# Weyl Type Theorem and the Fuglede Property 

M. H. M. Rashid


#### Abstract

Given $\mathcal{H}$ a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operator in $\mathcal{H}$, let $\delta_{A B}$ denote the generalized derivation defined by $A$ and $B$. The main objective of this article is to study Weyl type theorems for generalized derivation for $(A, B)$ satisfying a couple of Fuglede.


Keywords-Fuglede Property, Weyl's theorem, generalized derivation, Aluthge Transformation.

## I. Introduction

THe algebra of all bounded operators on a complex infinite dimensional Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$ . For $A, B \in \mathcal{B}(\mathcal{H})$, let $\delta_{A B}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ and $\Delta_{A B}$ : $\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ denote the itgeneralized derivation $\delta_{A B}=$ $A X-X B$ and the itelementary operator $\Delta_{A B}=A X B-X$. Let $d_{A B}=\delta_{A B}$ or $\Delta_{A B}$. If $T$ is a bounded linear operator on a normed linear space $\mathcal{X}$, we have

$$
\begin{aligned}
\operatorname{ker}\left(d_{A B}\right) \perp \Re\left(d_{A B}\right) \Rightarrow & \\
\operatorname{ker}\left(d_{A B}\right) \cap c l\left(\Re\left(d_{A B}\right)\right) & =\{0\} \\
\Rightarrow \operatorname{ker}\left(d_{A B}\right) \cap \Re\left(d_{A B}\right) & =\{0\} \\
\Leftrightarrow a s c\left(d_{A B}\right) & \leq 1
\end{aligned}
$$

[1, Page 25]. Here $\operatorname{asc}\left(d_{A B}\right)$ denotes the itascent of $d_{A B}, \operatorname{cl}\left(\Re\left(d_{A B}\right)\right)$ denote the closure of the range of $d_{A B}$ and $\operatorname{ker}\left(d_{A B}\right) \perp \Re\left(d_{A B}\right)$ denotes that the kernel of $d_{A B}$ is orthogonal to the range of $d_{A B}$ in the sense of G. Birkhoff. A normed linear subspace $\mathcal{M}$ is said to be orthogonal to a normed linear subspace $\mathcal{M}$ in the sense of Birkhoff, written as, $\mathcal{M} \perp \mathcal{N}$, if $\|m\| \leq\|m+n\|$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$. This concept of orthogonality is not symmetric, i.e., $\mathcal{M} \perp \mathcal{N}$ does not imply $\mathcal{N} \perp \mathcal{M}$, but the concept does agree with the usual concept of orthogonality in the case in which $\mathcal{X}=\mathcal{H}$. The range-kernel orthogonality of $d_{A B}$ has been considered by a number of authors, see ( [1], [2], [3], [4], [5] and [6]). A sufficient condition guaranteeing $\operatorname{ker}\left(d_{A B}\right) \perp \Re\left(d_{A B}\right)$ is that $\operatorname{ker}\left(d_{A B}\right) \subseteq \operatorname{ker}\left(d_{A^{*} B^{*}}\right)$ [3]. The inclusion $\operatorname{ker}\left(d_{A B}\right) \subseteq \operatorname{ker}\left(d_{A^{*} B^{*}}\right)$, known in the literature as the Putnam-Fuglede commutativity theorem.

Definition I. Let $A, B \in \mathcal{B}(\mathcal{H})$. We say that $(A, B)$ is a couple of Fuglede (Shortly, $(A, B) \in F P$ ) if $A X=X B$ implies that $A^{*} X=X B^{*}$, for all $X \in \mathcal{B}(\mathcal{H})$.

Lemma II. ([7]) Let $A, B \in \mathcal{B}(\mathcal{H})$. Then the following assertions equivalent.
(i) $(A, B)$ is a couple of Fuglede.
M. H. M. Rashid is with the Department of Mathematics, Faculty of Science P.O. Box(7), Mu'tah university, Al-Karak-Jordan (e-mail: malik_okasha@yahoo.com).
(ii) If $A X=X B$, then $\operatorname{cl}(\Re(X))$ reduces $A, \operatorname{ker}(X)^{\perp}$ reduces $B$, and $\left.A\right|_{c l(\Re(X))},\left.B\right|_{\operatorname{ker}(X)^{\perp}}$ are unitarily equivalent normal operators.
Lemma III. ([8]) Let $A, B \in \mathcal{B}(\mathcal{H})$. If $(A, B)$ is a couple of Fuglede, then $\operatorname{ker}\left(d_{A B}\right) \subseteq \operatorname{ker}\left(d_{A^{*} B^{*}}\right)$.

## II. Main Results and Their Proofs

Let $A \in \mathcal{B}(\mathcal{H})$ have the polar decomposition $A=U|A|$. Then the first Aluthge transform $\widetilde{\sim}$ $\widetilde{A}$ has the polar decomposition $\widetilde{A}=V|\widetilde{A}|$, then the second Aluthge $\widehat{A}=|\widetilde{A}|^{\frac{1}{2}} V|\widetilde{A}|^{\frac{1}{2}}$. It is known that $A, \widetilde{A}$ and $\widehat{A}$ have the same point spectrum, the same approximate point spectrum and the same spectrum. Furthermore, $\widehat{A}$ has a normal part if and only if $A$ has a normal part.
Lemma IV. Let $A, B \in \mathcal{B}(\mathcal{H})$. If $(A, B) \in F P$, then $(A-\lambda, B) \in F P$.

