International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:7, 2010

Hopf Bifurcation Analysis for a Delayed Predator–prey System with Stage Structure

Kejun Zhuang

Abstract—In this paper, a delayed predator–prey system with stage structure is investigated. Sufficient conditions for the system to have multiple periodic solutions are obtained when the delay is sufficiently large by applying Bendixson's criterion. Further, some numerical examples are given.

Keywords—Predator-prey system, Stage structure, Hopf bifurcation, Periodic solutions.

I. INTRODUCTION

SINCE Aiello and Freedman proposed and studied the well-known single species model with delay and stage structure in [1], people have payed great attention to stage-structured population dynamics and obtained significant results, see [2]–[4] and the references therein. This is not only because they are simpler than the models governed by partial differential equations, but also they can exhibit phenomena similar to those of partial differential models.

In 1997, Wang and Chen constructed a predator-prey system with stage structure for predator as follows,

$$\begin{cases} \dot{x}(t) = x(t)\left(r - ax(t-\tau_1) - by_2(t)\right), \\ \dot{y}_1(t) = kbx(t-\tau_2)y_2(t-\tau_2) - (D+\nu_1)y_1(t), \\ \dot{y}_2(t) = Dy_1(t) - \nu_2 y_2(t), \end{cases} \tag{1}$$

where $x(t)$ denotes the density of prey at time t, $y_1(t)$ denotes the density of immature predator at time t , $y_2(t)$ denotes the density of mature predator at time t, constant $\tau_1 \geq 0$ corresponds to the time delay in the feedback of prey's density and constant $\tau_2 \geq 0$ denotes the time delay due to gestation of mature predator. All coefficients are positive constants and the detailed ecological meanings can be found in [5].

For (1), Wang and Chen studied the permanence and global stability of positive equilibrium, and obtained the existence of orbitally asymptotically stable periodic solutions without time delays. And it shows that stage structure can be a cause of periodic oscillation and can make the population behavior more complex. Further, existence and global stability for the corresponding nonautonomous systems were derived in [6] by using Mawhin's continuation theorem and constructing a suitable Lyapunov functional respectively. However, there are few results about the properties of positive equilibrium as time delay varies. To reduce the complexity of the analysis, we mainly consider the following system,

$$\begin{cases} \dot{x}(t) = x(t)\left(r - ax(t) - by_2(t)\right), \\ \dot{y}_1(t) = kbx(t-\tau)y_2(t-\tau) - (D+\nu_1)y_1(t), \\ \dot{y}_2(t) = Dy_1(t) - \nu_2 y_2(t), \end{cases} \tag{2}$$

Kejun Zhuang is with the School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu 233030, China, email: zhkj123@163.com

if τ equals to 0 and each predator has the same vital rates, (2) can be reduced to the classical Lotka–Volterra model. So in this paper, to generalize the related results in [5], we shall concentrate on the local and global existences of bifurcating periodic solutions for (2) while τ is regarded as the parameter.

The paper is organized as follows. In the next section, sufficient conditions for existence of local Hopf bifurcation are obtained. In Section 3, global existence of multiple periodic solutions is discussed. In Section 4, a numerical example is given to illustrate the theoretical analysis.

II. STABILITY OF POSITIVE EQUILIBRIUM AND EXISTENCE OF LOCAL HOPF BIFURCATION

It is known that time delay does not change the location and number of positive equilibrium. According to the results in [5], we have the following lemma.

Lemma 2.1. Let

$$\frac{r}{a} > \nu_2 \frac{D+\nu_1}{kbD}. \tag{3}$$

Then (2) has the unique positive equilibrium $E^* = (x^*, y_1^*, y_2^*)$, where $x^* = \frac{(D+\nu_1)\nu_2}{kbD}$, $y_1^* = \frac{a\nu_2}{bD}\left[\frac{r}{a} - \frac{(D+\nu_1)\nu_2}{kbD}\right]$, $y_2^* = \frac{a}{b}\left[\frac{r}{a} - \frac{(D+\nu_1)\nu_2}{kbD}\right]$.

