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On the Flow of a Third Grade Viscoelastic Fluid in an Orthogonal Rheometer

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***Abstract*—**The flow of a third grade fluid in an orthogonal rheometer is studied. We employ the admissible velocity field proposed in [5]. We solve the problem and obtain the velocity field as well as the components for the Cauchy tensor. We compare the results with those from [9]. Some diagrams concerning the velocity and Cauchy stress components profiles are presented for different values of material constants and compared with the corresponding values for a linear viscous fluid.

***Keywords*—**Non newtonian fluid flow, orthogonal rheometer, third grade fluid.

I. Introduction

THE flow occuring in the orthogonal rheometer has been studied by many authors. For instance in [5] was investigated the flow of asecond grade fluid and in [9] was studied the flow of BKZ fluid in the same domain.

The apparatus has two parallel plates rotating with the same constant angular velocity Ω arround two parallel and different axes (d is the distance between the plates, see Fig. 1). The fluid to be tested fills the space between them (the distance between axes of rotation is a).

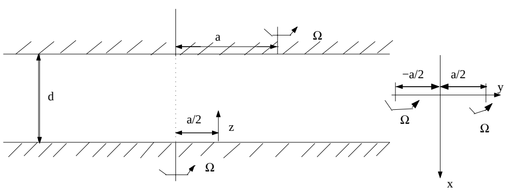

Fig. 1 Scheme of the orthogonal rheometer

In this paper we study the flow of an incompressible fluid of third grade. The boundary conditions arised from the adherence conditions on the two plates, and the bilocal problem obtained from the described mechanical problem is solved exactly. We calculate the hydrostatic pressure and the stresses on plates.

Some numerical experiments concerning the velocity field and Cauchy stress components are presented and discussed.

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II. Equations of Motion

We assume that the motion occuring in the orthogonal rheometer can be represented by:

$$\vec{v} = -\Omega(y - g(z))\vec{i} + \Omega(x - f(z))\vec{j}\,, \tag{1}$$

where (x, y, z) is a fixed cartesian co-ordinate system (see [5]).

It follows from (1) that the velocity gradient $\mathbf{L}$ has the following representation:

$$\mathbf{L} = \begin{pmatrix} 0 & -\Omega & \Omega g'(z) \\ \Omega & 0 & -\Omega f'(z) \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

The Cauchy's stress tensor $\mathbf{T}$ for the incompressible fluid of third grade is given by:

$$\begin{aligned} \mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1(\mathbf{A}_2 - \mathbf{A}_1^2) + \beta_1\mathbf{A}_3 \\ + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1 \end{aligned}. \tag{3}$$

where p is the hydrostatic pressure, μ, α_1, β_1, β_2, β_3 are constant constitutive coefficients, $\mathbf{I}$ is the identity tensor, $\mathrm{tr}(\cdot)$ is the trace operator, and $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are the Rivlin-Ericksen tensors ($\mathbf{A}_1 = \nabla\vec{v} + (\nabla\vec{v})^{\mathbf{T}}$, and for $n = 2,3,\ldots$ $\mathbf{A}_n = \overset{\circ}{\mathbf{A}}_{\mathbf{n-1}} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^{\mathbf{T}}\,\mathbf{A}_{n-1}$).

The components of the stress tensor are given by:

$$\begin{aligned} &T_{11} = -p - \alpha_1\Omega^2 g'^2 - 2\beta_2\Omega^3 f'g', \\ &T_{22} = -p - \alpha_1\Omega^2 f'^2 + 2\beta_2\Omega^3 f'g', \\ &T_{33} = -p + \alpha_1\Omega^2(f'^2 + g'^2), \\ &T_{12} = \alpha_1\Omega^2 f'g' + \beta_2\Omega^3(f'^2 - g'^2), \\ &T_{13} = \mu\Omega g' - \alpha_1\Omega^2 f' - \beta_1\Omega^3 g' + 2(\beta_2 + \beta_3)\Omega^3(f'^2 g' + g'^3), \\ &T_{23} = -\mu\Omega f' - \alpha_1\Omega^2 g' + \beta_1\Omega^3 f' - 2(\beta_2 + \beta_3)\Omega^3(f'^3 + f'g'^2). \end{aligned} \tag{4}$$