Proof. Suppose that $(A, B) \in F P$. Then

$$
(A-\lambda) X=A X-\lambda X=X B-X \lambda=X(B-\lambda)
$$

Now

$$
\begin{gathered}
(A-\lambda)^{*} X \quad=\left(A^{*}-\bar{\lambda}\right) X=A^{*} X-\bar{\lambda} X \\
=X B^{*}-X \bar{\lambda} \quad \text { since }(A, B) \in F P \\
=X(B-\lambda)^{*}
\end{gathered}
$$

Hence $(A-\lambda, B) \in F P$.
Lemma V. ([9]) Let $A, B \in \mathcal{B}(\mathcal{H})$ be normal. If there is a quasiaffinity $X \in \Delta_{A B}^{-1}(0)$, then $B$ is invertible and $X \in \delta_{A B}^{-1}(0)$.

Lemma VI.([9]) If $A, B \in \mathcal{B}(\mathcal{H})$ are normal, then $d_{A B}^{-2}(0)=d_{A B}^{-1}(0)$.

Lemma VII. Let $A, B \in \mathcal{B}(\mathcal{H})$. If $(A, B)$ is a couple of Fuglede, then $\operatorname{ker}\left(\delta_{A B}-\lambda\right) \subseteq \operatorname{ker}\left(\delta_{A^{*} B^{*}}-\bar{\lambda}\right)$ for all $\lambda \in \mathbf{C}$, where $\bar{\lambda}$ denote the complex conjugate of $\lambda$.

Proof. Since $(A, B)$ is a couple of Fuglede, it follows from Lemma I that $c l(\Re(X))$ reduces $A, \operatorname{ker}(X)^{\perp}$ reduces $B$, and $\left.A\right|_{c l(\Re(X))},\left.B\right|_{\operatorname{ker}(X)^{\perp}}$ are unitarily equivalent normal operators. Then with respect to the orthogonal decomposition $\mathcal{H}=\operatorname{cl}(\Re(X)) \oplus \overline{\Re(X)}^{\perp}$ and $\mathcal{H}=\operatorname{ker}(X) \oplus \operatorname{ker}(X)^{\perp}, A$ and $B$ can be respectively represented as $A=A_{n} \oplus A_{p}$ and $B=B_{n} \oplus B_{p}$, where $A_{n}, B_{n}$ are normal operators and $A_{p}, B_{p}$ are pure parts; now assume $X \in \operatorname{ker}\left(\delta_{A B}-\lambda\right)$,
$X: \operatorname{ker}(X)^{\perp} \oplus \operatorname{ker}(X) \rightarrow c l(\Re(X)) \oplus \operatorname{cl}(\Re(X))^{\perp}$, have the corresponding matrix representation $X=\left[X_{i j}\right]_{i, j=1}^{2}$. Then
$\operatorname{ker}\left(\delta_{A B}-\lambda\right)=\left(\begin{array}{ll}\left(\delta_{A_{n} B_{n}}-\lambda\right) X_{11} & \left(\delta_{A_{n} B_{p}}-\lambda\right) X_{12} \\ \left(\delta_{A_{p} B_{n}}-\lambda\right) X_{21} & \left(\delta_{A_{p} B_{p}}-\lambda\right) X_{22}\end{array}\right)=0$.
Since the operator $A_{n}-\lambda$ (resp., $B_{n}-\lambda$ ) is normal. Since $c l(\Re(X))$ reduces $A, \operatorname{ker}(X)^{\perp}$ reduces $B$, we have the pure parts of $A$ and $B$ are injective. It follows from an application of the Fuglede-Putnam property to $\left(\delta_{A_{n} B_{p}}-\lambda\right) X_{12}=\left(\delta_{A_{p} B_{n}}-\right.$ d) $X_{21}=0$ that $X_{12}=X_{21}=0$. Define the second Aluthge transforms as above. Then

$$
\left(\delta_{A_{p} B_{p}}-\lambda\right) X_{22}=0 \Longleftrightarrow\left(\delta_{A_{p} T_{p}}-\lambda\right) Y=0,
$$

where we have set $\widehat{B^{*}}=T_{p}$ and $Y=$ $\left|\widetilde{A_{p}}\right|^{\frac{1}{2}}\left|A_{p}\right|^{\frac{1}{2}} X_{22}\left|B_{p}^{*}\right|^{\frac{1}{2}}\left|\widetilde{B_{p}^{*}}\right|^{\frac{1}{2}}$. Since $\quad\left|\widetilde{A_{p}}\right|^{\frac{1}{2}},\left|A_{p}\right|^{\frac{1}{2}},\left|B_{p}^{*}\right|^{\frac{1}{2}}$ and $\left|\widetilde{B_{p}^{*}}\right|^{\frac{1}{2}}$ are quasiaffinities, which implies that $X_{22}=0$ and $X=X_{11} \oplus 0$. Since $\operatorname{ker}\left(\delta_{A_{n} B_{n}}-\lambda\right) \subseteq \operatorname{ker}\left(\delta_{A_{n}^{*} B_{n}^{*}}-\lambda\right)$ we have $\operatorname{ker}\left(\delta_{A B}-\lambda\right) \subseteq \operatorname{ker}\left(\delta_{A^{*} B^{*}}-\bar{\lambda}\right)$ for all $\lambda \in \mathbf{C}$.

