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Zeros of Bargmann analytic representation in the complex plane

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Abstract—The paper contains an investigation of zeros Of Bargmann analytic representation. A brief introduction to Harmonic oscillator formalism is given. The Bargmann analytic representation has been studied. The zeros of Bargmann analytic function are considered. The Q or Husimi functions are introduced. The The Bargmann functions and the Husimi functions have the same zeros. The Bargmann functions $f(z)$ have exactly q zeros. The evolution time of the zeros μ_n are discussed. Various examples have been given.

Keywords—Bargmann functions, Husimi functions, zeros.

I. Introduction

THIS Paper is devoted to study the zeros of Bargmann analytic representation in the complex plane. The Bargmann function is very important kind of analytic functions [1], [2], [3] in the complex plane [4], [5], [6]. The zeros of Bargmann functions and the zeros of the Q or Husimi function which are identical, have been used to consider of various models [7], [8], [9], [10], [11], [12], [13]. The analytic Bargmann functions $f(z)$ have exactly q zeros which subjected to the constraint.(30). The growth of an entire function $f(z)$ is described by the order ρ and type σ [14], [15], [16], [17]. The entire function $f(z)$ is polynomial of order q and has q zeros. The q zeros of the analytic functions $f(z)$ depends on the distribution of the coefficients $f_0, f_1, ..., f_n$. If the coefficients $f_0, f_1, ..., f_n$ are real then the zeros μ_n are real or appear as complex conjugate pairs and draw symmetric graph with respect to the z_r axis.

II. Harmonic oscillator formalism

Let $\mathcal{H}_q$ be the Hilbert space with number eigenstates $|n\rangle$. We consider a harmonic oscillator corresponding the Hamiltonian:

$$H = \frac{1}{2}(x^2 + p^2); \tag{1}$$

where x; p; the position and momentum operators with $[x, p] = i\mathbf{1}$.
Let $a, a^\dagger$ be the creation and annihilation operators:

$$a = \frac{x + ip}{\sqrt{2}}; \quad a^\dagger = \frac{x - ip}{\sqrt{2}}; \tag{2}$$

where

$$aa^\dagger|n\rangle = n|n\rangle. \tag{3}$$

These two operators obey the canonical commutation relation

$$[a, a^\dagger] = \mathbf{1}; \tag{4}$$

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and act on the number state as follows:

$$\begin{aligned} a^\dagger|n\rangle &= (n+1)^{1/2}|n+1\rangle; \\ a|n\rangle &= n^{1/2}|n-1\rangle; \end{aligned} \tag{5}$$

The displacement operators are defined as

$$D(z) = \exp(za^\dagger - z^*a); \quad z = (x + ip)/\sqrt{2}. \tag{6}$$

We consider the coherent states

$$|z\rangle = \exp\left(-\frac{1}{2}|z|^2\right)\sum_{n=0}^{\infty}(n!)^{-1/2}z^n|n\rangle. \tag{7}$$

The coherent states are defined as the eigenstate of the annihilation operator a

$$a|z\rangle = z|z\rangle; \tag{8}$$

and the position representation of the coherent state is a Gaussian function

$$f_z(x) = \pi^{-1/4}exp(-\frac{x^2}{2} + \sqrt{2}zx - zz_R); \quad z = z_R + iz_I. \tag{9}$$

The inner product of two coherent states $|z_1\rangle$ and $|z_2\rangle$ is

$$\langle z_1|z_2\rangle = exp(-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2 + z_1z_2^*). \tag{10}$$

III. Husimi functions

If we let $|\psi\rangle$ be an arbitrary state, the Husimi function is defined by

$$Q(\alpha) = \frac{|\langle\alpha|\psi\rangle|^2}{\pi} \tag{11}$$

with

$$\int_{\mathbb{C}} d^2\alpha Q(\alpha) = 1 \tag{12}$$

Example 1:
The Husimi function of the coherent state $|\beta\rangle$

$$Q(\alpha) = \frac{1}{\pi}|\langle\alpha|\beta\rangle| = \frac{1}{\pi}exp(-|\alpha - \beta|^2), \tag{13}$$