From the form of velocity field proposed results the acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = -\Omega^2(x - f(z))\,\vec{i} - \Omega^2(y - g(z))\,\vec{j} \tag{5}$$

We also assume that the specific body force $\vec{b}$ is conservative and hence derivable from a potential ϕ:

$$\vec{b} = -\operatorname{grad}\phi\,. \tag{6}$$

The local form of the balance of linear momentum is:

$$\rho\vec{a} = \rho\vec{b} + \operatorname{div}\mathbf{T} \tag{7}$$

and implies that:

$$\begin{aligned}
&p_x + \rho\phi_x - \rho\Omega^2(x - f(z)) = \mu\Omega g'' - \alpha_1\Omega^2 f'' - \beta_1\Omega^3 g'' \\
&\qquad + 2\Omega^3(\beta_2 + \beta_3)\cdot(2f'f''g' + f'^2 g'' + 3g'^2 g''),\\
&p_y + \rho\phi_y - \rho\Omega^2(y - g(z)) = -\mu\Omega f'' - \alpha_1\Omega^2 g'' + \beta_1\Omega^3 f'' \\
&\qquad - 2\Omega^3(\beta_2 + \beta_3)\cdot(3f'^2 f'' + f''g'^2 + 2f'g'g''),\\
&p_z + \rho\phi_z = 2\alpha_1\Omega^2(f'f'' + g'g''),
\end{aligned} \tag{8}$$

where we have denoted $\bullet_x, \bullet_y, \bullet_z$ the derivatives of the functions with respect to the indices. The boundary conditions for the velocity field arise from the adherence conditions upon the upper, at $z = d$, and lower, at $z = 0$, plates of the orthogonal rheometer:

$$\begin{aligned}
&\vec{v}\big|_{z=0} = -\Omega(a/2 + y)\vec{i} + \Omega x\vec{j},\\
&\vec{v}\big|_{z=d} = \Omega(a/2 - y)\vec{i} + \Omega x\vec{j},\\
&u = \vec{v}\cdot\vec{i} \to \mp\infty, v = \vec{v}\cdot\vec{j} \to \pm\infty, \vec{v}\cdot\vec{k} = 0, \text{ when } x, y \to \pm\infty.
\end{aligned} \tag{9}$$

From (9) it follows that:

$$f(0) = f(d) = 0,\ g(0) = -a/2,\ g(d) = a/2\,. \tag{10}$$

In order to use the curl operator the system (8) could be written as:

$$\begin{aligned}
&[\mu\Omega g'' - \alpha_1\Omega^2 f'' - \beta_1\Omega^3 g'' + 2\Omega^3(\beta_2 + \beta_3)\\
&\cdot(2f'f''g' + f'^2 g'' + 3g'^2 g'')]\vec{i} + [-\mu\Omega f'' - \alpha_1\Omega^2 g'' + \beta_1\Omega^3 f''\\
&- 2\Omega^3(\beta_2 + \beta_3)\cdot(3f'^2 f'' + f''g'^2 + 2f'g'g'')]\vec{j}\\
&+ 2\alpha_1\Omega^2(f'f'' + g'g'')\vec{k} = [p_x + \rho\phi_x - \rho\Omega^2(x - f(z))]\vec{i}\\
&\qquad + [p_y + \rho\phi_y - \rho\Omega^2(y - g(z))]\vec{j} + [p_z + \rho\phi_z]\vec{k}
\end{aligned} \tag{11}$$

