

# Using Non-Linear Programming Techniques in Determination of the Most Probable Slip Surface in 3D Slopes

M. M. Toufigh, A. R. Ahangarar, and A. Ouria

**Abstract**—Among many different methods that are used for optimizing different engineering problems mathematical (numerical) optimization techniques are very important because they can easily be used and are consistent with most of engineering problems. Many studies and researches are done on stability analysis of three dimensional (3D) slopes and the relating probable slip surfaces and determination of factors of safety, but in most of them force equilibrium equations, as in simplified 2D methods, are considered only in two directions. In other words for decreasing mathematical calculations and also for simplifying purposes the force equilibrium equation in 3rd direction is omitted. This point is considered in just a few numbers of previous studies and most of them have only given a factor of safety and they haven't made enough effort to find the most probable slip surface. In this study shapes of the slip surfaces are modeled, and safety factors are calculated considering the force equilibrium equations in all three directions, and also the moment equilibrium equation is satisfied in the slip direction, and using nonlinear programming techniques the shape of the most probable slip surface is determined. The model which is used in this study is a 3D model that is composed of three upper surfaces which can cover all defined and probable slip surfaces. In this research the meshing process is done in a way that all elements are prismatic with quadrilateral cross sections, and the safety factor is defined on this quadrilateral surface in the base of the element which is a part of the whole slip surface. The method that is used in this study to find the most probable slip surface is the non-linear programming method in which the objective function that must get optimized is the factor of safety that is a function of the soil properties and the coordinates of the nodes on the probable slip surface. The main reason for using non-linear programming method in this research is its quick convergence to the desired responses. The final results show a good compatibility with the previously used classical and 2D methods and also show a reasonable convergence speed.

**Keywords**—Non-linear programming, numerical optimization, slope stability, 3D analysis.

## I. INTRODUCTION

**M**ATHEMATICAL Optimization concepts and methods are not new. Indeed, optimization is fundamental for most of what we do. Whether we are engineers, athletes, or

businessmen, our goal is to be best in some way. Numerical optimization helps us for those cases where we are able to define the optimization problem in a consistent mathematical or numerical way. Interest in optimization originated with the simple linear programming model since it was practical and perhaps the only computationally tractable model at the time. Linear optimization models were soon adopted in numerous application areas and are perhaps the most widely used mathematical models in operations, research, and management science. Modelers have, however, found the assumption of linearity to be overly restrictive in expressing the real-world phenomena and problems in economics, finance, business, communication, engineering design, computational biology, and other areas that frequently demand the use of nonlinear expressions and discrete variables in optimization models. On the brighter side, recent advances in algorithmic and computing technology make it possible to revisit these problems with the hope of solving practically relevant problems in reasonable amounts of computational time.

Numerical optimization has traditionally been developed in the operations research community. The use of these techniques in engineering was popularized in 1960 when Schmit applied nonlinear optimization techniques to structural design.

A major advantage of using numerical optimization is the reduction in design time especially when the same computer program can be applied to many design projects, and also we can deal with a wide variety of design variables and constraints which are difficult to visualize using graphical or tabular methods.

Field observations of landslide failure surfaces typically display spatial variability. However, analyses of these slides are usually limited to two-dimensional (2D) approximations. The demand for practical, three-dimensional (3D) slope stability analysis methods and their associated user-friendly computer programs is high (Seed et al. 1990; Morgenstern 1992; Stark and Eid 1998).

There are a large number of publications that deal with 3D slope stability analysis. Duncan (1996) summarized the main aspects of 24 publications dealing with limit equilibrium approaches. This list could now be extended to include recent publications (e.g., Huang and Tsai 2000). All of these methods divide the failure mass into a number of columns with vertical interfaces and use the conditions for static equilibrium to find the factor of safety. Assumptions must be

M. M. Toufigh, Associate Professor, Civil Engineering Department, University of Kerman, Iran (e-mail: mmtoufigh@yahoo.com).

A. R. Ahangarar, Graduate Student, Civil Engineering Department, University of Kerman, Iran (e-mail: A\_Ahangarar@yahoo.com).