The linear part of (2) at E^* is

$$\begin{cases} \dot{x}(t) = -ax^*x(t) - bx^*y_2(t), \\ \dot{y}_1(t) = kby_2^*x(t-\tau) - (D+\nu_1)y_1(t) + kbx^*y_2(t-\tau), \\ \dot{y}_2(t) = Dy_1(t) - \nu_2 y_2(t), \end{cases} \tag{4}$$

and the corresponding characteristic equation is

$$\begin{aligned} &\lambda^3 + (ax^* + D + \nu_1 + \nu_2)\lambda^2 + [ax^*(D+\nu_1+\nu_2) \\ &+(D+\nu_1)\nu_2]\,\lambda + ax^*\nu_2(D+\nu_1) \\ &+ [kbDx^*(by_2^* - ax^*) - kbDx^*\lambda]\,e^{-\lambda\tau} = 0. \end{aligned} \tag{5}$$

Next, we shall investigate the distribution of roots of (5). When $\tau = 0$, (5) can be reduced to

$$\lambda^3+(ax^*+D+\nu_1+\nu_2)\lambda^2+ax^*(D+\nu_1+\nu_2)\lambda+kb^2Dx^*y_2^* = 0. \tag{6}$$

By Routh–Hurwitz criteria, if

$$a(ax^* + D + \nu_1 + \nu_2)(D + \nu_1 + \nu_2) > kb^2Dy_2^* \tag{7}$$

holds, then all roots of (6) have strictly negative real part. For simplicity, we denote (5) as follows,

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)e^{-\lambda\tau} = 0, \tag{8}$$

where $a_2 = ax^* + D + \nu_1 + \nu_2$, $a_1 = ax^*(D + \nu_1 + \nu_2) + (D + \nu_1)\nu_2$, $a_0 = ax^*\nu_2(D + \nu_1)$, $b_1 = -kbDx^*$, $b_0 =$

$kbDx^*(by_2^* - ax^*)$. Obviously, $\lambda = i\omega(\omega > 0)$ is a root of (8) if and only if

$$\omega^3 i + a_2\omega^2 - a_1\omega i - a_0 - (b_1\omega i + b_0)(\cos\omega\tau - i\sin\omega\tau) = 0. \quad (9)$$

Separating the real part and imaginary part, we can obtain

$$\begin{cases} a_2\omega^2 - a_0 = b_0\cos\omega\tau + b_1\omega\sin\omega\tau, \\ a_1\omega - \omega^3 = b_0\sin\omega\tau - b_1\omega\cos\omega\tau, \end{cases} \quad (10)$$

and

$$\omega^6 + p\omega^4 + q\omega^2 + s = 0, \quad (11)$$

where $p = a_2^2 - 2a_1$, $q = a_1^2 - 2a_0a_2 - b_1^2$, $s = a_0^2 - b_0^2$. Set $z = \omega^2$, then (11) takes the form

$$z^3 + pz^2 + qz + s = 0. \quad (12)$$

Set $h(z) = z^3 + pz^2 + qz + s$.

Lemma 2.2. [7] For (12), we have the following results.
(a) If $s < 0$, then (12) has at least one positive root.
(b) If $s \geq 0$ and $\Delta = p^2 - 3q \leq 0$, then (12) has no positive roots.
(c) If $s \geq 0$ and $\Delta = p^2 - 3q > 0$, then (12) has positive roots if and only if $z_1^* = \frac{1}{3}(-p + \sqrt{\Delta}) > 0$ and $h(z_1^*) \leq 0$.

The above lemma can be seen in [7]. Suppose that (12) has positive roots. Without loss of generality, we assume that it has three positive roots z_1, z_2 and z_3. Then (11) has three positive roots $\omega_1 = \sqrt{z_1}$, $\omega_2 = \sqrt{z_2}$, and $\omega_3 = \sqrt{z_3}$. By (10), we have

$$\cos\omega\tau = \frac{b_1\omega_k^4 + (a_2b_0 - a_1b_1)\omega_k^2 - a_0b_0}{b_0^2 + b_1^2\omega_k^2}.$$