III. Solution for Equations of Motion

Using the curl operator in (11) we find that:

$$\begin{aligned}
&g'''[\mu\Omega - \beta_1\Omega^3 g'' + 2\Omega^3(\beta_2 + \beta_3)\cdot(3g'^2 + f'^2)]\\
&\quad + f'''[-\alpha_1\Omega^2 + 4\Omega^3(\beta_2 + \beta_3)\cdot f'g']\\
&\quad + 4\Omega^3(\beta_2 + \beta_3)[2g''(f'f'' + g'g'') + g'(f''^2 + g''^2)] = \rho\Omega^2 f'\\
&- f'''[\mu\Omega - \beta_1\Omega^3 + 2\Omega^3(\beta_2 + \beta_3)\cdot(3f'^2 + g'^2)]\\
&\quad + g'''[-\alpha_1\Omega^2 - 4\Omega^3(\beta_2 + \beta_3)\cdot f'g']\\
&\quad - 4\Omega^3(\beta_2 + \beta_3)[2f''(f'f'' + g'g'') + f'(f''^2 + g''^2)] = \rho\Omega^2 g'.
\end{aligned} \tag{12}$$

We denote:

$$\begin{aligned}
&h_1(z) = g''[\mu\Omega - \beta_1\Omega^3 + 2\Omega^3(\beta_2 + \beta_3)\cdot(3g'^2 + f'^2)]\\
&\qquad + f''[-\alpha_1\Omega^2 + 4\Omega^3(\beta_2 + \beta_3)\cdot f'g'],\\
&h_2(z) = -f''[\mu\Omega - \beta_1\Omega^3 + 2\Omega^3(\beta_2 + \beta_3)\cdot(3f'^2 + g'^2)]\\
&\qquad + g''[-\alpha_1\Omega^2 - 4\Omega^3(\beta_2 + \beta_3)\cdot f'g'],\\
&h_3(z) = 2\alpha_1\Omega^2(f'f'' + g'g''),
\end{aligned} \tag{13}$$

and thus the system (12) can be expressed as:

$$h_1' = \rho\Omega^2 f', \quad h_2' = \rho\Omega^2 g' \tag{14}$$

and the system (8) as:

$$\begin{aligned}
&p_x + \rho\phi_x - \rho\Omega^2(x - f(z)) = h_1(z),\\
&p_y + \rho\phi_y - \rho\Omega^2(y - g(z)) = h_2(z),\\
&p_z + \rho\phi_z = h_3(z).
\end{aligned} \tag{15}$$

After integrating the system (14) we obtain:

$$h_1 = \rho\Omega^2 f + s, \quad h_2 = \rho\Omega^2 g + q, \tag{16}$$

with s and q constants.

We can write:

$$\begin{aligned}
\frac{\hat{p}}{\rho} &= \int[\Omega^2(x - f(z)) + \frac{h_1(z)}{\rho}]dx + \int[\Omega^2(y - g(z))\\
&\quad + \frac{h_2(z)}{\rho}]dy + \frac{1}{\rho}\int h_3(z)dz + C
\end{aligned} \tag{17}$$

for $\hat{p} = p + \rho\phi$, with $dp/\rho = (p_x dx + p_y dy + p_z dz)/\rho$.

From (16) we have:

$$\begin{aligned}
\frac{\hat{p}}{\rho} &= (\frac{\Omega^2 x^2}{2} + \frac{s}{\rho}x) + (\frac{\Omega^2 y^2}{2} + \frac{q}{\rho}y) + \frac{\alpha_1\Omega^2}{\rho}(f'^2 + g'^2) + C\\
&\equiv \frac{\Omega^2}{2}(x^2 + y^2) + (\frac{s}{\rho}x + \frac{q}{\rho}y) + \frac{\alpha_1\Omega^2}{\rho}(f'^2 + g'^2) + C.
\end{aligned}$$

In order to ensure the symmetry of the velocity distribution on the plane $z = d$ we set $s = q = 0$, therefore:

$$\hat{p} = \frac{\rho\Omega^2}{2}(x^2 + y^2) + \alpha_1\Omega^2(f'^2 + g'^2) + \rho C\,\cdot$$

or

$$p = \frac{\rho\Omega^2}{2}(x^2 + y^2) + \alpha_1\Omega^2(f'^2 + g'^2) + \rho(C - \phi) \cdot \qquad (17)$$