Lemma VIII. If $(A, B) \in F P$ and $\lambda \in \mathbf{C}$, then $\operatorname{asc}\left(\delta_{A B}-\lambda\right) \leq 1$.

Proof. Let $X \in\left(\delta_{A B}-\lambda\right)^{-1}(0)$. Then $A X-X(B+\lambda)=0=A^{*} X-X\left(B^{*}+\bar{\lambda}\right)$ implies that $\overparen{\Re(X)}$ reduces $A$ and $X^{-1}(0)^{\perp}$ reduces $B+\lambda$. Since $X \in\left(d_{A B}-\lambda\right)^{-1}(0)$, we have $A X$ and $X(B+\lambda) \in$ $\left(\delta_{A B}-\lambda\right)^{-1}(0), A^{*} A X=A X(B+\lambda)^{*}=A A^{*} X$ and $X(B+\lambda)^{*}(B+\lambda)=A^{*} X(B+\lambda)=X(B+\lambda)(B+\lambda)^{*}$. Hence $A_{1}=\left.A\right|_{\overline{\Re(X)}}$ and $B_{1}=\left.(B+\lambda)\right|_{X^{-1}(0)^{\perp}}$ are normal operators.

Suppose now that $Y \in\left(\delta_{A B}-\lambda\right)^{-2}(0)$. Set $\left(\delta_{A B}-\lambda\right) Y=X$, let $X_{1}: X^{-1}(0)^{\perp} \longrightarrow \overline{\Re(X)}$ be the quasiaffinity defined by setting $X_{1} h=X h$ for each $h \in \mathcal{H}$ and let $Y: X^{-1}(0)^{\perp} \oplus X^{-1}(0) \longrightarrow \overline{\Re(X)} \oplus \overline{\Re(X)}{ }^{\perp}$ have the matrix representation $Y=\left[Y_{i j}\right]_{i, j=1}^{2}$. Then $0=\delta_{A B}(X)=\delta_{A_{1} B_{1}}\left(X_{1}\right) \oplus 0=\delta_{A_{1} B_{1}}^{2}\left(Y_{11}\right) \oplus 0$. The operators $A_{1}$ and $B_{1}$ being normal, it follows from Lemma III that $\delta_{A_{1} B_{1}}\left(Y_{11}\right)=0$. Hence $X=\delta_{A_{1} B_{1}}\left(Y_{11}\right) \oplus 0=0$ implies $\left(\delta_{A B}-\lambda\right)(Y)=0$ and so $\operatorname{asc}\left(\delta_{A B}-\lambda\right) \leq 1$.

Let $\mathcal{X}$ be a complex Banach space. A Banach space operator $T \in \mathcal{B}(\mathcal{X})$ has the single-valued extension property, or SVEP, at a point $\lambda \in \sigma(T)$ if for every open disc $\mathcal{D}$ centered at $\lambda$ the only analytic function $f: \mathcal{D} \longrightarrow \mathcal{X}$ satisfying $(T-\mu) f(\mu)=0$ is the function $f \equiv 0 ; T$ has SVEP if it has SVEP at every $\lambda \in \sigma(T)$.
Corollary IX. If $(A, B) \in F P$, then $\delta_{A B}$ has SVEP.
Proof. The finite ascent property of $\left(\delta_{A B}-\lambda\right)$ implies SVEP [10].

Remark. Recall from [11] that $\sigma\left(\delta_{A B}\right)=\{\lambda \in$ $\sigma(A)-\sigma(B): \lambda=\alpha-\beta, \alpha \in \sigma(A)$ and $\beta \in \sigma(B)\}$.

Theorem X. If $(A, B) \in F P$, then $\Re\left(d_{A B}-\lambda\right)$ is closed for each $\lambda \in \operatorname{iso} \sigma\left(d_{A B}\right)$.

Proof. Let $\lambda \in \operatorname{iso} \sigma\left(\delta_{A B}\right)$. Then $0 \in \operatorname{iso} \sigma\left(\delta_{A B}-\lambda\right)$, where $\sigma\left(\delta_{A B}-\lambda\right)=\sigma(A)-\sigma(B+\lambda)$. Hence $\sigma(A) \cap \sigma(B+$ $\lambda)$ consists of points which are isolated in both $\sigma(A)$ and $\sigma(B+\lambda)$. In particular, $\sigma(A) \cap \sigma(B+\lambda)$ does not contain any limit points of $\sigma(A) \cap \sigma(B+\lambda)$. There exists a finite set $S=\left\{\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right\}$ of distinct values $\alpha_{i}$ such that $S=$ $\sigma(A) \cap \sigma(B+\lambda)$ and each $\alpha_{i}, 1 \leq i \leq n$, is an isolated point of both $\sigma(A)$ and $\sigma(B+\lambda$. Let

$$
\begin{gathered}
H_{1}=\bigvee_{i=1}^{n}\left(B-\alpha_{i}\right)^{*^{-1}}(0), \quad H_{1}^{\prime}=\bigvee_{i=1}^{n}\left(A-\alpha_{i}\right)^{-1}(0) \\
H_{2}=\mathcal{H} \ominus H_{1} \text { and } H_{2}^{\prime} \\
=\mathcal{H} \ominus H_{1}^{\prime}
\end{gathered}
$$