Thus, if

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\cos^{-1}\left(\frac{b_1\omega_k^4 + (a_2b_0 - a_1b_1)\omega_k^2 - a_0b_0}{b_0^2 + b_1^2\omega_k^2}\right) + 2j\pi\right\} \quad (13)$$

where $k = 1, 2, 3$, $j = 0, 1, \cdots$, then $\pm i\omega_k$ is a pair of purely imaginary roots of (8) with $\tau = \tau_k^{(j)}$. Suppose

$$\tau_0 = \tau_{k0}^{(0)} = \min_{k\in\{1,2,3\}}\left\{\tau_k^{(0)}\right\}, \quad \omega_0 = \omega_{k0}. \quad (14)$$

Thus, by Lemma 2.2 and Corollary 2.4 in [8], we can easily get the following results.

Lemma 2.3. (a) If $s \geq 0$ and $\Delta = p^2 - 3q \leq 0$, then for any $\tau \geq 0$, (5) and (6) have the same number of roots with positive real parts.
(b) If either $s < 0$ or $s \geq 0$, $\Delta = p^2 - 3q > 0$, $z_1^* > 0$, and $h(z_1^*) \leq 0$ is satisfied, then (5) and (6) have the same number of roots with positive real parts when $\tau \in [0, \tau_0)$.

Let

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$$

be the root of (5) satisfying

$$\alpha\left(\tau_k^{(j)}\right) = 0, \omega\left(\tau_k^{(j)}\right) = \omega_k.$$

Thus, the following transversality condition holds.

Lemma 2.4. If $z_k = \omega_k^2$ and $h'(z_k) \neq 0$, then

$$\frac{d\mathrm{Re}\lambda(\tau_k^{(j)})}{d\tau} \neq 0.$$

Further, $\frac{d\mathrm{Re}\lambda(\tau_k^{(j)})}{d\tau}$ and $h'(z_k) \neq 0$ have same sign.

Proof: By direct computation , we obtain

$$\begin{aligned} &\left\{3\lambda^2 + 2a_2\lambda + a_1 + [b_1 - \tau(b_1\lambda + b_0)]e^{-\lambda\tau}\right\}\frac{d\lambda}{d\tau} \\ = \;& \lambda(b_1\lambda + b_0)e^{-\lambda\tau}, \end{aligned}$$

and

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2a_2\lambda + a_1)e^{\lambda\tau}}{\lambda(b_1\lambda + b_0)} + \frac{b_1}{\lambda(b_1\lambda + b_0)} - \frac{\tau}{\lambda}. \quad (15)$$

By (10) ,

$$[\lambda(b_1\lambda + b_0)]_{\tau=\tau_k^{(j)}} = -b_1\omega_k^2 + ib_0\omega_k, \quad (16)$$

$$\begin{aligned} &[(3\lambda^2 + 2a_2\lambda + a_1)e^{\lambda\tau}]_{\tau=\tau_k^{(j)}} = [(a_1 - 3\omega_k^2)\cos\omega_k\tau_k^{(j)} \\ &-2a_2\omega_k\sin\omega_k\tau_k^{(j)}] + i[2a_2\omega_k\cos\omega_k\tau_k^{(j)} \\ &+(a_1 - 3\omega_k^2)\sin\omega_k\tau_k^{(j)}]. \end{aligned} \quad (17)$$

From (15) to (17) , we have

$$\left[\frac{d\mathrm{Re}\lambda(\tau)}{d\tau}\right]^{-1}_{\tau=\tau_k^{(j)}} = \frac{z_k}{\Lambda}h'(z_k),$$

where $\Lambda = \left(b_1\omega_k^2\right)^2 + (b_0\omega_k)^2$. Thus,

$$\mathrm{sign}\left[\frac{d\mathrm{Re}\lambda(\tau)}{d\tau}\right]_{\tau=\tau_k^{(j)}} = \mathrm{sign}\left[\frac{d\mathrm{Re}\lambda(\tau)}{d\tau}\right]^{-1}_{\tau=\tau_k^{(j)}} = \frac{z_k}{\Lambda}h'(z_k) \neq 0.$$

Because $\Lambda, z_k > 0$, the sign of $\frac{d\mathrm{Re}\lambda(\tau_k^{(j)})}{d\tau}$ is consistent with that of $h'(z_k) \neq 0$. This proves the lemma. ∎

By above analysis, we can obtain the following theorem about the stability of positive equilibrium and the existence of periodic solutions for (2) .