Following the procedure by [9], we put $s = q = 0$ in (16), then:

$$\begin{aligned}\rho\Omega^2 f &= g''[\mu\Omega - \beta_1\Omega^3 + 2\Omega^3(\beta_2 + \beta_3)\cdot(3g'^2 + f'^2)] \\ &\quad + f''[-\alpha_1\Omega^2 + 4\Omega^3(\beta_2 + \beta_3)\cdot f'g'], \\ \rho\Omega^2 g &= -f''[\mu\Omega - \beta_1\Omega^3 + 2\Omega^3(\beta_2 + \beta_3)\cdot(3f'^2 + g'^2)] \\ &\quad + g''[-\alpha_1\Omega^2 - 4\Omega^3(\beta_2 + \beta_3)\cdot f'g'].\end{aligned} \qquad (18)$$

We shall linearise the system (18) under the constitutive restrictions:

$$\mu \geq 0, \alpha_1 \geq 0, \beta_1 \leq 0, \beta_1 + 2(\beta_2 + \beta_3) \geq 0 .$$

We can also obtain a linear system if we make the hypothesis $\beta_2 + \beta_3 = 0$, that implies $\beta_1 = 0$. The solution will be similar with those obtained in [9], for the case of linear viscoelasticity, but with different coefficients.

The system (18) becomes:

$$\begin{aligned}\rho\Omega^2 f &= -\alpha_1\Omega^2 f'' + (\mu\Omega - \beta_1\Omega^3)g'', \\ \rho\Omega^2 g &= -(\mu\Omega - \beta_1\Omega^3)f'' - \alpha_1\Omega^2 g''.\end{aligned} \qquad (19)$$

If we write the corresponding dimensionless system we find:

$$\begin{aligned}\mathrm{Re}_m \bar{f} &= -\bar{\alpha}_m\bar{f}'' + \bar{g}'', \\ \mathrm{Re}_m \bar{g} &= -\bar{f}'' - \bar{\alpha}_m\bar{g}''.\end{aligned} \qquad (19)'$$

Here $\bar{\bullet}$ denote the dimensionless quantities, and Re_m the modified Reynolds number:

$$\mathrm{Re}_m = \frac{\rho\Omega d^2}{\mu - \beta_1\Omega^2}, \quad \bar{\alpha}_m = \frac{\alpha_1\Omega}{\mu - \beta_1\Omega^2}. \qquad (20)$$

The system (19)' could be written as:

$$\begin{aligned}\bar{f}'' &= \frac{\mathrm{Re}_m}{1 + \bar{\alpha}_m^2}(-\bar{\alpha}_m\bar{f} - \bar{g}), \\ \bar{g}'' &= \frac{\mathrm{Re}_m}{1 + \bar{\alpha}_m^2}(\bar{f} - \bar{\alpha}_m\bar{g}).\end{aligned} \qquad (19)''$$

The system (19)'', with dimensionless boundary conditions:

$$\bar{f}(0) = \bar{f}(1) = 0, \bar{g}(0) = -\frac{a}{2d}, \bar{g}(1) = \frac{a}{2d}, \qquad (10)'$$

is solved for $\bar{f}$ and $\bar{g}$ and leads to:

$$\begin{aligned}\bar{f}(\bar{z}) &= \frac{2a}{d\cdot\Delta}\{\sin\beta\cosh\alpha\cdot[\cos\beta\bar{z}\sinh\alpha\bar{z} \\ &\quad + \cos\beta(\bar{z}-1)\sinh\alpha(\bar{z}-1)] - \cos\beta\sinh\alpha \\ &\quad \cdot[\sin\beta\bar{z}\cosh\alpha\bar{z} + \sin\beta(\bar{z}-1)\cosh\alpha(\bar{z}-1)]\}, \\ \bar{g}(\bar{z}) &= \frac{2a}{d\cdot\Delta}\{\cos\beta\sinh\alpha\cdot[\cos\beta\bar{z}\sinh\alpha\bar{z} \\ &\quad + \cos\beta(\bar{z}-1)\sinh\alpha(\bar{z}-1)] + \sin\beta\cosh\alpha \\ &\quad \cdot[\sin\beta\bar{z}\cosh\alpha\bar{z} + \sin\beta(\bar{z}-1)\cosh\alpha(\bar{z}-1)]\}\end{aligned} \qquad (21)$$