Then $A$ and $B$ have the direct sum decompositions $A=$ $A_{1} \oplus A_{2}$ and $B=B_{1} \oplus B_{2}$, where $A_{1}=\left.A\right|_{H_{1}^{\prime}}$ and $B_{1}=\left.B\right|_{H_{1}}$ are normal operators. Let $X=\left[X_{i j}\right]_{i, j=1}^{2}$, it is seen that

$$
\left(\delta_{A B}-\lambda\right) X=\left(\begin{array}{cc}
\left(\delta_{A_{1} B_{1}}-\lambda\right) X_{11} & \left(\delta_{A_{1} B_{2}}-\lambda\right) X_{12} \\
\left(\delta_{A_{2} B_{1}}-\lambda\right) X_{21} & \left(\delta_{A_{2} B_{2}}-\lambda\right) X_{22}
\end{array}\right)
$$

where $A_{2}=\left.A\right|_{H_{2}^{\prime}}, B_{2}=\left.B\right|_{H_{2}}$ and $\sigma\left(A_{i}\right) \cap \sigma\left(B_{i}+\lambda\right)=\emptyset$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$ and so that $0 \notin \sigma\left(\delta_{A B}-\lambda\right)$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. Therefore, $\Re\left(\delta_{A B}-\lambda\right)$ is closed.

## III. The Operator $d_{A B}$ and Weyl's Theorem

Let us denote by $\alpha(T)$ the dimension of the kernel and by $\beta(T)$ the codimension of the range. Recall that the operator $T \in \mathcal{B}(\mathcal{X})$ is said to be itupper semi-Fredholm, $T \in S F_{+}(\mathcal{X})$, if the range of $T \in \mathcal{B}(\mathcal{X})$ is closed and $\alpha(T)<\infty$, while $T \in \mathcal{B}(\mathcal{X})$ is said to be lower semi-Fredholm, $T \in S F_{-}(\mathcal{X})$, if $\beta(T)<\infty$. An operator $T \in B(\mathcal{X})$ is said to be itsemi-Fredholm if $T \in S F_{+}(\mathcal{X}) \cup S F_{-}(\mathcal{X})$ and Fredholm if $T \in S F_{+}(\mathcal{X}) \cap S F_{-}(\mathcal{X})$. If $T$ is semi-Fredholm then the itindex of $T$ is defined by $\operatorname{ind}(T)=\alpha(T)-\beta(T)$. A bounded linear operator $T$ acting on a Banach space $\mathcal{X}$ is itWeyl if it is Fredholm of index zero and Browder if $T$ is Fredholm of finite ascent and descent. The Weyl spectrum $\sigma_{w}(T)$ and Browder spectrum $\sigma_{b}(T)$ of $T$ are defined by $\sigma_{w}(T)=\{\lambda \in \mathbf{C}: T-\lambda I$ is not Weyl $\}$ and $\sigma_{b}(T)=\{\lambda \in$ C : $T-\lambda I$ is not Browder $\}$. Let $E^{0}(T)=\{\lambda \in$ iso $\sigma(T):$ $0<\alpha(T-\lambda)<\infty\}$ and let $\pi_{0}(T):=\sigma(T) \backslash \sigma_{b}(T)$ all Riesz points of $T$. According to Coburn [12], Weyl's theorem holds for $T$ if $\sigma(T) \backslash \sigma_{w}(T)=E^{0}(T)$, and that Browder's theorem holds for $T$ if $\sigma(T) \backslash \sigma_{w}(T)=\pi^{0}(T) . \quad$ Let $S F_{+}^{-}(\mathcal{X})=$ $\left\{T \in S F_{+}: \operatorname{ind}(T) \leq 0\right\}$. The upper semi Weyl spectrum is defined by $\sigma_{S F_{+}^{-}}(T)=\left\{\lambda \in \mathbf{C}: T-\lambda \notin S F_{+}^{-}(\mathcal{X})\right\}$. According to Rakočević [13], an operator $T \in B(\mathcal{X})$ is said to satisfy $a$-Weyl's theorem if $\sigma_{a}(T) \backslash \sigma_{S F_{+}^{-}}(T)=E_{a}^{0}(T)$, where $E_{a}^{0}(T)=\left\{\lambda \in\right.$ iso $\left.\sigma_{a}(T): 0<\alpha\left(T{ }^{+} \lambda I\right)<\infty\right\}$. It is known [13] that an operator satisfying $a$-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