Example 2:
The Husimi function of the number state $|n\rangle$

$$Q(\alpha) = \frac{1}{\pi}|\langle\alpha|n\rangle| = \frac{1}{\pi}exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}. \tag{14}$$

IV. Bargmann analytic representation

We cosider an arbitrary $|f\rangle$ state

$$|f\rangle = \sum_{n=0}^{\infty} f_n |n\rangle; \quad \sum_{n=0}^{\infty} |f_n|^2 = 1. \tag{15}$$

In The Bargmann representation [2], [4], [5], [6] , the state $|f\rangle$ is represented by

$$f(z) = exp(\frac{|z|^2}{2})\langle z^*|f\rangle = \sum_{n=0}^{\infty} \frac{f_n z^n}{\sqrt{n!}}, \tag{16}$$

which is an entire function (i.e. analytic function in the complex plane $\mathbb{C}$) defined on a torus, satisfying the quasi-periodic condition [7]

$$\begin{aligned} f\left[z + 1/\sqrt{2}\right] &= \exp(q\pi(\frac{1}{2} + z\sqrt{2}))f(z), \\ f\left[z + i/\sqrt{2}\right] &= \exp(q\pi(\frac{1}{2} - iz\sqrt{2}))f(z). \end{aligned} \tag{17}$$

The inner product of the two states[2] is given by

$$\begin{aligned} \langle f|g\rangle &= \frac{1}{\pi}\int_{\mathbb{C}} [f(z)]^* g(z) exp(-|z|^2)\frac{d^2 z}{\pi} \\ &= \sum_n f_n^* g_n \quad , d^2 z = dz_R dz_I. \end{aligned} \tag{18}$$

We consider an arbitrary operator Ω

$$\Omega = \sum_{m,n=0}^{\infty} \Omega_{mn} |m\rangle\langle n|. \tag{19}$$

In the Bargmann analytic representation this operator can be represented by the two variable analytic functions [18]

$$\begin{aligned} \Omega(z,\mu^*) &= exp(\frac{1}{2}|z|^2 + \frac{1}{2}|\mu|^2)\langle z^*|\Omega|\mu^*\rangle \\ &= \sum_{m,n=0}^{\infty} \frac{\Omega_{mn} z^m \mu^{*n}}{\sqrt{m!n!}}. \end{aligned} \tag{20}$$

The operator Ω acts on a state $|f\rangle$ as following

$$\Omega|f\rangle \longrightarrow \int_{\mathbb{C}} d^2\zeta exp(-|\mu|^2)\Omega(z,\mu^*)f(z). \tag{21}$$

Therefore we can represent the creation and annihilation operators by the two variable analytic functions in the Bargmann analytic [18] representation (see 1)as following

$$a \longrightarrow \mu^* \exp(z\mu^*), \quad a^\dagger \longrightarrow z\exp(z\mu^*) \tag{22}$$

The Bargmann analytic representation of the creation and annihilation operator is

$$a \longrightarrow \partial_z, \quad a^\dagger \longrightarrow z. \tag{23}$$

A. The growth of Bargmann analytic functions

The growth of an entire function $f(z)$ is described by the order ρ and type σ [14], [15], [16], [17], [18].

$$\rho = \lim_{R\to\infty} sup\frac{lnlnM(R)}{lnR}, \; \sigma = \lim_{R\to\infty} sup\frac{lnM(R)}{R^\rho}, \tag{24}$$

where $M(R)$ is the maximum value of $f(z)$ on $|z| = R$. The space $H(\rho,\sigma)$ is a subspace of $H(\rho^{'},\sigma^{'})$ if $\rho < \rho^{'}$ or if $\rho = \rho^{'}$; $\sigma < \sigma^{'}$.
We can now derive the Bargmann analytic representation of some quantum states as examples.
• The number state $|n\rangle$ is represented as

$$f(z) = \frac{z^n}{\sqrt{n!}}. \tag{25}$$

It is of order 0.
• The coherent state $|\alpha\rangle$ is represented as

$$f(z) = exp(\alpha z - \frac{1}{2}|\alpha|^2). \tag{26}$$

It is of order $\rho = 1$ and type $|\alpha|$.