Theorem 2.5. If (3) and (7) are satisfied, then the following results hold.
(a) If $s \geq 0$ and $\Delta = p^2 - 3q \leq 0$, then for any $\tau \geq 0$, all roots of (5) have negative real parts. Further, positive equilibrium of (2) is absolutely stable for $\tau \geq 0$.
(b) If either $s < 0$ or $z_1^* > 0$, $h(z_1^*) \leq 0$, $r \geq 0$ and $\Delta = p^2 - 3q > 0$ holds , then $h(z)$ has at least one positive root z_k , and when $\tau \in [0, \tau_k^{(0)})$, all roots of (5) have negative real parts. So the positive equilibrium of (2) is asymptotically stable for $\tau \in [0, \tau_k^{(0)})$.
(c) If conditions in (b) hold and $h'(z_k) \neq 0$, then Hopf bifurcation for (2) occurs at positive equilibrium when $\tau = \tau_k^{(j)}$ $(j = 0, 1, 2, \cdots)$, which means that small amplified periodic solutions will bifurcate from positive equilibrium.

III. Existence of Global Hopf Bifurcation

Next, we shall establish the existence of global periodic solutions of (2) by ODE's Bendixson criterion. Define $X = C([-\tau, 0], R^3)$, $\sum = Cl\{(u(t), \tau, p) \in X \times R \times R^+; u(t)$ is $p-$periodic solution of (2)$\}$, $\ell(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0})$ is the connected component in $\sum$ of an isolated center $(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0})$, and $\ell(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0})$ is nonempty, where $\tau_0^{(j)} = \frac{1}{\omega_0}\left\{\cos^{-1}\left(\frac{b_1\omega_0^4 + (a_2b_0 - a_1b_1)\omega_0^2 - a_0b_0}{b_0^2 + b_1^2\omega_0^2}\right) + 2j\pi\right\}, j = 0, 1, 2, \cdots$.

Lemma 3.1. [9] Let $D \subset R^n$ be a simply connected region. Assume that the family of linear systems

$$z'(t) = \frac{\partial f^{[2]}}{\partial x}(x(t, x_0))z(t), \quad x_0 \in D$$

is equi-uniformly asymptotically stable. Then
(a) D contains no simple invariant curves including periodic obits, homoclinic orbits, heteroclinic cycles.
(b) Each semi-orbit in D converges to a single equilibrium.
In particular, If D is positively invariant and contains an unique equilibrium $\bar{x}$, then $\bar{x}$ is globally asymptotically stable in D.

When $\tau = 0$, (2) is equivalent to

$$\begin{cases} \dot{x}(t) = x(t)\left(r - ax(t) - by_2(t)\right), \\ \dot{y}_1(t) = kbx(t)y_2(t) - (D+\nu_1)y_1(t), \\ \dot{y}_2(t) = Dy_1(t) - \nu_2 y_2(t), \end{cases} \tag{18}$$

We make the following assumptions.
$(H1)$ There exist $\alpha, \beta > 0$, such that

$$\sup_{x,y_1,y_2 \in R}\Big\{r - D - \nu_1 - b|y_2(t)| - 2a|x(t)| + \frac{\alpha}{\beta}kb|x(t)|, \\ +\alpha b|x(t)|\frac{\beta}{\alpha}D + r - \nu_2 - b|y_2(t)| - 2a|x(t)|, \\ \frac{1}{\beta}kb|y_2(t)| - (D+\nu_1+\nu_2)\Big\} < 0. \tag{19}$$

Lemma 3.2. If $(H1)$ is satisfied, then (18) has no non-constant periodic solution.