In the following we prove that if $(A, B) \in F P$, then $\delta_{A B}$ satisfies the property that its quasinilpotent part $H_{0}\left(d_{A B}-\lambda\right)$,

$$
\begin{gathered}
H_{0}\left(\delta_{A B}-\lambda\right) \quad=\{X \in \mathcal{B}(\mathcal{H}): \\
\left.\lim _{n \rightarrow \infty}\left\|\left(\delta_{A B}-\lambda\right)^{n} X\right\|=0\right\}
\end{gathered}
$$

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equals $\left(\delta_{A B}-\lambda\right)^{-1}(0)$ for all $\lambda \in$ iso $\sigma\left(\delta_{A B}\right)$. This implies that $\delta_{A B}$ satisfies Weyl's theorem, $\delta_{A B}^{*}$ satisfies $a$-Weyl's theorem.
Lemma XI. If $(A, B) \in F P$, then $H_{0}\left(\delta_{A B}-\lambda\right)=$ $\left(\delta_{A B}-\lambda\right)^{-1}(0)$ for all $\lambda \in$ iso $\sigma\left(\delta_{A B}\right)$.

Proof. Evidently, the non-zero points $\alpha_{i}$ (resp. $\bar{\beta}$ ), $1 \leq i \leq$ $m$, are normal eigenvalues of $A$ (resp., $B^{*}$ ). Let $M_{1 i}=(A-$ $\left.\alpha_{i}\right)^{-1}(0), N_{1 i}=\left(B-\mathcal{B}_{i}\right)^{-1}(0)\left(=\left(B-\beta_{i}\right)^{*^{-1}}\right), M_{1}=$ $\bigoplus_{i=1}^{m} M_{1 i}, N_{1}=\bigoplus_{i=1}^{m}, M_{2}=M_{1}^{\perp}$ and $N_{2}=N_{1}^{\perp} ;$ let $A=$ $A_{1} \oplus A_{2} \in \mathcal{L}\left(M_{1} \oplus M_{2}\right)$ and $B=B_{1} \oplus B_{2} \in \mathcal{L}\left(N_{1} \oplus N_{2}\right)$. Then $\sigma\left(A_{2}\right)=\sigma(A) \backslash\left\{\alpha_{1}, \cdots, \alpha_{m}\right\}$ and $\sigma\left(B_{2}\right)=\sigma(B) \backslash$ $\left\{\beta_{1}, \cdots, \beta_{m}\right\} \lambda \notin \sigma\left(\delta_{A_{k} B_{t}}\right)$ for all $1 \leq k, t \leq 2$ other than $k=t=1$.

Let $X \in H_{0}\left(\delta_{A B}-\lambda\right)$, and let $X \in \mathcal{L}\left(N_{1} \oplus N_{2}, M_{1} \oplus M_{2}\right)$ have the representation $X=\left[X_{i j}\right]_{i, j=1}^{2}$. Then
$\left(\delta_{A B}-\lambda\right)^{n} X=\left(\begin{array}{lc}* & * \\ & \left(\delta_{A_{2} B_{2}}-\lambda\right)^{n} X_{22}\end{array}\right)$
(for some, as yet, non specified entries *). Since $\quad \lim _{n \rightarrow \infty}\left\|\left(\delta_{A B}-\lambda\right)^{n} X\right\|^{\frac{1}{n}}=0 \quad$ implies $\lim _{n \rightarrow \infty}\left\|\left(\delta_{A_{2} B_{2}}-\lambda\right)^{n} X_{22}\right\|^{\frac{1}{n}}=0$, and since $\delta_{A_{2} B_{2}}-\lambda$ is invertible, we have $X_{22}=0$, and then
$\left(\delta_{A B}-\lambda\right)^{n} X=\left(\begin{array}{cc}* & \left(\delta_{A_{1} B_{2}}-\lambda\right)^{n} X_{12} \\ \left(\delta_{A_{2} B_{1}}-\lambda\right)^{n} X_{21} & 0\end{array}\right)$
(for some, as yet, non specified entry and
*). Again, $\quad \lim _{n \rightarrow \infty}\left\|\left(\delta_{A B}-\lambda\right)^{n} X\right\|^{\frac{1}{n}}=0$
implies

$$
\lim _{n \rightarrow \infty}\left\|\left(\delta_{A_{1} B_{2}}-\lambda\right)^{n} X_{12}\right\|^{\frac{1}{n}}
$$

$=$
$\lim _{n \rightarrow \infty}\left\|\left(\delta_{A_{2} B_{1}}-\lambda\right)^{n} X_{21}\right\|^{\frac{1}{n}}=0$, and since $\delta_{A_{1} B_{2}}-\lambda$ and $\delta_{A_{2} B_{1}}^{n \rightarrow \lambda}-\lambda$ are invertible, we have $X_{12}=0=X_{21}$. Hence, $\left(\delta_{A B}-\lambda\right)^{n} X=\left(\delta_{A_{1} B_{1}}-\lambda\right)^{n} X_{11}$. Let $X_{11}=\left[Y_{i j}\right]_{1 \leq i, j \leq m} \in \mathcal{L}\left(\bigoplus_{i=1}^{m} N_{1 i}, \bigoplus_{i=1}^{m} M_{1 i}\right)$. Then, for $1 \leq i, j \leq m$,

