ,  $\tilde{v} = v\pi^{-1}$ . In addition,  $\beta$  and  $\lambda$  in (17) remained the same.

**Remark 3.5: (Investment Value Process):** The results of Lemma 3.4 together with (16) gives us the investor's total investment value process of the form

$$B_t^\pi = \ell \exp \left\{ ((1 - \pi)r^T + \bar{\mu})T + \tilde{L}_t \right\}, \quad t \geq 0, \quad (19)$$

where

$$\bar{\mu} = \pi\mu \quad \text{and} \quad \tilde{L}_t = \tilde{W}(t) - \frac{1}{2}\tilde{\sigma}^2 \int_0^t \tilde{W}^{2\beta}(s) ds - \pi w \quad (20) \quad t \geq 0.$$

From Definition 3.1, we see that the  $\gamma$ -quantile of  $\tilde{L}_T$  needs to be determined. Moreover, the usefulness of  $SaR(\ell, \pi, T)$  stems from the fact that it is independent of moments of  $B_T^\pi$ . Therefore, it can be defined for finite or infinite moments of  $B_T^\pi$ . Despite the fact that  $SaR$  is not dependent on the moments, in order to solve Problem 3.2, we require the existence of a first moment of  $B_T^\pi$  that has to be finite. However, it is not possible to decide if  $B_T^\pi$  has a finite mathematical expectation. In this regard, we assume that  $W(T)$  or  $\tilde{W}(T)$  have the stationary distribution of the process  $W$  or  $\tilde{W}$  respectively, for time horizon  $T$  which is sufficiently large.

**Proposition 3.6: (Finite Mean of  $B_T^\pi$ ):** Suppose that  $W(T)$  and  $\tilde{W}(T)$  are GIG diffusion processes with constants  $\varpi, v, \lambda, \beta$  and  $\tilde{\varpi}, \tilde{v}, \lambda, \beta$  respectively. Let  $\pi > 0$  be the proportion of investor funds invested in subprime SMP. Then  $B_T^\pi$  has a finite mean if  $\tilde{v} = v\pi^{-1} > 2$ .

In line with Proposition 3.6, the optimization Problem 3.2 is well-defined and solveable. However, the question is whether the approach can be achieved via analytical or numerical means. In the next corollary we discuss the analytic approach for solving Problem 3.2.

**Corollary 3.7: (Analytic Approach for Solving Problem 3.2)** Suppose that the dynamic of  $W(t)$  is given by (17). Then for  $\beta = 1$ ,  $v = 0$ , we have

$$dW(t) = \left( \frac{1}{4}\sigma^2 \varpi + \frac{1}{2}\sigma^2(1 + \lambda)W(t) \right) dt + \sigma W(t) dZ_t, \quad W(0) = w, \quad (21)$$

which is interpreted as mean-reverting model. Furthermore, the solution of (21) is given by

$$W(t) = \exp \left( \frac{1}{2}\sigma^2 \lambda t + \sigma Z_t \right) \left\{ w \int_0^t \exp \left( -\frac{1}{4}\sigma^2 \varpi \left( -\frac{1}{2}\sigma^2 \lambda s - \sigma Z_s \right) ds \right) \right\},$$

and its expected value is written as

$$E[W(t)] = \begin{cases} \exp \left( \frac{1}{2}\sigma^2 t(\lambda + 1) \right) \left\{ w + \frac{1}{4\lambda} \varpi \sigma^2 \lambda \left( 1 - \exp(-t) \right) \right\} & \text{if } \lambda \neq -1; \\ w - \frac{1}{4} \varpi \left( 1 - \exp(\sigma^2 t) \right) & \text{if } \lambda = -1. \end{cases}$$

Also, we have that

$$L_t = \frac{1}{4}\sigma^2 \varpi t + \frac{1}{2}(1 + \lambda)\sigma^2 \int_0^t W(s) ds + \sigma \int_0^t W(s) dZ_s - \frac{1}{2}\sigma^2 \int_0^t W^2(s) ds. \quad (22)$$

We obtain the same representations for  $\tilde{W}(t)$  and  $\tilde{L}_t$  if we substitute  $\varpi$  by  $\tilde{\varpi} = \pi\varpi$ .