A. Ouria, PhD Candidate, Civil Engineering Department, University of Kerman, Iran (e-mail: Aouria@mail.com).

introduced to render the problem statically determinate and to facilitate the numerical procedures. A number of the methods (Hung et al. 1989; Huang and Tsai 2000) neglect the vertical shear force components of the inter-column force and project the forces applied on a column in the vertical direction. The normal force of the column base can then be readily determined without the knowledge of the unknown inter-column forces. Force or moment equilibrium equations are subsequently established to calculate the factor of safety. This kind of treatment can be traced back to 2D analysis, where Bishop (1955) established his simplified method for circular slip surfaces (although complete satisfaction of force equilibrium conditions for an individual slice or for the whole failure mass were not considered). Hung et al. (1989) discussed the limitations involved in their 3D Bishop and simplified Janbu methods. These do not satisfy the overall force equilibrium condition in the lateral direction. Huang and Tsai (2000) employ moment equilibrium conditions around two co-ordinate axes. However, since the force equilibrium equations are not fully satisfied in their method, the moment equilibrium conditions are dependent upon the location of the axes around which the moments are calculated. Their method is therefore only applicable to spherical slip surfaces in which the location of the center is known and allows the establishment of the moment equilibrium conditions [1] [7] [5] [14] [15] [16].

## II. OPTIMIZATION PROCESS

### A. Basic Optimization Concepts

Mathematical programming provides a very general framework for scarce resource allocation and the basic algorithms originate in the operations research community. Engineering applications include chemical process design, aerodynamic optimization, nonlinear control system design, mechanical component design, structural design and a variety of others. Because the statement of the numerical optimization problem is so close to the traditional statement of engineering design problems, the variety of tasks to which it can be applied is inexhaustible, [1].

In the most general sense, numerical optimization solves the nonlinear, constrained problem; Finds the set of design variables,  $X_i$ ,  $i=1, N$ , contained in vector  $\mathbf{X}$ , that will Minimize

$$F(\mathbf{X}) \quad (1)$$

Subject to;

$$g_j(\mathbf{X}) \leq 0 \quad j = 1, M \quad (2)$$

$$h_k(\mathbf{X}) = 0 \quad k = 1, L \quad (3)$$

$$X_i^L \leq X_i \leq X_i^U \quad i = 1, N \quad (4)$$

Equation (1) defines the objective function which depends on the values of the design variables,  $\mathbf{X}$ . Equations (2) and (3) are inequality and equality constraints respectively, and

equation (4) defines the region of search for the minimum. The bounds defined by equation (4) are referred to as side constraints. A clear understanding of the generality of this formulation makes the breadth of problems that can be addressed apparent [1] [7] [2] [3] [4].

### B. General Optimizing Steps

The symbol  $\nabla$  is called the gradient operator.

$$\nabla F(\mathbf{X}) = \left\{ \begin{array}{c} \frac{\partial F(\mathbf{X})}{\partial X_1} \\ \frac{\partial F(\mathbf{X})}{\partial X_2} \\ \dots \\ \dots \\ \frac{\partial F(\mathbf{X})}{\partial X_n} \end{array} \right\} \quad (5)$$

Where, in general,  $F(\mathbf{X})$  can be the objective or any constraint function.

This is a vector direction. The slope is the direction you might choose to search since this will move you towards the optimum response at the fastest rate. This we call the "search" direction. Mathematically, this gradient of the objective is referred to as a direction of "steepest ascent." Because we wish to minimize  $F(\mathbf{X})$ , we would move in the negative gradient, or "steepest descent" direction. You can now move in this direction until you reach the optimum answer or encounter a constraint. Note that the number of steps you take in this direction is a scalar parameter (partial steps are allowed so it will usually not be an integer number). We will call the number of steps in a given search direction  $\alpha$ . Now define the point at which you started as  $\mathbf{X}^0$ . In this case,  $\mathbf{X}^0$  contains two entries, being the longitude and latitude of your starting point. You move in a vector search direction we will call  $\mathbf{S}$ . Also, this is the first iteration in the process of optimizing your objective function so it is iteration 1. In general we will use the letter "q" to indicate the iteration number. Finally, you moved in a steepest descent direction so, mathematically,  $\mathbf{S} = -\nabla F(\mathbf{X})$ . Remember that, because  $F(\mathbf{X})$  is the **negative** of the objective function, this is actually the steepest **ascent** direction towards the optimum answer. Since this is the first iteration, the direction you move is designated as  $\mathbf{S}_1$ . Upon encountering a constraint or the desired optimum in direction  $\mathbf{S}_1$ , we can update the description of the variable  $\mathbf{X}$  by the simple mathematical expression;