$$
\begin{aligned}
\left(d_{A_{1} B_{1}}-\lambda\right)^{n}\left(X_{11}\right) & =\left(\left(L_{A_{1}-\alpha_{i}}-R_{B_{1}-\beta_{j}}\right)+\right. \\
& \left.\left(\alpha_{i}-\beta_{j}-\lambda\right)\right)^{n}\left[Y_{i j}\right]_{1 \leq i, j \leq m} \\
= & \left(\sum_{k=0}^{n}\binom{n}{k}\left(L_{A_{1}-\alpha_{i}}-R_{B_{1}-\beta_{j}}\right)^{k}\right. \\
\times & \left.\left(\alpha_{i}-\beta_{j}-\lambda\right)^{n-k}\right)\left[Y_{i j}\right]_{1 \leq i, j \leq m}
\end{aligned}
$$

where we have set $L_{A_{1}-\alpha_{i}} R_{B_{1}}+\alpha_{i} R_{B_{1}-\beta_{j}}=T$. Since $\left.\left(A-\alpha_{i}\right)\right|_{M_{1 i}}=0=\left.\left(B_{1}-\beta_{i}\right)\right|_{N_{1 i}}$, it follows that

$$
\left.\left(\delta_{A_{1} B_{1}}-\lambda\right)^{n}\left(X_{11}\right)=\left(\alpha_{i}-\beta_{j}-\lambda\right)\right)^{n}\left[Y_{i j}\right]_{1 \leq i, j \leq m}
$$

Recall, $\quad \lim _{n \rightarrow \infty}\left\|\left(\delta_{A_{1} B_{1}}-\lambda\right)^{n} X_{11}\right\|^{\frac{1}{n}}=0 ; \quad$ hence $\lim _{n \rightarrow \infty}\left|\alpha_{i}-\beta_{j}-\lambda\right|\left\|Y_{i j}\right\|^{\frac{1}{n}}=0$. Thus $\quad Y_{i j}=0$ ${ }_{\text {for }}^{n \rightarrow \infty}$ all $i, j$ such that $i \neq j$. This implies that $X=X_{11}=\bigoplus_{i=1}^{m} Y_{i j} \in\left(\delta_{A B}-\lambda\right)^{-1}(0)$. Hence $H_{0}\left(\delta_{A B}-\lambda\right) \subset\left(\delta_{A B}-\lambda\right)^{-1}(0)$. Since the reverse inclusion holds for every operator, we must have $H_{0}\left(\delta_{A B}-\lambda\right)=\left(\delta_{A B}-\lambda\right)^{-1}(0)$.

For an operator $T \in \mathcal{B}(\mathcal{X})$, the analytic core $\mathcal{K}(T-\lambda)$ of $T-\lambda$ is defined by

$$
\begin{gathered}
\mathcal{K}(T-\lambda) \quad=\{x \in \mathcal{X}: \text { there exists a sequence } \\
\left\{x_{n}\right\} \subset \mathcal{X} \text { and } \delta>0 \text { for which } \\
x=x_{0},(T-\lambda) x_{n+1}=x_{n} \text { and } \\
\left.\left\|x_{n}\right\| \leq \delta^{n}\|x\| \text { for all } n=1,2, \cdots\right\} .
\end{gathered}
$$

We note that $H_{0}(T-\lambda)$ and $\mathcal{K}(T-\lambda)$ are generally non-closed hyperinvariant subspaces of $T-\lambda$ such that $(T-\lambda)^{-q}(0) \subseteq H_{0}(T-\lambda)$ for all $q=0,1,2, \cdots$ and $(T-\lambda) \mathcal{K}(T-\lambda)=\mathcal{K}(T-\lambda)$ [14]. Recall from [14] that if $0 \in$ iso $\sigma(T)$, then $H_{0}(T)$ and $\mathcal{K}(T)$ are closed and $\mathcal{X}=H_{0}(T) \oplus \mathcal{K}(T)$.
Theorem XII. If $(A, B) \in F P$, then $\delta_{A B}$ satisfies Weyl's theorem and $\delta_{A B}^{*}$ satisfies $a$-Weyl's theorem.

Proof. Let $\lambda \in$ iso $\sigma\left(d_{A B}\right)$. Then by Lemma III, $H_{0}\left(\delta_{A B}-\right.$ $\lambda)=\left(\delta_{A B}-\lambda\right)^{-1}(0)$ implies

$$
\begin{aligned}
\mathcal{B}(\mathcal{H})=H_{0}\left(\delta_{A B}-\lambda\right) & \oplus \mathcal{K}\left(\delta_{A B}-\lambda\right) \\
=\left(\delta_{A B}-\lambda\right)^{-1}(0) & \oplus \mathcal{K}\left(\delta_{A B}-\lambda\right) .
\end{aligned}
$$