V. Zeros of Bargmann function

We denote as μ_n the zeros of $f(z)$, i.e. $f(\mu_n) = 0$. Let ℓ be the boundary of the fundamental domain of analyticity, $S = [0, 1/\sqrt{2}] \times [0, 1/\sqrt{2}]$. We consider the integrals

$$I = \oint_\ell \frac{dz}{2\pi i}\frac{f(z)^{'}}{f(z)}, \quad J = \oint_\ell \frac{dz}{2\pi i}\frac{f(z)^{'}}{f(z)} z. \tag{27}$$

I is equal to the number of zeros of this function (with the multiplicities taken into account), inside the contour ℓ. J is equal to the sum of these zeros. Using the quasi-periodicity of Eq. (17) we prove that the integral I, for a contour along the boundary ℓ, is equal to q. Therefore the analytic functions $f(z)$ have exactly q zeros [7], [8].

$$\oint_\ell \frac{dz}{2\pi i}\frac{f(z)^{'}}{f(z)} = q. \tag{28}$$

Using the quasi-periodicity of Eq. (17) we also prove that [7], [8]

$$\oint_\ell \frac{dz}{2\pi i}\frac{\partial_z f(z)}{f(z)} z \; = 2^{-3/2} q(1+i); \tag{29}$$

giving the sum of the zeros μ_n of $f(z)$. Therefore the analytic functions $f(z)$ [7], [8] have exactly q zeros subjected to the constraint

$$\sum_{n=1}^{q} \mu_n = 2^{-3/2} q(1+i). \tag{30}$$

The Husimi function and Bargmann function $f(z)$ are related to each other and it easy to see that there zeros are identical (i.e μ is a zero of $f(z)$ providing ζ is a zero of the Husimi function). The Weierstrass-Hadamard factorization allows the reconstruction of entire functions from their zeros [2], [18]. We suppose that q zeros μ_n of $f(z)$ are given, and that they

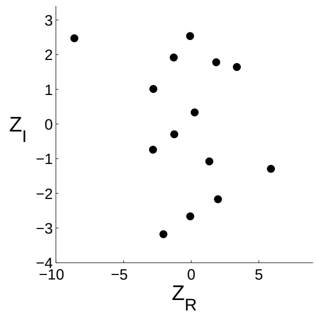

Fig. 1. The distributions of zeros of function $f(z)$ of Eq.(33). The $|f(t)\rangle$ at $t = 0$ is described through the coefficients f_n in table.I

satisfy the constraint of Eq. (30). The Weierstrass-Hadamard reconstructs the Bargmann functions $f(z)$ as following [2]

$$f(z) = z^m \prod_{n=1}^{q} \exp(Q_p(z)E(\mu_n, d), \tag{31}$$

where

$$E(\mu_n, d) = (1 - \frac{z}{\mu}) \exp\left(\frac{z}{\mu} + \frac{z^2}{\mu^2} + ... + \frac{z^d}{\mu^d}\right); \tag{32}$$

m is the multiplicity of the zero, $Q_p(z)$ is polynomial of degree p and d is a positive number.

Example 3:
As an example we consider the function

$$f(z) = \sum_{n=0}^{14} \frac{f_n z^n}{\sqrt{n!}}, \tag{33}$$

The coefficients f_n are given in Table. I. In Fig.1 we show

i	$\mathrm{f}_i(0)$	i	$\mathrm{f}_i(0)$
0	0.1-0.2i	8	0.1+0.01i
1	0.3+0.3i	9	0.1-0.2i
2	0.3+0.2i	10	0.1-0.1
3	0.01-0.3i	11	-0.1+0.2
4	0.1-0.01i	12	0.2+0.3i
5	0.3-0.2i	13	-0.01-0.1i
6	0.9-0.03i	14	-0.01-0.1i
7	0.3+0.01i		

TABLE I
THE COEFFICIENTS f_n OF FUNCTION IN EQ.(33)

the distribution of zeros of function $f(z)$ of Eq.(33) which is polynomial of order 14 and has 14 zeros. The q zeros of the analytic functions $f(z)$ depends on the distribution of the coefficients $f_0, f_1, ..., f_n$. This coefficients subjected to the constraint

$$\sum_{n=0}^{q} f_n^2 = 1. \tag{34}$$

which comes from the normalization.