$$\mathbf{X}^1 = \mathbf{X}^0 + \alpha \times \mathbf{S}^1 \quad (6)$$

This completes the first iteration in the "search" process. If you have your desired answer, we could just repeat the process of finding a new steepest descent direction and moving again. In practice we will see that, in this case, there is

a better choice of directions, called a conjugate direction. In case that we didn't get at the optimum answer we continue this process by changing direction until we get at the expected optimum [1].

### C. Different Optimization Methods

There are many different methods that almost all of them follow the process that mentioned above to get to the desired result. At this part some of these methods that are used in this study are presented. First one is the Broydon-Fletcher-Goldfarb-Shano (BFGS) Variable metric method which usually considered best on theoretical grounds. The second one is the Fletcher-Reeves (F.R.) conjugate gradient method which uses very little computer memory and has been found to be reliable, and the third method is the Modified Method of Feasible Directions (MMFD) which is reliable and uses the least computer memory. These three methods are used to justify results and find the quickest and the most suitable method in determination of the most probable slip surface in slopes [1] [6] [8].

## III. SLOPE ANALYSIS METHOD

The three dimensional method that is used in this study to analyze slopes to find the safety factor is proposed by Zuyu Chen (2003). This method is a 3D limit equilibrium method.

### A. Assumptions made about the Internal Shear Forces

As with other 3D limit equilibrium methods, the failure mass are divided into a number of columns with vertical interfaces (Fig. 1). The conventional definition for factor of safety  $F$  reduces the available shear strength parameters  $c'$  and  $\phi'$  by the following equations to bring the slope to a limiting state.

$$c'_e = \frac{c'}{F} \quad (7)$$

$$\tan \phi'_e = \frac{\tan \phi'}{F} \quad (8)$$

Throughout this paper, the subscript "e" is used to indicate the variables that are determined based on the reduced shear strength parameters  $c'_e$  and  $\phi'_e$ .

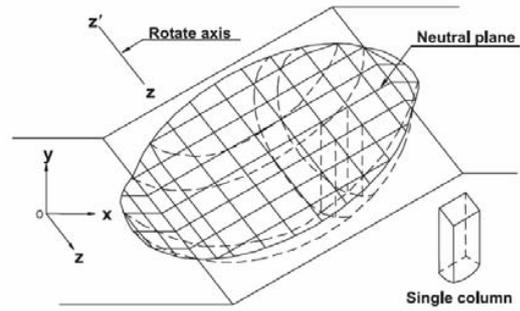


Fig. 1 Discretization of a failure mass [9]

The following assumptions are made in the establishment of the force and moment equilibrium equations (Fig. 2).

(1) The horizontal shear forces,  $H$ , on the row-interfaces (ABFE and DCGH in Fig. 2a) are neglected, i.e., the inter-column forces with inclinations of  $\beta$  to the  $x$ -axis and designated  $G$ , are assumed to be parallel to the  $xoy$  plane. It is further assumed that  $\beta$  is constant for all columns. This treatment is therefore equivalent to that used in Spencer's (1967) method in two dimensions. Ignoring the horizontal components of shear forces on the row-interfaces is a common assumption made to almost all of the 3D methods appearing in the literature.

(2) Shear forces,  $P$  and  $V$ , applied to the column-interfaces (ADHE and BCGF in Fig. 2) are neglected. Similar assumptions have been made by other researchers (e.g., Hungr et al. 1989; Huang and Tsai 2000).

(3) The shear force applied to any column base,  $T$ , is assumed to be inclined at an angle of  $\rho$  measured from the  $xoy$  plane to the positive  $z$ -axis. For prisms in any column direction (i.e., those with constant  $z$  values),  $\rho$  is taken to be constant. In the  $z$ -direction,  $\rho$  varies according to the following two modes:

(a) Mode I: the direction of the shear forces on all of the column bases is the same, i.e.,  $|\rho| = \kappa = \text{constant}$  (Fig. 3a).