Hence

$$
\begin{gathered}
\Re\left(\delta_{A B}-\lambda\right)=0 \oplus\left(\delta_{A B}-\lambda\right)\left(\mathcal{K}\left(\delta_{A B}-\lambda\right)\right) \\
=\mathcal{K}\left(\delta_{A B}-\lambda\right)
\end{gathered}
$$

and

$$
\mathcal{B}(\mathcal{H})=\left(\delta_{A B}-\lambda\right)^{-1}(0) \oplus \Re\left(\delta_{A B}-\lambda\right) .
$$

Thus, isolated points of $\sigma\left(\delta_{A B}\right)$ are simple poles of the resolvent of $\delta_{A B}$ (i.e., $\delta_{A B}$ is simply polaroid). Observe from the argument above that if we let $\pi^{0}\left(\delta_{A B}\right)$ denote the set of finite rank poles of the resolvent of $\delta_{A B}$, then $\pi^{0}\left(\delta_{A B}\right)=E^{0}\left(\delta_{A B}\right)$. Since $\delta_{A B}$ has SVEP by Lemma II , it follows that $\delta_{A B}$ satisfies Weyl's theorem [15, Theorem 3.85], i.e., $\sigma\left(\delta_{A B}\right) \backslash \sigma_{w}\left(\delta_{A B}\right)=E^{0}\left(\delta_{A B}\right)$.

Since $\delta_{A B}^{*}$ has SVEP, $\sigma\left(\delta_{A B}\right)=\sigma\left(\delta_{A B}^{*}\right)=\sigma_{a}\left(\delta_{A B}^{*}\right)$ (see [10, Proposition 1.3.2]), which implies $E^{0}\left(\delta_{A B}^{*}\right)=E_{a}^{0}\left(\delta_{A B}^{*}\right)$. Again, since $\lambda \notin \sigma_{S F_{+}^{-}}\left(\delta_{A B}^{*}\right)$ if and only if $\delta_{A B}^{*}-\lambda$ is upper semi-Fredholm and ind $\left(\delta_{A B}^{*}-\lambda\right) \leq 0$, and since $\delta_{A B}$ has SVEP and $\delta_{A B}^{*}-\lambda$ is upper semi-Fredholm implies $\operatorname{ind}\left(\delta_{A B}^{*}-\lambda\right) \geq 0, \lambda \notin \sigma_{S F_{+}^{-}}\left(\delta_{A B}^{*}\right)$ implies $\lambda \notin \sigma_{w}\left(\delta_{A B}^{*}\right)$, and this, since $\sigma_{S F_{+}^{-}}(T) \subseteq{ }^{+} \sigma_{w}(T)$ for every operator $T \in$ $\mathcal{B}(\mathcal{X})$, implies $\sigma_{S F_{+}^{-}}\left(\delta_{A B}^{*}\right)=\sigma_{w}\left(\delta_{A B}^{*}\right)=\sigma_{w}\left(\delta_{A B}\right)$. As seen above, $\delta_{A B}$ is simply polaroid; hence $\delta_{A B}^{*}$ is simply polaroid, with $\pi^{0}\left(\delta_{A B}\right)=\pi^{0}\left(\delta_{A B}^{*}\right)$. This implies that $E^{0}\left(\delta_{A B}\right)=$ $\pi^{0}\left(\delta_{A B}\right)=\pi^{0}\left(\delta_{A B}^{*}\right) \subseteq E^{0}\left(\delta_{A B}^{*}\right)=E^{0}\left(\delta_{A B}^{*}\right)$. Since $\lambda \in$ $E^{0}\left(\delta_{A B}^{*}\right)$ implies $\lambda \in$ iso $\sigma\left(\delta_{A B}\right)$ and so $\lambda \in \pi^{0}\left(\delta_{A B}\right)$, it follows that $E^{0}\left(\delta_{A B}\right)=E_{a}^{0}\left(\delta_{A B}^{*}\right)$. Hence,

$$
\begin{gathered}
E^{0}\left(\delta_{A B}\right)=\begin{array}{c}
\sigma\left(\delta_{A B}\right) \backslash \sigma_{w}\left(\delta_{A B}\right) \\
=\sigma_{a}\left(\delta_{A B}^{*}\right) \backslash \sigma_{S F_{+}^{-}}\left(\delta_{A B}^{*}\right) \\
=E_{a}^{0}\left(\delta_{A B}^{*}\right)
\end{array}, \$ \text {. }
\end{gathered}
$$

i.e., $\delta_{A B}^{*}$ satisfies $a$-Weyl's theorem.

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Recall from [13] that an operator $T \in \mathcal{B}(\mathcal{X})$ is said to be obeys property $(w)$ if $\sigma_{a}(T) \backslash \sigma_{S F_{+}^{-}}(T)=E^{0}(T)$. From the proof of Theorem III, we have

Corollary XIII. If $(A, B) \in F P$, then $\delta_{A B}^{*}$ satisfies property $(w)$.

For a Banach space operator $T$, let $H(\sigma(T))$ denote the set of functions which are holomorphic on an open neighborhood of $\sigma(T)$.

Corollary XIV. If $(A, B) \in F P$, then $f\left(\delta_{A B}\right)$ satisfies Weyl's theorem and $f\left(\delta_{A B}^{*}\right)$ satisfies $a$-Weyl's theorem for every $f \in H\left(\sigma\left(\delta_{A B}\right)\right)$.