Real coefficients

let $\mu_1, \ \mu_2 \in \Re$. If $f(x)$ is a polynomial with real coefficients, then:
$\mu_1 + i\mu_2$ is a zero of $f(x) \Longleftrightarrow \mu_1 - i\mu_2$ is a zero of $f(x)$.
In our case if the coefficients $f_0, f_1, ..., f_n$ are real or imaginary numbers, then the zeros μ_n are real or appear as complex conjugate pairs. Below we give examples with real and imaginary coefficients. zeros μ_n are real or appear as complex conjugate pairs and draw symmetric graph with respect to the z_r axis.

Example 4:
We consider the function in Eq.33. The coefficients $f_0, f_1, ..., f_n$ is described through the real part of the coefficients in table.(I), where the zeros at $t = 0$ are given in the table. II In this case the eighth zeros μ_n are real or appear

i	$\mu_i(0)$	i	$\mu_i(0)$
0	-22.3	7	-0.8-i2.7
1	13.5	8	0.2+i2.5
2	2.9+i2.9	9	0.2-i2.5
3	2.9-i2.9	10	1.6+i1.7
4	-2.86+i0.87	11	1.6-i1.7
5	-2.9-i0.9	12	-0.9
6	-0.8+i2.7	13	0.17

TABLE II
THE DISTRIBUTION OF THE ZEROS $\mu_n(t)$ OF FUNCTION IN EQ.(33), WHERE THE COEFFICIENTS ARE THE REAL PART OF THE COEFFICIENTS IN TABLE.(I)

as complex conjugate pairs and draw symmetric graph with respect to the z_r axis. Here each zero has its own complex conjugate. It is easy to see that

$$\begin{aligned} \mu_2\,(0) &= (\mu_3\,(0))^*, \ \ \mu_4(0) = (\mu_5(0))^* \\ \mu_6\,(0) &= (\mu_7\,(0))^*, \ \ \mu_8(0) = (\mu_9(0))^* \\ \mu_{10}(0) &= (\mu_{11}(0))^*. \end{aligned} \tag{35}$$

In Fig. 2 we present the distribution of this zeros. Therefore whether the coefficients real or imaginary numbers the zeros μ_n appear as complex conjugate pairs or lie on on z_r axis. In addition we found numerically that the zeros of function $f(x)$ with coefficients $f_0, f_1, ..., f_n$ are equal the zeros of function $g(x)$ with coefficients $if_0, if_1, ..., if_n$.

Example 5:
We consider the function

$$f(z) = \sum_{n=0}^{8} \frac{f_n z^n}{\sqrt{n!}}. \tag{36}$$

The coefficients $f_0, f_1, ..., f_n$ is described through the imaginary part of the coefficients The corresponding zeros In this it is seen that the zeros of the function with the coefficients $if_0, if_1, ..., if_n$ are the same zeros in table.IV.
In Fig. 3 we show the distribution of this zeros.

VI. MOTION OF THE ZEROS

Using the Hamiltonian H the state $|f(0)\rangle$ at $t = 0$ evolves at time t into

$$|f(t)\rangle = \exp(itH)|f(0)\rangle, \tag{37}$$

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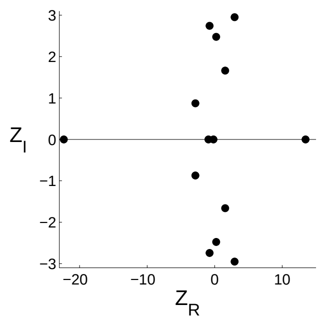

Fig. 2. The distributions of zeros in table.(II). The coefficients are real part of the coefficients in table.I. Each zero has its own complex conjugate or lies on z_r axis $(z_i = 0)$.

