

# Yield Onset of Thermo-Mechanical Loading of FGM Thick Walled Cylindrical Pressure Vessels

S. Ansari Sadrabadi, G. H. Rahimi

**Abstract**—In this paper, thick walled Cylindrical tanks or tubes made of functionally graded material under internal pressure and temperature gradient are studied. Material parameters have been considered as power functions. They play important role in the elastoplastic behavior of these materials. To clarify their role, different materials with different parameters have been used under temperature gradient. Finally, their effect and loading effect have been determined in first yield point. Also, the important role of temperature gradient was also shown. At the end the study has been results obtained from changes in the elastic modulus and yield stress. Also special attention is also given to the effects of this internal pressure and temperature gradient in the creation of tensile and compressive stresses.

**Keywords**—FGM, Cylindrical pressure tubes, Small deformation theory, Yield onset, Thermal loading.

## I. INTRODUCTION

FUNCTIONALLY GRADED MATERIALS (FGM) are heterogeneous materials in which the elastoplastic and thermal properties change from one surface to the other, gradually and continuously. The material is constructed by smoothly changing the volume fraction of its constituent materials. Since ceramic has good heat-resistance and metal has high strength, Ceramic–Metal FGM may work at super high-temperatures or under high-temperature gradient field.

Classical method of analysis is to combine the equilibrium and compatibility equations with the stress–strain relations to arrive at the governing equations in terms of the stress components.

Elastic and elastic-plastic analyses of thick-walled cylindrical pressure vessels subjected to internal pressure and temperature gradient are important in solid mechanics and engineering applications.

Obata and Noda [1] presented the steady thermal stresses in a hollow circular cylinder and a hollow sphere made of a functionally gradient material, in order to understand the effect of the composition on stresses and to design the optimum FGM hollow circular cylinder and hollow sphere. Tutuncu and Ozturk [2] presented the closed-form solutions for stresses and displacements in functionally graded cylindrical and spherical vessels subjected to internal pressure alone, using the infinitesimal theory of elasticity. Eraslan et al [3] presented

Plane strain analytical solutions to functionally graded elastic and elastic–plastic pressurized tube problems were obtained in the framework of small deformation theory. The modulus of elasticity and the uniaxial yield limit of the tube material were assumed to vary radially according to two parametric parabolic forms.

It was used the Airy stress function to derive exact solutions for plane strain deformations of a functionally graded (FG) hollow cylinder with the inner and the outer surfaces subjected to different boundary conditions, and the cylinder composed of an isotropic and incompressible linear elastic material. For the shear modulus given by either a power law or an exponential function of the radius  $r$ , it was derived explicit expression for stresses, the hydrostatic pressure and displacements [4]. In investigation [5], the effect of material nonhomogeneity on the mechanical behaviors of a thick-walled sandwich cylindrical structure embedded with a functionally graded interlayer is investigated. The inner of the sandwich cylindrical structure is a homogeneous and isotropic layer. The outer is a homogeneous and orthotropic layer and the middle is a transition layer. Sadeghi et al [6] studied Strain gradient elasticity formulation for analysis of FG micro-cylinders. The material properties are assumed to obey a power law in radial direction. A power series solution for stresses and displacements in FG micro-cylinders subjected to internal and external pressures is obtained. Numerical examples are presented to study the effect of the characteristic length parameter and FG power index on the displacement field and stress distribution in FG cylinders. Also, several thermal stress problems in a hollow cylinder were analytically solved [7]–[10].

It is found that, unlike the deformation behavior of homogeneous cylindrical pressure tubes, the plastic deformation may commence at the inner surface or at the outer surface or simultaneously at both surfaces. Heat conductivity of FGM has a great effect on the temperature distribution. As a result, the thermal loads and material parameters affect on the deformations and stresses in the tubes.

## II. HEAT CONDUCTION SOLUTION

Consider a thick cylindrical tube of inside radius  $a$  and outside radius  $b$  made of FGM. The geometry of the cylinder in relation with the coordinate axes is shown in Fig. 1. It is assume that the mechanical and thermal loads and their associated boundary conditions are such that the stress field is a function of variables  $r$ ,  $\theta$  and  $z$ .