Proof: Denote $u = (x, y_1, y_2)^T$ and $f(x, y_1, y_2) = (x(r - ax - by_2), kbxy_2 - (D+\nu_1)y_1, Dy_1 - \nu_2 y_2)^T$, then we have

$$\frac{\partial f}{\partial x} = \begin{pmatrix} r - by_2 - 2ax & 0 & -bx \\ kby_2 & -D-\nu_1 & kbx \\ 0 & D & -\nu_2 \end{pmatrix},$$

and by [10],

$$\frac{\partial f^{[2]}}{\partial x} = \begin{pmatrix} a_{11} & kbx & bx \\ D & a_{22} & 0 \\ 0 & kby_2 & a_{33} \end{pmatrix}.$$

where $a_{11} = r - D - \nu_1 - by_2 - 2ax$, $a_{22} = r - \nu_2 - by_2 - 2ax$ and $a_{33} = -D - \nu_1 - \nu_2$. For the following second compound system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \frac{\partial f^{[2]}}{\partial x}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix},$$

and

$$\begin{cases} \dot{z}_1 = [r - D - \nu_1 - by_2(t) - 2ax(t)]z_1 + kbx(t)z_2 + bx(t)z_3, \\ \dot{z}_2 = Dz_1 + [r - \nu_2 - by_2(t) - 2ax(t)]z_2, \\ \dot{z}_3 = kby_2(t)z_2 - (D+\nu_1+\nu_2)z_3, \end{cases} \tag{20}$$

where $u(t) = (x(t), y_1(t), y_2(t))^T$ is a solution of (18) when $u(0) = u_0 \in R^3$. Set

$$W(z) = \max\{\alpha|z_1|, \beta|z_2|, |z_3|\}, \tag{21}$$

where α and β are both positive constants. Then we can get

$$\begin{aligned} &\frac{d^+}{dt}\alpha|z_1(t)| \le [r - D - \nu_1 - b|y_2(t)| - 2a|x(t)|]\,\alpha|z_1| \\ &\qquad + \frac{\alpha}{\beta}kb|x(t)|\beta|z_2| + \alpha b|x(t)||z_3|, \\ &\frac{d^+}{dt}\beta|z_2(t)| \le \frac{\beta}{\alpha}D\alpha|z_1| + [r - \nu_2 - b|y_2(t)| - 2a|x(t)|]\,\beta|z_2|, \\ &\frac{d^+}{dt}|z_3(t)| \le \frac{1}{\beta}kb|y_2(t)|\beta|z_2| - (D+\nu_1+\nu_2)|z_3|, \end{aligned}$$

where $\frac{d^+}{dt}$ denotes the right-hand derivative. Therefore,

$$\frac{d^+}{dt}W(z(t)) \le \mu(t)W(z(t)), \tag{22}$$

where $\mu(t) = \max\{r - D - \nu_1 - b|y_2(t)| - 2a|x(t)| + \frac{\alpha}{\beta}kb|x(t)|, +\alpha b|x(t)|\frac{\beta}{\alpha}D + r - \nu_2 - b|y_2(t)| - 2a|x(t)|, \frac{1}{\beta}kb|y_2(t)| - (D+\nu_1+\nu_2)\}$. If $(H1)$ holds, then there exists $\delta > 0$, such that $\mu(t) < -\delta < 0$, thus

$$W(z(t)) \le W(z(s))e^{-\delta(t-s)}, \quad t \ge s > 0.$$

So (20) is equi-uniform asymptotic stability and hence the conclusion of Lemma 3.2 follows. ■

Lemma 3.3. All the periodic solutions of (2) are uniformly bounded.

Proof: Denote the solutions of (2) as follows:

$$\begin{aligned} x(t) &= \phi(0)\exp\left\{\int_0^t [r - ax(s) - by_2(s)]ds\right\}, \\ y_1(t) &= \psi_1(0)\exp\left\{\int_0^t [kbx(s-\tau)y_2(s-\tau)/y_1(s) - D - \nu_1]\,ds\right\}, \\ y_2(t) &= \psi_2(0)\exp\left\{\int_0^t [Dy_1(s)/y_2(s) - \nu_2]\,ds\right\}, \end{aligned} \tag{23}$$

as a result, it is impossible for the solutions of (2) to pass through coordinate surface. If $u(t) = (x(t), y_1(t), y_2(t))^T$ is any nontrivial periodic solution of (2), we first define

$$\begin{aligned} &\max\{x(t)\} = x(t_1), && \min\{x(t)\} = x(t_2), \\ &\max\{y_1(t)\} = y_1(t_3), && \min\{y_1(t)\} = y_1(t_4), \\ &\max\{y_2(t)\} = y_2(t_5), && \min\{y_2(t)\} = y_2(t_6). \end{aligned} \tag{24}$$