i	$\mathrm{f}_i(0)$	i	$\mathrm{f}_i(0)$
0	0.04-i0.02	5	0.28+i0.21
1	0.09+i0.02	6	0.32+i0.25
2	0.13-i0.07	7	0.37+i0.30
3	0.23+i0.16	8	0.42+i0.35
4	0.23+i0.16		

TABLE III
THE COEFFICIENTS f_n OF FUNCTION IN EQ.(36)

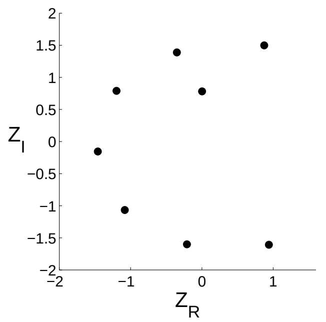

Fig. 3. The distributions of zeros in table.(IV).

the corresponding zeros also evolves in time.

Example 6:

Let

$$\begin{aligned}\mu_0(0) &= -0.1 - 1.2,\ \mu_1(0) = -1.1 - 0.4i, \\ \mu_2(0) &= -0.7 + 0.8i,\ \mu_3(0) = 0.3 + 0.96i, \end{aligned} \tag{38}$$

be the zeros at $t = 0$ and let

$$H = \begin{bmatrix} 1 & i & 0 & 0 & 0 \\ -i & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

be the Hamiltonian with eigenvalues $0.38, 1, 1, 2.3$ In Fig.4 we plot the motion of this zeros.

Example 7:

i	$\mu_i(0)$	i	$\mu_i(0)$
0	0.93-i1.60	4	-1.46-i0.15
1	-0.21-i1.60	5	-1.19+i0.79
2	-1.08-i1.06	6	-0.35+i1.38
3	0.87+i1.49	7	-0.01+i0.78

TABLE IV
THE DISTRIBUTION OF THE ZEROS $\mu_n(t)$ OF FUNCTION IN EQ.(**??**), WHERE THE COEFFICIENTS ARE THE REAL PART OF THE COEFFICIENTS IN TABLE.(III)

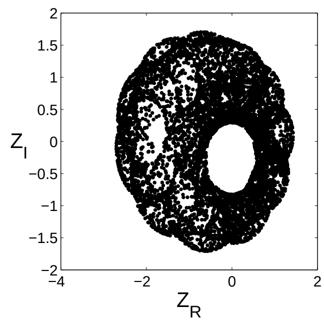

Fig. 4. The motion of zeros in Eq. (38) using the Hamiltonian of Eq.(39).

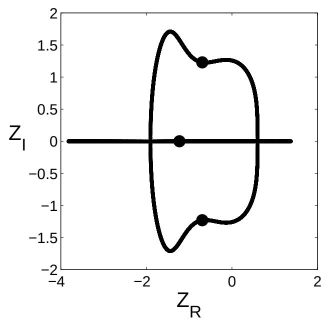

Fig. 5. The motion of zeros in Eq. (40) using the Hamiltonian of Eq.(41).

Let

$$\begin{aligned}\mu_0(0) &= -0.7 + 1.2i,\ \mu_1(0) = -0.7 - 1.2i, \\ \mu_2(0) &= -1.2, \end{aligned} \tag{40}$$

be the zeros at $t = 0$ and let

$$H = \begin{bmatrix} 1 & i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{41}$$

be the Hamiltonian with eigenvalues $0, 1, 1, 2$ We seen that the zeros of the function with real coefficients are real or appear as complex conjugate pairs and draw symmetric graph with respect to the z_r axis. Here we found numerically that the motion of the zeros also draw symmetric graph with respect to the z_r axis. In Fig.5 we plot the motion of this zeros.

VII. CONCLUSION

We have studied the Bargmann analytic representation. The zeros Bargmann function and there time evolution have been considered. We have derived some examples to consider the motion of various zeros for various Hamiltonians. A brief discussion to the Husimi functions are given. The Husimi function and Bargmann function $f(z)$ are related to each other and there zeros are identical. The analytic functions $f(z)$ have exactly q zeros. If the coefficients $f_0, f_1, ..., f_n$ are real then the zeros μ_n are real or appear as complex conjugate pairs and draw symmetric graph with respect to the z_r axis.

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