Proof. It follows from Theorem III that $\delta_{A B}$ satisfies Weyl's theorem. That is,

$$
\sigma\left(\delta_{A B}\right) \backslash E^{0}\left(\delta_{A B}\right)=\sigma_{w}\left(\delta_{A B}\right) .
$$

Now the polaroid property of $\delta_{A B}$ implies, [15, Lemma 3.89], that

$$
f\left(\sigma\left(\delta_{A B}\right) \backslash E^{0}\left(\delta_{A B}\right)\right)=\sigma\left(f\left(\delta_{A B}\right)\right) \backslash E^{0}\left(f\left(\delta_{A B}\right)\right)
$$

for every $f \in H\left(\sigma\left(\delta_{A B}\right)\right)$. Also, since $\delta_{A B}$ has SVEP,

$$
f\left(\sigma_{w}\left(\delta_{A B}\right)\right)=\sigma_{w}\left(f\left(\delta_{A B}\right)\right)
$$

for every $f \in H\left(\sigma\left(\delta_{A B}\right)\right)$. Hence,

$$
\begin{aligned}
\sigma\left(f\left(\delta_{A B}\right)\right) \backslash E^{0}\left(f\left(\delta_{A B}\right)\right)= & f\left(\sigma\left(\delta_{A B}\right)\right) \backslash f\left(E^{0}\left(\delta_{A B}\right)\right) \\
& =f\left(\sigma\left(\delta_{A B}\right) \backslash E^{0}\left(\delta_{A B}\right)\right) \\
= & f\left(\sigma_{w}\left(\delta_{A B}\right)\right)=\sigma_{w}\left(f\left(\delta_{A B}\right)\right) .
\end{aligned}
$$

That is, $f\left(\delta_{A B}\right)$ satisfies Weyl's theorem.
To prove that $f\left(\delta_{A B}^{*}\right)$ satisfies $a$-Weyl's theorem, we recall that $\delta_{A B}$ is polaroid if and only $\delta_{A B}^{*}$ is polaroid, $\delta_{A B}^{*}$ is isoloid. Applying [15, Lemma 3.89], it thus follows that

$$
f\left(\sigma\left(\delta_{A B}^{*}\right)\right) \backslash f\left(E^{0}\left(\delta_{A B}^{*}\right)\right)=\sigma\left(f\left(\delta_{A B}^{*}\right)\right) \backslash E^{0}\left(f\left(\delta_{A B}^{*}\right)\right)
$$

Again, since $\delta_{A B}$ has SVEP implies $f\left(\delta_{A B}\right)$ has SVEP [15, Theorem 2.39],

$$
\begin{gathered}
\sigma\left(f\left(\delta_{A B}\right)\right) \quad=\sigma\left(f\left(\delta_{A B}^{*}\right)\right)=\sigma_{a}\left(f\left(\delta_{A B}^{*}\right)\right), \\
E^{0}\left(f\left(\delta_{A B}^{*}\right)\right)=E_{a}^{0}\left(\delta_{A B}^{*}\right) \\
\sigma_{w}\left(f\left(\delta_{A B}\right)\right)=\sigma_{w}\left(f\left(\delta_{A B}^{*}\right)\right)=\sigma_{S F_{+}^{-}}\left(f\left(\delta_{A B}^{*}\right)\right) \\
\text { and } \sigma_{S F_{+}^{-}}\left(f\left(\delta_{A B}^{*}\right)\right)=f\left(\sigma_{S F_{+}^{-}}\left(\delta_{A B}^{*}\right)\right)
\end{gathered}
$$

[15, Corollary 3.72]. Thus $\delta_{A B}^{*}$ satisfies $a$-Weyl's theorem, we have that

$$
\begin{aligned}
\sigma_{a}\left(f\left(\delta_{A B}^{*}\right)\right) \backslash E_{a}^{0}\left(f\left(\delta_{A B}^{*}\right)\right)= & f\left(\sigma\left(\delta_{A B}^{*}\right) \backslash E^{0}\left(\delta_{A B}^{*}\right)\right) \\
& =f\left(\sigma_{w}\left(\delta_{A B}^{*}\right)\right) \\
= & \sigma_{S F_{+}^{-}}\left(f\left(\delta_{A B}^{*}\right)\right) .
\end{aligned}
$$

That is, $f\left(\delta_{A B}^{*}\right)$ satisfies $a$-Weyl's theorem.

## IV. Conclusion

In this paper we investigated Weyl's type theorems for generalized derivation for $(A, B)$ satisfying a couple of Fuglede. Many questions are raised by this work. First one such, the examination of the conditions which enable one to easily apply the Weyl types theorems and Fuglede-Putnam theorem, we have discussed, which are mostly stated as purely mathematical results. The second question of which our theorems can give constructive proofs. Other questions can be posed and indeed all are under investigation and will be considered elsewhere.

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Mohammad H.M. Rashid is a Full Professor in the Department of Mathematics and Statistics of the Faculty of Science at the University of Mut'ah, Jordan. The major field of his research interests are Functional Analysis/Operator Theory, Fuzzy mathematics and inequalities.

