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Block homotopy perturbation method for solving fuzzy linear systems

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Abstract—In this paper, we present an efficient numerical algorithm, namely block homotopy perturbation method, for solving fuzzy linear systems based on homotopy perturbation method. Some numerical examples are given to show the efficiency of the algorithm.

Keywords—Homotopy perturbation method, Fuzzy linear systems, Block linear system, Fuzzy solution, Embedding parameter.

I. Introduction

MANY engineering problems, such as equilibrium and steady-state problems, a mechanism using the kinetostatic approach, require the solution of simultaneous algebraic linear equations. However, many real-world engineering systems are too complex to be defined in precise terms, therefore, imprecision is often involved. Fuzzy linear systems, which can formulate uncertainty in actual environment, play an essential role in such cases [12], [18], [11] since the concept of fuzzy number and arithmetic operations with these numbers are first introduced by Zadeh [24]. It is immensely important to development mathematical model and numerical procedure that would appropriately treat general fuzzy linear systems and solve them.

Friedman et al. [12] proposed a general model for solving an $n \times n$ fuzzy linear system whose coefficient matrix is a crisp matrix and the right hand column is an arbitrary fuzzy number vector. Using the embedding method given in [21], Friedman et al. [12] replace the original fuzzy linear system by an $2n \times 2n$ crisp linear system. Then solving $n \times n$ fuzzy linear system is equal to solving $2n \times 2n$ crisp linear system. Followed Friedman et al. [12], the point Jacobi, Gauss-Seidel, SOR and steepest descent methods and conjugate gradient methods have been presented for solving $2n \times 2n$ crisp fuzzy linear system, see for example [6], [7], [19], [3], [1] and references therein. The direct methods based on LU decomposition have been proposed and analyzed by Abbasbandy, R. Ezzati and A. Jafarian [2].

However, if there is an diagonal element of the coefficient matrix being zero, then classic point iterative methods are not working. The block method has been studied in [20], [17], including block Jacobi, block Gauss-Seidel and block SOR methods.

Recently, an analytic approach based on the basic ideas of homotopy, which is called homotopy perturbation method (HPM), is provided by He [14] for nonlinear problems. The HPM, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. In most cases, using HPM, gives a very rapid convergence of the solution series, and usually only a few iterations leading to very accurate solutions. The HPM has been used to solve various types of nonlinear problems, see [23], [5], [4], [10], [9] and references therein.

In [15] and [22], the HPM has been used to solve the linear systems. Especially, the point HMP method for solving $n \times n$ nonsingular fuzzy linear system have been studied in [8] while the block HMP method for solving full fuzzy linear system have been considered in [16].

In this paper, we consider the block HMP method for solving $n \times n$ fuzzy linear system, which is efficient and practical because the procedure only require the nonsingularity of the coefficient matrix of $n \times n$ fuzzy linear system while the point HMP method require the diagonal entries of the coefficient matrix are nonzero (see [8]).

The structure of this paper is organized as follows: In Sect. II, we introduce the notation, the definitions, and preliminary results that will be used throughout the paper. In Sect. III, the block HMP method for solving fuzzy linear system is proposed. The proposed model is illustrated by solving some examples in Sect. IV and conclusions are drawn in Sect. V.

II. Preliminaries

Zadeh [24] defined a fuzzy number as follows:

Definition 1. A fuzzy number is a fuzzy set like $\tilde{u} : R \to I = [0, 1]$, which satisfies

1. $\tilde{u}$ is upper semi-continuous,
2. $\tilde{u}(x) = 0$ outside some interval $[c, d]$,
3. There are real numbers a, b such that $c \le a \le b \le d$ and
   3.1 $\tilde{u}(x)$ is monotonic increasing on $[c, a]$,
   3.2 $\tilde{u}(x)$ is monotonic decreasing on $[b, d]$,
   3.3 $\tilde{u}(x) = 1$ when $a \le x \le b$.

The set of all these fuzzy numbers is denoted by E. An equivalent parametric of a fuzzy number is given in [13] as

Definition 2. A fuzzy number in parametric form is an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0, 1]$,
3. $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \le r \le 1$.