where

$$\begin{aligned}\Delta &= 4\cdot(\sinh^2\alpha + \sin^2\beta), \\ \alpha^2 &= \frac{\mathrm{Re}_m}{2(1 + \bar{\alpha}_m^2)}(\sqrt{1 + \bar{\alpha}_m^2} - \bar{\alpha}_m), \\ \beta^2 &= \frac{\mathrm{Re}_m}{2(1 + \bar{\alpha}_m^2)}(\sqrt{1 + \bar{\alpha}_m^2} + \bar{\alpha}_m).\end{aligned} \qquad (22)$$

If we evaluate the difference between the normal stresses (in fixed $\vec{x}$ points) on the two plates we have:

$$\Delta\mathbf{T}\vec{n}\cdot\vec{n} = -\Delta p + \Delta\mathbf{T}_E\vec{n}\cdot\vec{n} ,$$

where $\vec{n}$ is the normal versor on the plates and for Δ we understand $\Delta\bullet = \bullet(x, y, 0) - \bullet(x, y, l)$.

Using the pressure given by (17) and the stress tensor (4) we obtain:

$$\mathbf{T}\vec{n}\cdot\vec{n} = -p + \mathbf{T}_{E(33)} = -\rho[\frac{\Omega^2}{2}(x^2 + y^2) + \phi - C]. \qquad (23)$$

We consider the stress vector field $\vec{t} = \mathbf{T}\vec{n}$ and we obtain for its components in a (x,y) plane:

$$\begin{aligned}t_x &= T^{13} = \mu\Omega\bar{g}' - \alpha_1\Omega^2\bar{f}' - \beta_1\Omega^3\bar{g}', \\ t_y &= T^{23} = -\mu\Omega\bar{f}' - \alpha_1\Omega^2\bar{g}' + \beta_1\Omega^3\bar{f}'\end{aligned} \qquad (24)$$

We simply see that:

$$\begin{aligned}t_x &= \frac{\rho\Omega^2 d^2}{\mathrm{Re}_m}(-\bar{\alpha}_m\bar{f}' + \bar{g}'), \\ t_y &= \frac{\rho\Omega^2 d^2}{\mathrm{Re}_m}(-\bar{f}' - \bar{\alpha}_m\bar{g}').\end{aligned} \qquad (24)'$$

The jumps $\Delta t_x, \Delta t_y$ will be:

$$\begin{aligned}\Delta t_x &= \frac{\rho\Omega^2 d^2}{\mathrm{Re}_m}(-\bar{\alpha}_m\Delta\bar{f}' + \Delta\bar{g}'), \\ \Delta t_y &= \frac{\rho\Omega^2 d^2}{\mathrm{Re}_m}(-\Delta\bar{f}' - \bar{\alpha}_m\Delta\bar{g}').\end{aligned}$$

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IV. Numerical Experiments

For numerical representations we consider $d = 1.5 \cdot 10^{-2}\,\mathrm{m}$, $a = d$, $\rho = 1000\,\mathrm{kg \cdot m^{-3}}$. For modified Reynolds number we use different values: $\mathrm{Re_m} = 0.01$, 1 or 10 for the dimensionless functions $\bar{f}$ and $\bar{g}$ and $\mathrm{Re_m} = 0.001$ for the traction $\bar{t}_x$. The constant $\bar{\alpha}_m$ has also different values: 1, 0.95, 0.5 or 0 (for the linear viscous fluid).

V. Conclusion

In Fig. 2 and Fig. 3 we represent the dimensionless functions $\bar{f}$ and $\bar{g}$ respectively, for various values of the modified Reynolds number.