There are several cases to be considered:
(a) $x(t) > 0, y_1(t) > 0, y_2(t) > 0$, from (24) and (2),

$$r - ax(t_1) - by_2(t_1) = 0, \tag{25}$$

$$r - ax(t_5) - by_2(t_5) = 0, \tag{26}$$

$$kbx(t_3 - \tau)y_2(t_3 - \tau) - (D+\nu_1)y_1(t_3) = 0. \tag{27}$$

By (25), $ax(t_1) = r - by_2(t_1) < r$,

$$0 < x(t_1) < \frac{r}{a}. \tag{28}$$

Similarly, we can also get

$$0 < y_2(t_5) < \frac{r}{b}. \tag{29}$$

By (27)–(29), we have $y_1(t_3) = \frac{kbx(t_3-\tau)y_2(t_3-\tau)}{D+\nu_1} < \frac{kb}{D+\nu_1}\frac{r}{a}\frac{r}{b}$, and

$$0 < y_1(t_3) < \frac{kr^2}{a(D+\nu_1)}. \tag{30}$$

(b) When $x(t) > 0, y_1(t) > 0, y_2(t) < 0$, the following inequality holds,

$$kbx(t_3 - \tau)y_2(t_3 - \tau) - (D+\nu_1)y_1(t_3) < 0,$$

which is contradictory to $\dot{y}_1(t_3) = 0$. So, in this case, (2) has no nontrivial periodic solutions.

By using the same method, we can prove that when $x(t) > 0, y_1(t) < 0, y_2(t) < 0$, or $x(t) > 0, y_1(t) < 0, y_2(t) < 0$, or $x(t) < 0, y_1(t) > 0, y_2(t) > 0$, or $x(t) < 0, y_1(t) > 0, y_2(t) < 0$, or $x(t) < 0, y_1(t) < 0, y_2(t) < 0$, or

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:7, 2010

$x(t) < 0, y_1(t) < 0, y_2(t) > 0$, (2) has no nontrivial periodic solutions, either. However, when $x(t) \equiv 0, y_1(t) \neq 0, y_2(t) \neq 0$, by the second equation of (2) , $\dot{y}_1(t) \neq 0$, in this case there is no nontrivial periodic solutions. When $x(t) \neq 0, y_1(t) \equiv 0, y_2(t) \neq 0$ or $x(t) \neq 0, y_1(t) \neq 0, y_2(t) \equiv 0$, the same results can be established.

From above, if $(x(t), y_1(t), y_2(t))$ is the nontrivial periodic solutions of (2), then $0 < x(t) < \frac{r}{a}$, $0 < y_2(t) < \frac{r}{b}$, and $0 < y_1(t) < \frac{kr^2}{a(D+\nu_1)}$. Hence the periodic solutions of (2) are uniformly bounded. ∎

Lemma 3.4. The periods of periodic solutions of (2) are uniformly bounded.

Proof: Note that if $u(t) = (x(t), y_1(t), y_2(t))^T$ is a $\tau-$periodic solution of (2), then $u(t)$ is a periodic solution of (18) and this contradicts Lemma 3.2. So (2) has no nontrivial periodic solutions. By the definition of $\tau_0^{(j)}$, when $j \geq 1$, we have $\frac{2\pi}{\omega_0} \leq \tau_0^{(j)}$.For $\tau > \tau_0^{(j)}$, there exists an integer m, such that $\frac{\tau}{m} < \frac{2\pi}{\omega_0} < \tau$. As system (1.2) has no nontrivial $\tau-$periodic solution, for any integer n, (2) has no $\frac{\tau}{n}-$periodic solution. This implies that the period p of a periodic solution on the connected component $\ell(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0})$ satisfies $\frac{\tau}{m} < p < \tau$. So we can know that the periods of the periodic solutions of (2) on $\ell(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0})$ are uniformly bounded. ∎

Theorem 3.5. Assume that $(H1)$ and hypothesis (c) in Theorem 2.5 are satisfied. Then (2) still has periodic solutions when $\tau > \tau_0^{(j)} (j \geq 1)$.