The addition and scalar multiplication of fuzzy numbers can be described as follows, for arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$, $0 \le r \le 1$, and real number k,

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1. $u+v=(\underline{u}(r)+\underline{v}(r),\overline{u}(r)+\overline{v}(r))$,
2. $ku=\begin{cases}(k\underline{u}(r),k\overline{u}(r)), & k\geq 0,\\ (k\overline{u}(r),k\underline{u}(r)), & k<0.\end{cases}$

Definition 3. The $n\times n$ linear system

$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=y_1,\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=y_2,\\ \qquad\vdots\\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=y_n,\end{cases} \tag{1}$$

or briefly

$$Ax=y,$$

where the coefficient matrix $A=(a_{ij}),\ 1\leq i,\ j\leq n$ is a crisp matrix, $y=(y_1,y_2,\cdots,y_n)^T$ is known with $y_i\in E$ and $x=(x_1,x_2,\cdots,x_n)^T$ is unknown with $x_i\in E,\ 1\leq i\leq n$, is called a fuzzy linear system (FLS).

Definition 4. A fuzzy number vector $X=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}$ given by $x_j=(\underline{x}_j(r),\overline{x}_j(r)),\ 1\leq j\leq n,\ 0\leq r\leq 1$, is called a solution of the fuzzy linear system (1) if

$$\begin{cases} \underline{\sum_{j=1}^{n}a_{ij}x_j}=\sum_{j=1}^{n}\underline{a_{ij}x_j}=\underline{y}_i,\\ \overline{\sum_{j=1}^{n}a_{ij}x_j}=\sum_{j=1}^{n}\overline{a_{ij}x_j}=\overline{y}_i,\end{cases} \quad i=1,\cdots,n. \tag{2}$$

By (2) and the operation of fuzzy numbers, Friedman et al. [12] replace the original fuzzy linear systems (1) by an $2n\times 2n$ crisp function linear system

$$\mathbf{SX}=\mathbf{Y}\quad\text{or}\quad \begin{bmatrix} B & C\\ C & B\end{bmatrix}\begin{bmatrix}\underline{X}\\ -\overline{X}\end{bmatrix}=\begin{bmatrix}\underline{Y}\\ -\overline{Y}\end{bmatrix}, \tag{3}$$

where $\mathbf{S}=(s_{kl}),\ 1\leq k,l\leq 2n,\ s_{kl}$ are determined as follows

$$\begin{array}{lll} a_{ij}\geq 0 \ \Rightarrow & s_{ij}=a_{ij}, & s_{i+m,\ j+n}=a_{ij},\\ a_{ij}<0 \ \Rightarrow & s_{i,\ j+n}=-a_{ij}, & s_{i+m,\ j}=a_{ij},\end{array}$$

and any s_{kl} which is not determined by the above items is zero and

$$\mathbf{X}=\begin{bmatrix}\underline{X}\\ -\overline{X}\end{bmatrix}=\begin{bmatrix}\underline{x}_1\\ \vdots\\ \underline{x}_n\\ -\overline{x}_1\\ \vdots\\ -\overline{x}_n\end{bmatrix},$$

$$\mathbf{Y}=\begin{bmatrix}\underline{Y}\\ -\overline{Y}\end{bmatrix}=\begin{bmatrix}\underline{y}_1\\ \vdots\\ \underline{y}_n\\ -\overline{y}_1\\ \vdots\\ -\overline{y}_n\end{bmatrix},$$

B contains the positive entries of A, C the absolute of the negative entries of A and $A=B-C$. Then solving the fuzzy linear system (1) is equal to solving crisp linear system (3). The crisp linear system (3) can be uniquely solved for $\mathbf{X}$ if and only if the coefficient matrix $\mathbf{S}$ is nonsingular. The following theorem tell us when $\mathbf{S}$ is nonsingular.

Theorem 1. [12] The matrix $\mathbf{S}$ is nonsingular if and only if $A=B-C$ and $B+C$ are both nonsingular.

If the matrix $\mathbf{S}$ is nonsingular, then the solution vector $\mathbf{X}$ represent a solution fuzzy vector to the fuzzy system (1) if and only if $(\underline{x}_j(r),\overline{x}_j(r))$ is a fuzzy number for all j.