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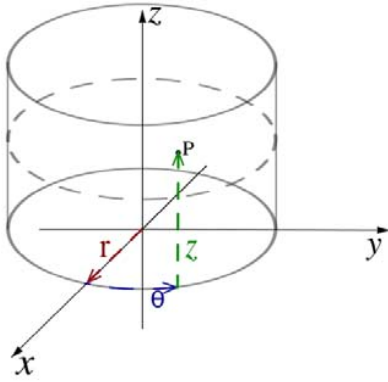


Fig. 1 Cylindrical (r, θ, z) and Cartesian (x,y,z) coordinates

It is well-known that the modulus of elasticity  $E$  and the yield limit  $\sigma_y$  are the most important material properties in determining the strength of structural elements during operation. Also, the thermal conduction coefficient  $k$  and the coefficient of thermal expansion  $\alpha$  are the most important material properties in heat conduction during operation. For this reason, in this work, these material properties are assumed to vary in the radial direction according to the relations

$$k(r) = k_0 \left( \frac{r}{b} \right)^\gamma, \alpha(r) = \alpha_0 \left( \frac{r}{b} \right)^\mu \quad (1)$$

$$E(r) = E_0 \left( \frac{r}{b} \right)^\xi, \sigma_y(r) = \sigma_0 \left( \frac{r}{b} \right)^\eta \quad (2)$$

where  $k_0, \alpha_0, E_0$  and  $\sigma_0$  are the reference values of  $k, \alpha, E$  and  $\sigma_y$  in other order, and  $r$  the radial coordinate, and  $\gamma, \mu, \xi$  and  $\eta$  are the grading parameters. By considering the variation of  $k(r), \alpha(r), E(r)$  and  $\sigma_y(r)$ , an elastic and a plastic region is formulated in the framework of small deformation theory.

The first law of thermodynamics for energy equation in the steady-state condition for the functionally graded cylinder is

$$\frac{1}{r} \frac{\partial}{\partial r} \left( kr \frac{\partial T}{\partial r} \right) = 0 \quad (3)$$

where  $T(r)$  is temperature distribution. The thermal boundary conditions are assumed as

$$T(a) = T_a, T(b) = T_b \quad (4)$$

Using the definition for the material properties, the temperature equation becomes

$$T = c_1 r^{-\gamma} + c_2 \quad (5)$$

Using the boundary conditions (4) to determine the constants  $c_1$  and  $c_2$

$$c_1 = \frac{T_b - T_a}{b^{-\gamma} - a^{-\gamma}} \quad (6)$$

$$c_2 = T_a - a^{-\gamma} \frac{T_b - T_a}{b^{-\gamma} - a^{-\gamma}} \quad (7)$$

### III. ELASTIC SOLUTION

Small deformations are assumed in the long cylindrical symmetric case without support ( $\sigma_z = 0, \varepsilon_z = 0$ ). Using the formal notation for the stresses and strains [1]

$$\varepsilon_r = \varepsilon_r^e + \varepsilon_r^p + \varepsilon^T \quad (8)$$

$$\varepsilon_\theta = \varepsilon_\theta^e + \varepsilon_\theta^p + \varepsilon^T \quad (9)$$

where strain in relations 8 and 9 can be written by Hooke's law as follows.

$$\varepsilon_r^e = \frac{1}{E} (\sigma_r - \nu \sigma_\theta) \quad (10)$$

$$\varepsilon_\theta^e = \frac{1}{E} (\sigma_\theta - \nu \sigma_r) \quad (11)$$

$$\varepsilon^T = \alpha T \quad (12)$$

Here  $\varepsilon_i$  the total strain components,  $\varepsilon^e$  the elastic strain components,  $\varepsilon^p$  the plastic strain components,  $\nu$  the Poisson's ratio, and  $\sigma_i$  are the stress components in radial and circumferential directions. For elastic stress state  $\varepsilon_i^p = 0$ .