Although this corollary provides us with some analytical properties of the process that is important for solving the

mean-SaR optimization problem, the closed form solution cannot be determined. This brings us to the presentation of the following numerical algorithm that can be used to determine a solution to Problem 3.2.

2) *Numerical algorithm for Problem 3.2:* This subsection discusses a numerical approximation of the solution(s) to Problem 3.2. In particular, we discuss an iterative method that can be implemented in order to find the optimal investment strategy for an investor subject to SaR. We note that the Monte-Carlo method may be helpful to achieve the solution. In the sequel, we first present the properties of Problem 3.2. The properties of the expectation operator together with (20) allow us to derive the inequality

$$\begin{aligned} \mathbf{E}[B_T^\pi] &\geq \ell \exp \left\{ ((1-\pi)r^T + \bar{\mu})T + \mathbf{E}[\tilde{L}_T] \right\} \\ &= \ell \exp \left\{ ((\mu - r^T)\pi + r^T)T + \frac{1}{4}\sigma^2 \varpi \pi T \right. \\ &\quad \left. + \frac{1}{2}\pi^2 \lambda \sigma^2 \int_0^T \mathbf{E}[\tilde{w}^2(s)] ds \right\} \\ &= \ell \exp(r^T T) \exp(\pi\gamma + \pi^2 \lambda \alpha). \end{aligned} \quad (23)$$

If the expected rate of return on subprime SMP is greater than Treasuries rate, i.e.,  $\mu > r^T$ , the constants  $\gamma$  and  $\alpha$  are both positive. In addition, whether the right-hand side of (23) increases depends on the intensity,  $\lambda$ . On the other hand, if the right-hand side of (23) increases in  $\pi$  and is large, so is the left-hand side. More precisely, for a non-negative  $\lambda$  the right-hand side can be made arbitrarily large by increasing the proportion of investor funds invested in subprime SMP,  $\pi$ . Thus, in solving Problem 3.2 numerically, we replace the value of the investor's expected terminal investment by the right-hand side of (23). As a consequence, we only have to find the largest proportion of the funds invested in subprime SMP which complies with the protection against possible losses from that investment measured by  $SaR(\ell, \pi, T)$ . In the sequel, if we use the inequality

$$\tilde{w}(t) \geq \tilde{W}(t) - \frac{1}{2}\sigma^2 \int_0^t \tilde{w}(s) ds$$

and consider the stationary distribution for  $\tilde{w}(T)$  as a better approximation of its exact distribution then we can solve the SaR-constraint to obtain the optimal investment in subprime SMP such that the constraint is still satisfied. Take note that in this case the stationary distribution of  $\tilde{w}(T)$  is an *inverse gamma distribution*.

A numerical iterations that can be used to solve Problem 3.2 are given below. For  $i = 1, \dots, K$  with  $K$  being large.

**Step 1:** Simulate the trajectories  $(Z_t^i)_{0 \leq t \leq T}$  of the Brownian motion  $(Z_t)_{0 \leq t \leq T}$ .

**Step 2:** Determine the numerical value for  $w_i(t)$  and  $\int_0^T w_i^2(t) dt$  of  $w(T)$  and  $\int_0^T w^2(t) dt$ , respectively, from the path followed by  $(Z_t^i)_{0 \leq t \leq T}$ .