Definition 6. Let $X=\{(\underline{x}_j(r),\overline{x}_j(r))\},\ 1\leq j\leq n$ denote the solution of (3). The fuzzy number vector $U=\{(\underline{u}_j(r),\overline{u}_j(r)),\ 1\leq j\leq n\}$ defined by

$$\begin{aligned}\underline{u}_j(r) &= \min\left\{\underline{x}_j(r),\overline{x}_j(r),\underline{x}_j(1),\overline{x}_j(1)\right\},\\ \overline{u}_j(r) &= \max\left\{\underline{x}_j(r),\overline{x}_j(r),\underline{x}_j(1),\overline{x}_j(1)\right\}\end{aligned}$$

is called the fuzzy solution of (3). If $(\underline{x}_j(r),\overline{x}_j(r)),1\leq j\leq n$ are all fuzzy numbers then $\underline{u}_j(r)=\underline{x}_j(r),\ \overline{u}_j(r)=\overline{x}_j(r)$, $1\leq j\leq n$ and U is called a strong fuzzy solution. Otherwise, U is called a weak fuzzy solution.

III. Analysis of the HPM

Consider the crisp linear system (3) and let

$$L(\mathbf{U})=\mathbf{SU}-\mathbf{Y},\quad F(\mathbf{U})=\mathbf{QU}-\mathbf{Y},$$

where $\mathbf{Q}$ is nonsingular. We define homotopy $H(\mathbf{U},p)$ by

$$H(\mathbf{U},0)=F(\mathbf{U}),\quad H(\mathbf{U},1)=L(\mathbf{U}).$$

We may choose a convex homotopy

$$H(\mathbf{U},p)=(1-p)F(\mathbf{U})+pL(\mathbf{U})=0, \tag{4}$$

and continuously trace an implicitly defined curve from a starting point $H(\mathbf{U},0)$ to a solution $H(\mathbf{U},1)$. The embedding parameter p monotonically increases from zero to one as the trivial problem $F(\mathbf{U})=0$ is continuously deformed to the original problem $L(\mathbf{U})=0$. The embedding parameter $p\in[0,\ 1]$ can be considered as an expanding parameter [14]

$$\mathbf{U}=\mathbf{U}_0+p\mathbf{U}_1+p^2\mathbf{U}_2+\cdots, \tag{5}$$

when $p\rightarrow 1$, (4) corresponds to $L(\mathbf{U})=0$ and (5) becomes the approximate solution of (3), i.e.,

$$\mathbf{X}=\lim_{p\rightarrow 1}(\mathbf{U}_0+p\mathbf{U}_1+p^2\mathbf{U}_2+\cdots)=\sum_{k=0}^{\infty}\mathbf{U}_k.$$

Substituting (5) into (4) and equating the terms with identical powers of p, we have

$$\begin{array}{l} p^0:\ \mathbf{QU}_0-\mathbf{Y}=0,\\ p^k:\ \mathbf{QU}_k+(\mathbf{S}-\mathbf{Q})\mathbf{U}_{k-1}=0,\ k=1,2,\cdots.\end{array}$$

This implies that

$$\begin{array}{l}\mathbf{U}_0=\mathbf{Q}^{-1}\mathbf{Y},\\ \mathbf{U}_k=(\mathbf{I}-\mathbf{Q}^{-1}\mathbf{S})\mathbf{U}_{k-1},\ k=1,2,\cdots,\end{array}$$

where $\mathbf{I}$ is an indent matrix with order $2n$. Moreover, we can rewrite $\mathbf{U}_k$ in terms of the vector $\mathbf{Y}$ as

$$\mathbf{U}_k=(\mathbf{I}-\mathbf{Q}^{-1}\mathbf{S})^k\mathbf{Q}^{-1}\mathbf{Y},\ k=1,2,\cdots.$$