(b) Mode II: the basal shear forces on the left and right side of the central  $xoy$  plane take opposite directions and vary linearly with respect to the  $z$ -axis, i.e., (Fig. 3b),

$$\begin{aligned} \rho_R &= kz & z \geq 0 \\ \rho_L &= -\eta kz & z < 0 \end{aligned} \quad (9)$$

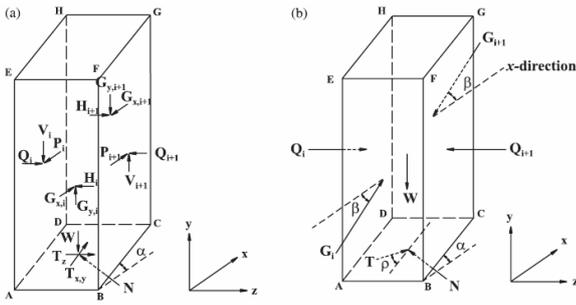


Fig. 2 Forces applied on a prism (a) before introducing the assumptions and (b) assumptions made for the shear forces on the prism

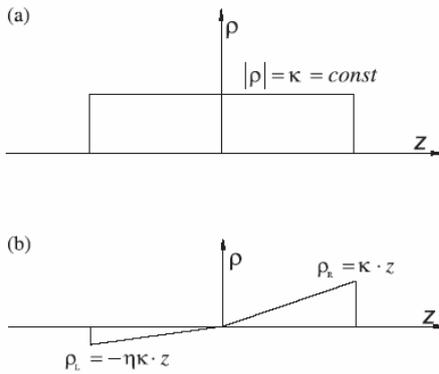


Fig. 3 Assumptions made for the distribution of r: (a) mode I, (b) mode II

The subscripts R and L indicate the right and left sides of the  $xoy$  plane, respectively,  $\eta$  is a coefficient of asymmetry, and  $\kappa$  is an unknown involved in the force and moment equilibrium equations. It determines the magnitude of  $\rho$  for each column after the solution is obtained.

The direction cosines of the shear force  $T$ , designated  $m_x, m_y, m_z$ , can be readily determined by the following equations:

$$\begin{cases} m_x^2 + m_y^2 + m_z^2 = 1 \\ m_x n_x + m_y n_y + m_z n_z = 0 \end{cases} \quad (10)$$

And:  $m_z = \sin(\rho)$

Where  $n_x, n_y, n_z$  are direction cosines of the normal to the column base. There are two solutions to  $m_x$ . The negative solution should be rejected [9] [11] [12] [13].

**B. The Force and Moment Equilibrium Equations**

By projecting all the forces in directions showed in Fig. 4 and calculating the moment equilibrium around Z axis, force and moment equilibrium equations can be determined as the following equations:

$$\begin{aligned} N &= -W_i \cos \beta + N_i (-n_x \sin \beta + n_y \cos \beta) + T_i (-m_x \sin \beta + m_y \cos \beta) = 0 \\ S &= \sum [N_i (n_x \cos \beta + n_y \sin \beta) + T_i (m_x \cos \beta + m_y \sin \beta) - W_i \sin \beta] = 0 \\ Z &= \sum (N_i \cdot n_z + T_i \cdot m_z) = 0 \\ M &= \sum [-W_i \cdot x - N_i \cdot n_x \cdot y + N_i \cdot n_y \cdot x - T_i \cdot m_x \cdot y + T_i \cdot m_y \cdot x] = 0 \\ T_i &= (N_i - uA_i) \tan \phi'_e + c'_e A_i \end{aligned}$$

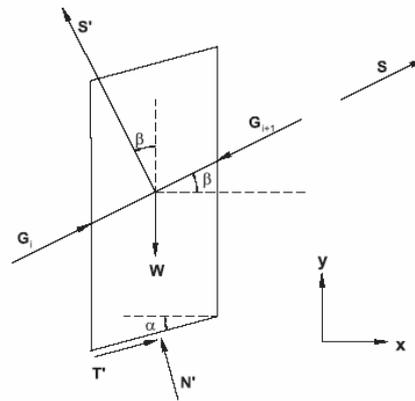


Fig. 4 The force and moment equilibriums

**IV. STUDY RESULTS**

At this part the results of this study are mentioned using two example problems that have been solved using the previously presented methods [9] [10].