Using the equilibrium and compatibility equations

$$\frac{d\sigma_r}{dr} + \frac{1}{r} (\sigma_r - \sigma_\theta) = 0 \quad (13)$$

$$\frac{d\varepsilon_\theta}{dr} + \frac{1}{r} (\varepsilon_\theta - \varepsilon_r) = 0 \quad (14)$$

the stresses then reduces to

$$\sigma_r = c_3 r^{\lambda_1} + c_4 r^{\lambda_2} + e_1 r^{l_1 + \lambda_1} + e_2 r^{l_2 + \lambda_1} + e_3 r^{l_3 + \lambda_2} + e_4 r^{l_4 + \lambda_2} \quad (15)$$

$$\sigma_\theta = c_3 (1 + \lambda_1) r^{\lambda_1} + c_4 (1 + \lambda_2) r^{\lambda_2} + e_1 (1 + l_1 + \lambda_1) r^{l_1 + \lambda_1} + e_2 (1 + l_2 + \lambda_1) r^{l_2 + \lambda_1} + e_3 (1 + l_3 + \lambda_2) r^{l_3 + \lambda_2} + e_4 (1 + l_4 + \lambda_2) r^{l_4 + \lambda_2} \quad (16)$$

where  $\lambda_1, \lambda_2, l_1$  to  $l_4$  and  $e_1$  to  $e_4$  are given in Appendix and  $c_3$  and  $c_4$  are the arbitrary integration constants.

### IV. NUMERICAL RESULTS AND DISCUSSION

The analytical solution obtained for spherical vessel in the same way is validated with the results of the published paper in [11].

A thick FGM Cylinder with inside and outside radii  $a=0.06$  and  $b=0.1$  is considered. The material properties shown in Table I are assumed. The results of behavior of this tube under different boundary condition with different material parameters are shown in Table II.

TABLE I  
MATERIAL PROPERTIES (OUTER SURFACE IS ASSUMED PURE STEEL)

$E_0$ (GPa)	200
$\alpha_0$ ( $1/^\circ\text{C}$ )	11.7
$\sigma_0$ (MPa)	415
$\nu$	0.3

In Table II, positive sign means tensile stresses and negative sign means compressive stresses. The striped surface shows progress of the plastic zone in the section.

As seen in Table II, material parameters have much influence of the behavior of FGM in Cylindrical structures. The temperature gradient has the opposite effect in creating stresses towards inner pressure loading.

As seen in Fig. 3, internal pressure causes tensile stresses. Compressive stress is created by adding the thermal loading. This causes the yield point to be reached at a later stage of the thermal loading.

The second and third rows in Table II demonstrate the important role of the parameter related to the modulus of elasticity. The fourth and fifth rows in Table II also demonstrate the role of the parameter related to the coefficient of expansion in changing the plastic region.

TABLE II  
RESULTS FOR THE BEHAVIOR OF THE CYLINDRICAL TUBE

N	$\gamma$	$\xi$	$\mu$	$\eta$	$p_y$ (MPa)		$\Delta T_y$ ( $^\circ\text{C}$ )		
					$\Delta T=0^\circ\text{C}$ ( $p_y$ )	$\Delta T=0.5$ $\Delta T_{y_0}$ ( $p_y$ )	$P=0$ MPa ( $\Delta T_{y_0}$ )	$P=0.5$ $p_{y_0}$ MPa	$P=p_{y_0}$ MPa
1	3	-1.1	2	2	39.6	59.4	352	528	704
2	-2.1	1.1	-1.4	3	34.9	52.5	83	124	165
							349	327	305
3	-2.1	-1.5	-1.4	3	22.2	11.1	$r = 70.5\text{mm}$ 31	$r = 68.5\text{mm}$ 15.5	$r = 66.5\text{mm}$ 0
							324	343	361
							$r_1 = 69.0\text{mm}$ $r_2 = 83.5\text{mm}$ $r_3 = 84.5\text{mm}$	$r_1 = 69.5\text{mm}$ $r_2 = 84.0\text{mm}$ $r_3 = 85.5\text{mm}$	$r_1 = 70.0\text{mm}$ $r_2 = 84.5\text{mm}$ $r_3 = 86.0\text{mm}$
4	-3	0.5	-0.5	-0.5	187.3	161.1	92	66	39
							147	220	293
5	-3	0.5	-0.5	0.5	112.4	168.5	$r = 93.0\text{mm}$ 88	$r = 85.5\text{mm}$ 76	$r = 81.0\text{mm}$ 60
						Both of Surface together	92	132	176
							$r = 60.5\text{mm}$	$r = 90.5\text{mm}$	$r = 84.5\text{mm}$