Hence, the solution of (3) can be of the form

$$\mathbf{X}=\mathbf{U}_0+\mathbf{U}_1+\mathbf{U}_2+\cdots$$

or

$$\begin{aligned}\mathbf{X} &= [\mathbf{Q}^{-1} + (\mathbf{I} - \mathbf{Q}^{-1}\mathbf{S})\mathbf{Q}^{-1} + (\mathbf{I} - \mathbf{Q}^{-1}\mathbf{S})^2\mathbf{Q}^{-1} + \cdots]\mathbf{Y} \\ &= \sum_{k=0}^{\infty}(\mathbf{I} - \mathbf{Q}^{-1}\mathbf{S})^k\mathbf{Q}^{-1}\mathbf{Y}. \qquad (6)\end{aligned}$$

In practice, all terms of series (6) cannot be determined and so we use an approximation of the solution by the following truncated series:

$$\mathbf{X} = \sum_{k=0}^{m-1}(\mathbf{I} - \mathbf{Q}^{-1}\mathbf{S})^k\mathbf{Q}^{-1}\mathbf{Y}.$$

The following theorem gives the convergent result of the above series.

Theorem 2. The sequence

$$\mathbf{U} = [\sum_{k=0}^{m-1}(\mathbf{I} - \mathbf{Q}^{-1}\mathbf{S})^k\mathbf{Q}^{-1}\mathbf{Y}$$

is convergent if

$$\|(\mathbf{I} - \mathbf{Q}^{-1}\mathbf{S})^k\| < 1,$$

where $\|\cdot\|$ denotes any norm of a matrix.

To find the solution of linear system (3), we should choose a nonsingular matrix $\mathbf{Q}$. From Theorem 2, the matrix $\mathbf{Q}$ can be selected as

$$\mathbf{Q} = \begin{bmatrix} B - C & 0 \\ 0 & B - C \end{bmatrix}$$

or

$$\mathbf{Q} = \begin{bmatrix} B + C & 0 \\ 0 & B + C \end{bmatrix}$$

or other block forms, see for example [17].

If the matrix $\mathbf{Q}$ is selected as

$$\mathbf{Q} = \begin{bmatrix} B - C & 0 \\ 0 & B - C \end{bmatrix},$$

then we have

$$\mathbf{I} - \mathbf{Q}^{-1}\mathbf{S} = \begin{bmatrix} I - (B-C)^{-1}B & -(B-C)^{-1}C \\ -(B-C)^{-1}C & I - (B-C)^{-1}B \end{bmatrix},$$

where I is an indent matrix with order n.

IV. Numerical examples

For a given parameter $r \in [0,\ 1]$, we consider the following two examples.

Example 1. Consider the 3×3 fuzzy system

$$\begin{cases} 2x_1 + 3x_2 - x_3 = (-1 + 4r,\ 6 - 3r), \\ 3x_1 - x_2 + 2x_3 = (4 + 2r,\ 12 - 6r), \\ x_1 + 2x_2 + 3x_3 = (4 + 5r,\ 13 - 4r). \end{cases}$$

The exact solution is

$$\begin{cases} x_1 = (\underline{x}_1(r),\ \overline{x}_1(r)) = (1,\ 2 - r), \\ x_2 = (\underline{x}_2(r),\ \overline{x}_2(r)) = (r,\ 1), \\ x_3 = (\underline{x}_3(r),\ \overline{x}_3(r)) = (1 + r,\ 3 - r), \end{cases}$$

which is a strong fuzzy solution.

The extended linear system is the form of (3), where

$$\mathbf{S} = \left(\begin{array}{ccc|ccc} 2 & 3 & 0 & 0 & 0 & 1 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right),$$

$$\mathbf{Y} = \begin{pmatrix} -1 + 4r \\ 4 + 2r \\ 4 + 5r \\ -6 + 3r \\ -12 + 6r \\ -13 + 4r \end{pmatrix}$$

and

$$\mathbf{X} = \begin{pmatrix} \underline{x}_1(r) \\ \underline{x}_2(r) \\ \underline{x}_3(r) \\ -\overline{x}_1(r) \\ -\overline{x}_2(r) \\ -\overline{x}_3(r) \end{pmatrix}.$$

By HPM and using nine iterations, we obtain the approximation to the solution of this extended system as

$$\mathbf{X} = \begin{pmatrix} 1.00000 + 0.00000r \\ 0.00000 + 1.00000r \\ 1.00000 + 1.00000r \\ -2.00000 + 1.0000r \\ -1.00000 + 0.00000r \\ -3.00000 + 1.00000r \end{pmatrix}.$$