**A. Problem Definition Process**

To define the problem to be solved using this method we consider a 3D slope with an arbitrary slip surface. As to calculate the factor of safety we need to have soil properties like cohesion and internal friction angle and also the shape of the slip surface, so we introduce these data to the computer code. To define the slip surface we just need to enter the coordinates of six points on the slip surface and the computer code, itself, will calculate the coordinates of the nodes on the slip surface (Fig. 5&6).

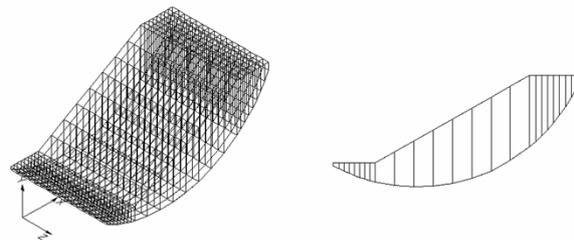


Fig. 5 The initial slip surface

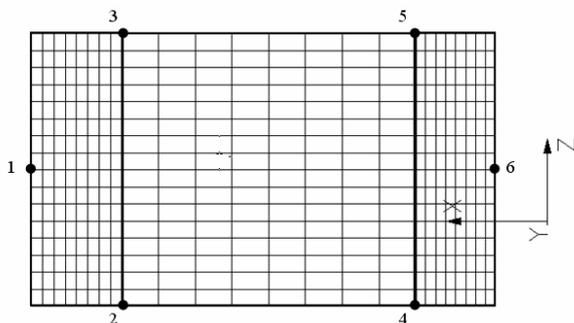


Fig. 6 Points on the slope surface that must be introduced

**B. Soil and Slope Properties in the Following Examples**

Table I shows the defined properties of soils in following examples.

TABLE I  
SLOPE AND SOIL PROPERTIES

Soil properties : $\gamma - \phi - c$	18.8 kN/m <sup>3</sup> , 20°, 29 kN/m <sup>2</sup>
Soil mass Dimensions	33.664 × 12.1 × 20
Coordinates of the entrance points (x, y, z) m	27.09, 12.1, 0 27.09, 12.1, 20 5.786, 0, 0 5.786, 0, 20

m = meter

**C. Example 1**

In this example the slip surface is assumed to be circular. The optimization parameters, which are considered, are: 1) coordinates of the center of the cross sectional circle in the middle of the slope mass, 2) radius of failure slip circle (cylinder for 3D). The initial values for these parameters are introduced to the optimization program in order to start the operation. These initial values and also the upper and lower limits of these variables are presented in Table II.

TABLE II  
PRIMITIVE VALUES AND RANGES OF THE VARIABLES

Primitive R- m	14
Primitive X <sub>c</sub> -m	13
Primitive Y <sub>c</sub> -m	14
R, Range-m	0-30
X <sub>c</sub> , Range-m	12-30
Y <sub>c</sub> , Range-m	12-30

m = meter

Using the optimization methods that are mentioned in part II of this paper (BFGS, F.R., MMFD method), the similar results of this these three methods are shown in Table III.

TABLE III  
FINAL VALUES OF THE OPTIMIZATION VARIABLES

Final R- m	23.7392
Final X <sub>c</sub> -m	12.1423
Final Y <sub>c</sub> -m	20.9957

m = meter

The optimum value that is determined using presented mathematical methods for the minimum factor of safety in this example is:

$$\text{Minimum factor of safety (FOS)} = 1.9721$$

The cross section of the soil mass which shows the shape of the cylindrical slip surface is shown in Fig. 7.

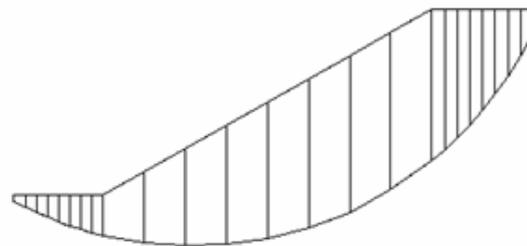


Fig. 7 The final and optimized circular cross section

**D. Example 2**

In this example the slip surface is assumed to be an ellipsoidal one. The optimization parameters, which are considered, are a, b, and c parameters of the ellipsoid. The initial values for these parameters are introduced to the optimization program in order to start operation. These initial values and also the upper and lower limits of these variables are presented in Table IV.