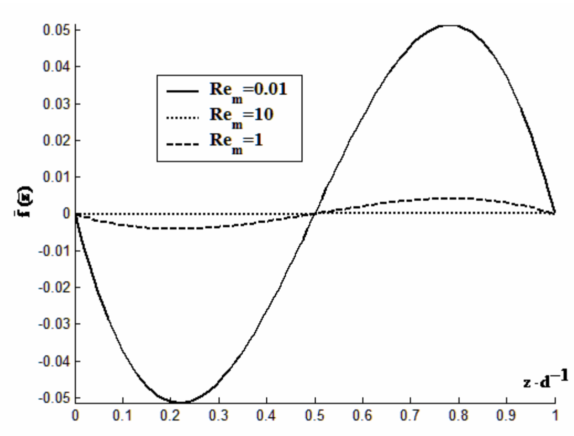

Fig. 2 The dimensionless function $\bar{f}(\bar{z})$. $\Omega = 40\,\mathrm{rad \cdot s^{-1}}$, $\bar{\alpha}_m = 1$

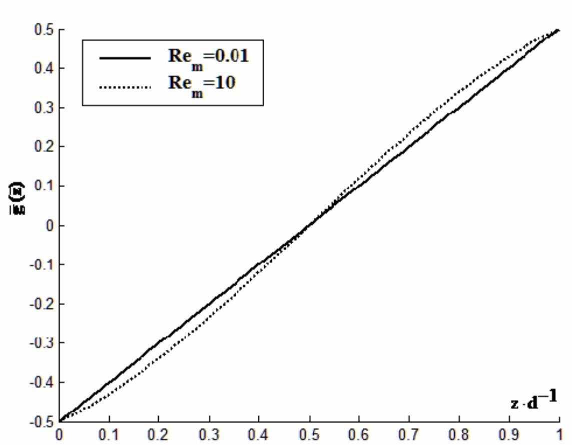

Fig. 3 The dimensionless function $\bar{g}(\bar{z})$. $\Omega = 40\,\mathrm{rad \cdot s^{-1}}$, $\bar{\alpha}_m = 1$

In Fig. 4 we represent the dimensionless component $\bar{t}_x$ of the pressure vector $\vec{t}$ for the third grade fluid and for the linear viscous fluid.

In Fig. 5 the same comparision is made for $\Omega = 80\,\mathrm{rad \cdot s^{-1}}$.

Similar comparisions can be made with a second grade fluid, but there are no relevant conclusions (distinct from most concerning linear viscous fluids).

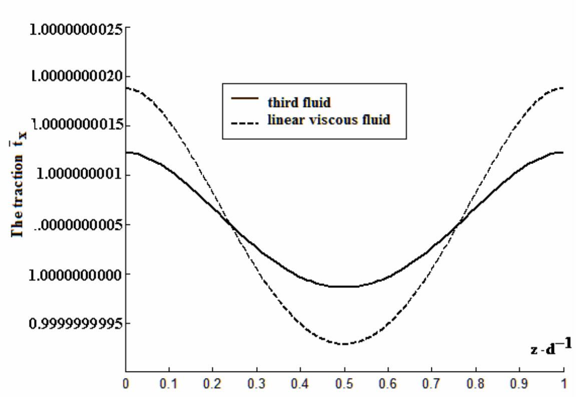

Fig. 4 The dimensionless component $\bar{t}_x$. $\Omega = 40\,\mathrm{rad \cdot s^{-1}}$, $\mathrm{Re_m} = 0.001$

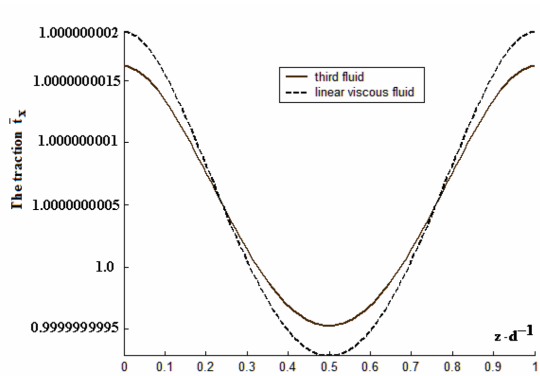

Fig. 5 The dimensionless component $\bar{t}_x$. $\Omega = 80\,\mathrm{rad \cdot s^{-1}}$, $\mathrm{Re_m} = 0.001$

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