Therefore, the approximation fuzzy solution of this example is

$$\begin{cases} x_1 = (\underline{x}_1(r),\ \overline{x}_1(r)) = (1,\ 2 - r), \\ x_2 = (\underline{x}_2(r),\ \overline{x}_2(r)) = (r,\ 1), \\ x_3 = (\underline{x}_3(r),\ \overline{x}_3(r)) = (1 + r,\ 3 - r). \end{cases}$$

Example 2. Consider the 3×3 fuzzy system

$$\begin{cases} -2x_2 + 5x_3 = (-3,\ -2 - r), \\ x_1 + 2x_2 = (r,\ 2 - r), \\ 3x_1 - x_3 = (1 + r,\ 3 - r). \end{cases}$$

The exact solution is

$$\begin{cases} x_1 = (\underline{x}_1(r),\ \overline{x}_1(r)) \\ \quad = (-0.18750 + 0.31250r,\ 0.87500 - 0.37500r), \\ x_2 = (\underline{x}_2(r),\ \overline{x}_2(r)) \\ \quad = (-0.09375 + 0.34375r,\ 0.56250 - 0.31250r), \\ x_3 = (\underline{x}_3(r),\ \overline{x}_3(r)) \\ \quad = (-0.37500 - 0.12500r,\ 0.43750 - 0.06250r). \end{cases}$$

In fact x_3 is not a fuzzy number, therefore the fuzzy solution is

$$\begin{cases} u_1 = (\underline{u}_1(r),\ \overline{u}_1(r)) \\ \quad = (-0.18750 + 0.31250r,\ 0.87500 - 0.37500r), \\ u_2 = (\underline{u}_2(r),\ \overline{u}_2(r)) \\ \quad = (-0.09375 + 0.34375r,\ 0.56250 - 0.31250r), \\ u_3 = (\underline{u}_3(r),\ \overline{u}_3(r)) \\ \quad = (-0.50000,\ -0.37500 - 0.12500r), \end{cases}$$

which is a weak fuzzy solution.

The extended linear system is the form of (3), where

$$\mathbf{S} = \left(\begin{array}{ccc|ccc} 0 & 0 & 5 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \end{array}\right),$$

$$\mathbf{Y} = \begin{pmatrix} -3 \\ r \\ 1+r \\ 2+r \\ -2+r \\ -3+r \end{pmatrix}$$

and

$$\mathbf{X} = \begin{pmatrix} \underline{x}_1(r) \\ \underline{x}_2(r) \\ \underline{x}_3(r) \\ -\overline{x}_1(r) \\ -\overline{x}_2(r) \\ -\overline{x}_3(r) \end{pmatrix}.$$

By HPM and using six iterations, we obtain the approximation to the solution of this extended system as

$$\mathbf{X} = \begin{pmatrix} -0.18750 + 0.31250r \\ -0.09375 + 0.34375r \\ -0.37500 - 0.12500r \\ -0.87500 + 0.37500r \\ -0.56250 + 0.31250r \\ -0.43750 + 0.06250r \end{pmatrix}.$$

Therefore, the approximation weak fuzzy solution of this example is

$$\begin{cases} x_1 = (\underline{x}_1(r),\ \overline{x}_1(r)) \\ \quad = (-0.18750 + 0.31250r,\ 0.87500 - 0.37500r), \\ x_2 = (\underline{x}_2(r),\ \overline{x}_2(r)) \\ \quad = (-0.09375 + 0.34375r,\ 0.56250 - 0.31250r), \\ x_3 = (\underline{x}_3(r),\ \overline{x}_3(r)) \\ \quad = (-0.37500 - 0.12500r,\ 0.43750 - 0.06250r). \end{cases}$$

V. Conclusion

In this paper, we consider the block HPM for finding an approximation solution of fuzzy linear systems. The block HMP method is efficient and practical because the procedure only require the nonsingularity of the coefficient matrix of $n \times n$ fuzzy linear system while the point HMP method require the diagonal entries of the coefficient matrix are nonzero (see [8]). The numerical results show that the block HPM converges to the exact solution of fuzzy linear systems.

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