TABLE IV  
PRIMITIVE VALUES AND RANGES OF THE VARIABLES

Primitive a- m	11
Primitive b-m	9
Primitive c-m	12
a, Range-m	0-40
b, Range-m	0-30
Y <sub>c</sub> , Range-m	0-40

m = meter

The formula standing for the ellipsoidal surfaces in general, is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0 \tag{11}$$

Using the optimization methods that are mentioned in part II of this paper (BFGS, F.R., MMFD method), the similar results of this these three methods are shown in Table V.

TABLE V  
FINAL VALUES OF THE OPTIMIZATION VARIABLES

Final a- m	14.6110
Final b-m	10.9889
Final c-m	14.9977

m = meter

The optimum value that is determined using presented mathematical methods for the minimum factor of safety in this example is:

$$\text{Minimum factor of safety (FOS)} = 2.432$$

The 3D shape of the slope which shows a very clear picture of the ellipsoidal slip surface is shown in Fig. 8.

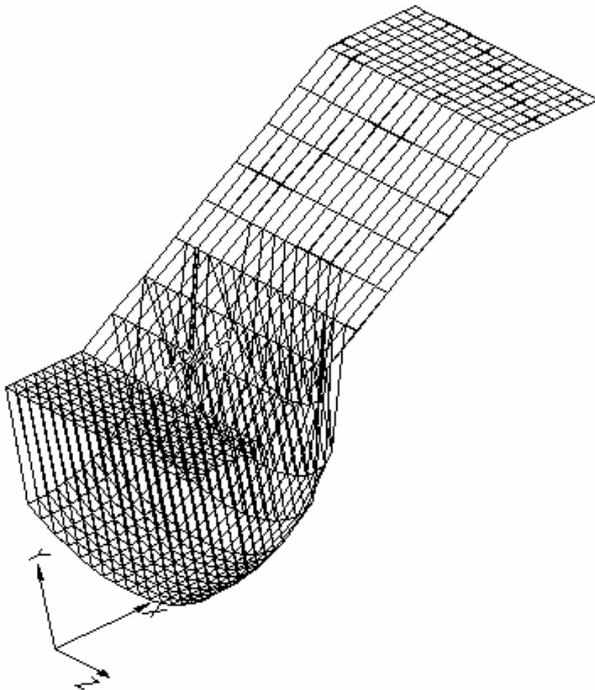


Fig. 8 3D view of the ellipsoidal slip surface

The cross section of the soil mass that indicates the shape of the ellipsoidal slip surface is depicted in Fig. 9.

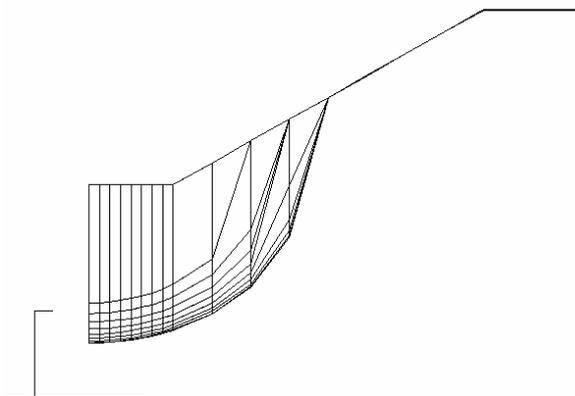


Fig. 9 Cross section of the ellipsoidal slip surface

## V. CONCLUSION

Generally it can be concluded that, mathematical or numerical optimization techniques can be very effective and also very efficient in 3D slope stability optimization problems. The proposed optimization methods are easy to use and more applicable in comparison with some other optimization methods that are based on the trial and error procedure. Based on the presented method, results of this study show that finding of the most probable slip surface in 3D slopes is less time consuming and more accurate. In order to find the minimum factor of safety, this particular problem can be solved by implemented mathematical techniques, results show quicker convergence compared to other techniques. Verification of this 3D method was justified for simple 2D problems based on classical methods for the most probable slip surface.

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