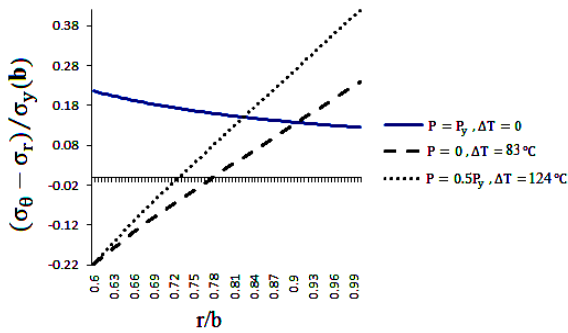


Fig. 3 Differences of radial and tangential stresses of the 2<sup>nd</sup> row pressure vessel of Table II

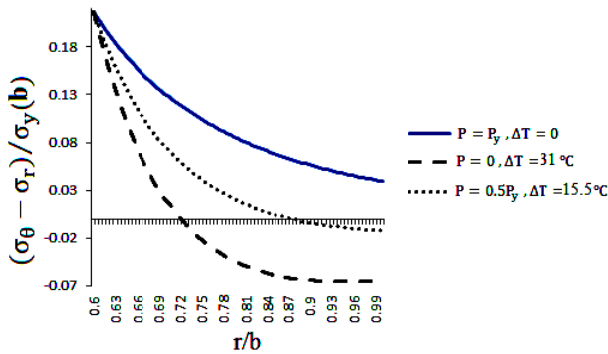


Fig. 4 Hoop stress minus radial stress for third row particulars in Table II in arriving at yield point

## V. CONCLUSIONS

An analytical method of solution is presented to obtain the thermo-mechanical response of a cylindrical tube made of FGM. Thermal and mechanical boundary conditions are assumed to be functions of the variables  $r$ . The power law model is considered for the variations of the FGM profiles.

From the results of the current investigation of FGM of different compositions subjected to a prescribed severe thermal loading with different internal pressure using the small deformation theory, the following conclusions can be drawn;

Heat conductivity of FGM has a great effect on the temperature distribution. As a result, the thermal loads and material parameters affect on the deformations and stresses in the tubes.

For FGM Cylindrical pressure tubes, the plastic deformation may commence at the inner surface, at the outer surface, simultaneously at both surfaces, or somewhere inside the cylindrical tube depending on the radial variation of FGM parameters.

By choosing the parameters of the first row of Table II, it can be seen that under internal pressure and temperature gradient the onset of plastic deformation is at inside of cylinder and gradually, with loading extended the yield zone and continue to outer of cylinder.

Comparing the two rows 2 and 3 in Table II and Figs. 3 and 4, which is used to compare changes in the elastic modulus. It can be seen internal pressure loading and temperature gradient

have opposite effects when the modulus of elasticity is positive and with increasing inner pressure, yield temperature increases that it is mentioned in reason of this in the text. The temperature decreases with increasing pressure when the modulus of elasticity is negative.

It also has a significant effect on the process of creating plastic deformation under higher temperature differences in the second row, plastic deformation starts initially from the inside and then from outside of cylinder whereas in the third row plastic deformation starts initially from the inside and then among the thickness.

By comparing rows 4 and 5 in Table II, we find the effect the yield stress changes

With a suitable parameter of material properties in the radial direction the hoop stress can be made to be compressive on the inner surface and tensile on the outer surface, or uniform through the cylinder thickness.