Proof: The characteristic equation of (2) at positive equilibrium E^* is

$$\Delta(E^*, \tau, p)(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)e^{-\lambda\tau}, \quad (31)$$

and the characteristic equation of (2) at zero is

$$\lambda[\lambda^2 + (D + \nu_1 + \nu_2)\lambda + \nu_2(D + \nu_1)] = 0,$$

this equation has no pure imagine root. By the definition of isolated center in [11], we can easily verify that $(E^*, \tau_0^{(j)}, p)$ is the unique isolated center. There exist $\varepsilon > 0$, $\delta > 0$ and a smooth curve $\lambda(\tau) : (\tau_0^{(j)} - \delta, \tau_0^{(j)} + \delta) \to C$, such that for any $\tau \in [\tau_0^{(j)} - \delta, \tau_0^{(j)} + \delta]$, $\Delta(\lambda(\tau)) = 0$, $|\lambda(\tau) - \omega_0 i| < \varepsilon$, and $\lambda(\tau_0^{(j)}) = i\omega_0$, $\left.\frac{dRe\lambda(\tau)}{d\tau}\right|_{\tau=\tau_0^{(j)}} \neq 0$.

Let $\Omega_{\varepsilon, \frac{2\pi}{\omega_0}} = \{(\eta, p) : 0 < \eta < \varepsilon, |p - \frac{2\pi}{\omega_0}| < \varepsilon\}$. If $|\tau - \tau_0^{(j)}| \leq \delta$ and $(\eta, p) \in \partial\Omega_{\varepsilon, 2\pi/\omega_0}$ are satisfied, then $\Delta(E^*, \tau, p)(\eta + \frac{2\pi}{p} i) = 0$ if and only if $\eta = 0$, $\tau = \tau_0^{(j)}$, $p = \frac{2\pi}{\omega_0}$.

If we put

$$H^{\pm}(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0})(\eta, p) = \Delta(E^*, \tau_0^{(j)} \pm \delta, p)(\eta + i\frac{2\pi}{p}),$$

then we have

$$\begin{aligned} \gamma(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0}) &= \deg_B(H^-(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0}), \Omega_{\varepsilon, \frac{2\pi}{\omega_0}}) \\ &\quad -\deg_B(H^+(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0}), \Omega_{\varepsilon, \frac{2\pi}{\omega_0}}) \\ &= -1. \end{aligned}$$

According to Theorem 3.3 in [11], connected component $\ell(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0})$ are unbounded. From Lemma 3.3 and Lemma 3.4, the projection of $\ell(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0})$ onto $\tau-$space are unbounded. As $\tau = 0$, (2) has no nontrivial periodic solution, this implies that projection of $\ell(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0})$ onto $\tau-$space must be positive and has a lower bound. ∎

IV. Numerical Simulation

Finally, we shall give a numerical example:

$$\begin{cases} \dot{x}(t) = x(t)\left(2 - x(t) - 0.3y_2(t)\right), \\ \dot{y}_1(t) = 0.24x(t-\tau)y_2(t-\tau) - 0.95y_1(t), \\ \dot{y}_2(t) = 0.8y_1(t) - 0.1y_2(t), \end{cases} \quad (32)$$

then (3) and (7) hold and (32) has the unique positive equilibrium $E^* = (0.494792, 0.62717, 5.01736)$. The corresponding characteristic equation has a pair of purely imaginary roots $\lambda = \pm 0.486661i$, and $\tau_0^{(0)} = 7.93665$, $\tau_0^{(1)} = 20.8476$, $\tau_0^{(2)} = 33.7584\cdots$. The following figures explicit the solutions of (32) with the initial value $(1, 1, 1)$.