**Step 3:**  $\forall \pi \in \mathbb{R}$  compute

$$\tilde{L}_T^{\pi i} = \pi w_i(T) - \frac{1}{2}\sigma^2 \pi \int_0^T w_i^2(t) dt - \pi w.$$

**Step 4:** Find the approximations  $\tilde{\Psi}(\pi)$  for  $\mathbf{E}[B_T^\pi]$  and  $\tilde{\Upsilon}(\ell, \pi, T)$  for  $SaR(\ell, \pi, T)$ :

$$\begin{aligned} \tilde{\Psi}(\pi) &= \frac{\ell}{K} \sum_{i=1}^K \exp \left( (r^T + (\mu - r^T)\pi)T + \tilde{L}_T^{\pi i} \right), \\ \tilde{\Upsilon}(\ell, \pi, T) &= \ell \exp(r^T T) \left[ 1 - \exp \left( \pi(\mu - r^T)T + \tilde{q}_\gamma \right) \right], \end{aligned}$$

where  $\tilde{q}_\gamma$  is the  $\gamma$ -quantile of the empirical distribution of  $\tilde{L}_T^{\pi i}$ .

**Step 5:** Finally, select a proportion of investor funds invested in subprime SMP with the largest value of  $\tilde{\Psi}(\pi)$  such that  $\tilde{\Upsilon}(\ell, \pi, T)$  is below or equal to a constant  $A$  for the SaR.

#### IV. CONCLUSIONS AND FUTURE INVESTIGATIONS

In this conference paper, we investigated the optimal investment strategy in subprime SMPs and Treasuries under a jump-diffusion process. In this regard, we constructed the stochastic dynamics of investment in subprime SMPs and Treasuries in a Lévy-process setting. This model enabled us to state and solve an optimization problem for investment with a SaR constraint. In Section III, we conclude that this problem cannot be solved by an analytical approach. However, we have found that the optimal solution of Problem 3.2 is the investor's largest investment strategy  $\pi \in [0, 1]$  that satisfies the SaR constraint in (14) and condition (9). We also found an association between our optimization problem and the SMC. An open problem involves the implementation of the numerical algorithm in Subsection III-D2. In order to accomplish this, we require real banking data to determine the optimal portfolio for investors before and during the SMC.

#### REFERENCES

- [1] D. Cuoco and H. Liu, An analysis of VaR-based capital requirements. *Journal of Financial Intermediation*, 15, 362–394 (2006).
- [2] F.J. Fabozzi, X. Cheng and R.-R. Chen, Exploring the components of credit risk in credit default swaps, *Finance Research Letters*, 4, 10–18 (2007).
- [3] Gorton, G. B. (2008). The subprime panic. Yale ICF Working Paper, 08, 25, Available: <http://ssrn.com/abstract=1276047>.
- [4] J. Hull, M. Predescu and A. White, The relationship between credit default swap spreads, bond yields and credit rating announcements, *Journal of Banking and Finance*, 28, 2789–2811 (2004).
- [5] MA. Petersen, MP. Mulaudzi, IM. Schoeman and J. Mukuddem-Petersen, A note on the subprime mortgage crisis: Dynamic modeling of bank leverage profit under loan securitization. *Applied Economic Letters*, 17(15), 1469–1474 (2010).
- [6] MA. Petersen, MC. Senosi and J. Mukuddem-Petersen, *Subprime Mortgage Models*, New York: Nova Science Publishers, ISBN: 978-1-61728-694-0 (2011).
- [7] MA. Petersen, MC. Senosi, J. Mukuddem-Petersen, MP. Mulaudzi and IM. Schoeman, Did bank capital regulation exacerbate the subprime mortgage crisis? *Discrete Dynamics in Nature and Society*, 2009, Article ID 742968, 37 pages (2009).
- [8] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*. Chapman & Hall/CRC (2004).
- [9] RC. Merton, *Continuous-Time Finance* (2nd Edition). Cambridge, Massachusetts: Blackwell Publishers (1992).
- [10] Y. Demyanyk and O. Van Hemert, Understanding the subprime mortgage crisis. Available at SSRN: <http://ssrn.com/abstract=1020396> (19 August 2008).