## LIST OF SYMBOLS

$a, b$	Inner and outer radius
$E$	Modulus of elasticity
$E_0$	Reference value of $E$
$F$	Yield function
$k$	Thermal conduction coefficient
$k_0$	Reference value of $k$
$P$	Internal pressure
$P_y$	Yield pressure
$r$	Radius
$T$	Thermal gradient
<i>Greek symbols</i>	
$\nu$	Poisson ratio
$\mu, \xi, \eta, \gamma$	Grading parameters
$\alpha$	Expansion coefficient
$\alpha_0$	Reference value of $\alpha$
$\epsilon_i$	Strain
$\epsilon_i^e$	Elastic strain
$\epsilon_i^p$	Plastic strain
$\epsilon^T$	Thermal strain
$\sigma_i$	Stress
$\sigma_y$	Yield stress
$\sigma_0$	Reference value of $\sigma_y$
$d\lambda$	Infinitesimal scale factor
$\Delta T$	Temperature difference
$\Delta T_y$	Yield temperature difference

## APPENDIX

$$\lambda_1, \lambda_2 = \frac{1}{2} \left( \xi - 2 \pm \sqrt{(2 - \xi)^2 + 4(1 - \nu)\xi} \right) \quad (22)$$

$$l_1 = \xi + \mu - \gamma - \lambda_1 \quad (23)$$

$$l_2 = \xi + \mu - \lambda_1 \quad (24)$$

$$l_3 = \xi + \mu - \gamma - \lambda_2 \quad (25)$$

$$l_4 = \xi + \mu - \lambda_2 \quad (26)$$

$$e_1 = \frac{E_0 \alpha_0 c_1 (\mu - \gamma)}{(\lambda_2 - \lambda_1) b^{\xi + \mu} l_1} \quad (27)$$

$$e_2 = \frac{E_0 \alpha_0 c_2 \mu}{(\lambda_2 - \lambda_1) b^{\xi + \mu} l_2} \quad (28)$$

$$e_3 = -\frac{E_0 \alpha_0 c_1 (\mu - \gamma)}{(\lambda_2 - \lambda_1) b^{\xi + \mu} l_3} \quad (29)$$

$$e_4 = -\frac{E_0 \alpha_0 c_2 \mu}{(\lambda_2 - \lambda_1) b^{\xi + \mu} l_4} \quad (30)$$

## REFERENCES

- [1] Obata. Y, A., and Noda. N, B., 1994. "Steady thermal stresses in a hollow circular cylinder and a hollow sphere of a functionally gradient material". *J Thermal Stresses*, 17(3), pp. 471–87.
- [2] Tutuncu. N, A., and Ozturk. M, B., 2001. "The exact solution for stresses in functionally graded pressure vessels". *Composites:Part B*, 32, pp. 683–686.
- [3] Eraslan. A.N, A., Akis. T, B., 2006. "Plane strain analytical solutions for a functionally graded elastic-plastic pressurized tube". *International Journal of Pressure Vessels and Piping*, 83, pp. 635–644.
- [4] Nie. G.J, A., Batra. R.C, B., 2010. "Material tailoring and analysis of functionally graded isotropic and incompressible linear elastic hollow cylinders". *Composite Structures*, 92, pp. 265–274.
- [5] Wang. H.M, A., Wei. Y.K, B., 2012. "Effect of material nonhomogeneity on the mechanical behaviors of a thick-walled functionally graded sandwich cylindrical structure". *Results in Physics*, 2, pp. 118–122.
- [6] Sadeghi. H, A., Baghani. M, B., Naghdabadi. R, C., 2012. "Strain gradient elasticity solution for functionally graded micro-cylinders". *International Journal of Engineering Science*, 50, pp. 22–30.
- [7] Tutuncu. N, A., 2007. "Stresses in thick-walled FGM cylinders with exponentially-varying properties". *Engineering Structures*, 29, pp. 2032–2035.
- [8] Jabbari. M, A., Bahtui. A, B., Eslami. M.R, C., 2009. "Axisymmetric mechanical and thermal stresses in thick short length FGM cylinders". *International Journal of Pressure Vessels and Piping*, 86, pp. 296–306.
- [9] Keles. I, A., Conker. C, B., 2011. "Transient hyperbolic heat conduction in thick-walled FGM cylinders and spheres with exponentially-varying properties". *European Journal of Mechanics A/Solids*, 30, pp. 449–455.
- [10] Wang. H.M, A., 2013. "An effective approach for transient thermal analysis in a functionally graded hollow cylinder". *International Journal of Heat and Mass Transfer*, 67, pp. 499–505.
- [11] Tolga. A, A., 2009. "Elastoplastic analysis of functionally graded spherical pressure vessels". *Computational Materials Sci*, 46, May, pp. 545–554.