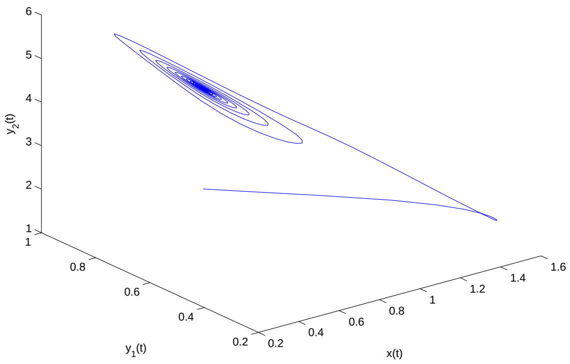

Figure 1. When $\tau = 7$, the positive equilibrium of (32) is asymptotically stable.

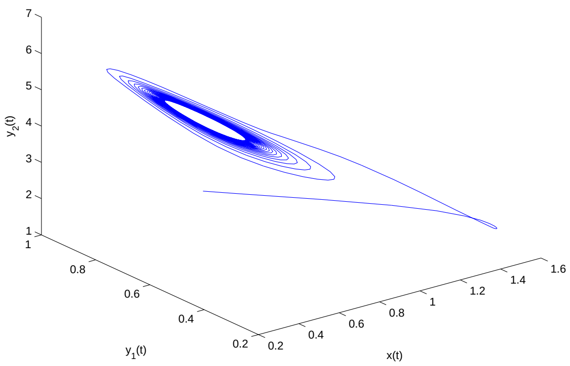

Figure 2. When $\tau = 9$, the positive equilibrium of (32) is unstable, and small amplified periodic solutions exist.

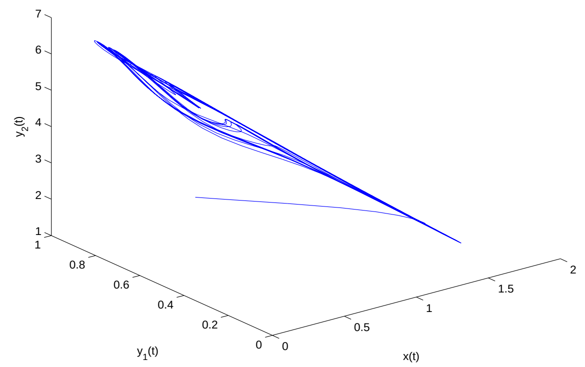

Figure 3. When $\tau = 34$, periodic solutions still exist.

ACKNOWLEDGMENT

This work is supported by Anhui Province Natural Science Foundation of China (No.090416222) and Natural Science Research Project of colleges and universities in Anhui Province (No.KJ2009B076Z).

REFERENCES

[1] W. Aiello and H. Freedman, A time-delay model of single-species growth with stage structure, *Math. Bios.* **101** (1990) 139–153.
[2] Z. Q. Zhang and X. W. Zheng, A periodic stage-structure model, *Appl. Math. Lett.* **16** (2003) 1053–1061.
[3] S. Gourley and Y.Kuang, A stage structured predator-prey model and its dependence on maturation delay and death rate, *J. Math. Biol.* **49** (2004) 188–200.
[4] J. Al-Omari, Gourley S, Dynamics of a stage-structured population model incorporating a state-dependent maturation delay, *Nonl. Anal.: RWA* **6** (2005) 13–33.
[5] W. D. Wang and L. S. Chen, A predator-prey system with stage-structure for predator, *Comp. Math. Appl.* **33** (1997) 83–91.
[6] R. Xu, M. Chaplain and F. Davidson, Periodic solutions of a delayed predator-prey model with stage structure for predator, *J. Appl. Math.* **3** (2004) 255–270.
[7] Y. L. Song, M. A. Han and J. J. Wei, Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays, *Phys. D* **200** (2005) 184–204.
[8] S. G. Ruan and J. J. Wei, On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, *Dyna. Cont. Disc. Impu. Syst. Series A: Math. Anal.* **10** (2003) 863–874.
[9] M. Y. Li and J. Muldowney, On Bendixson's criterion, *J. Diff. Eqns.* **106** (1993) 27–39.
[10] J. Muldowney, Compound matrices and ordinary differential equations, *Rocky Mount. J. Math.* **20** (1990) 857–872.
[11] J. H. Wu, Symmetric functional differential equations and neural networks with memory, *Trans. Amer. Math. Soc.* **350** (1998) 4799–4838.