

# Using Linear Quadratic Gaussian Optimal Control for Lateral Motion of Aircraft

A. Maddi, A. Guessoum, and D. Berkani

**Abstract**—The purpose of this paper is to provide a practical example to the Linear Quadratic Gaussian (LQG) controller. This method includes a description and some discussion of the discrete Kalman state estimator. One aspect of this optimality is that the estimator incorporates all information that can be provided to it. It processes all available measurements, regardless of their precision, to estimate the current value of the variables of interest, with use of knowledge of the system and measurement device dynamics, the statistical description of the system noises, measurement errors, and uncertainty in the dynamics models.

Since the time of its introduction, the Kalman filter has been the subject of extensive research and application, particularly in the area of autonomous or assisted navigation. For example, to determine the velocity of an aircraft or sideslip angle, one could use a Doppler radar, the velocity indications of an inertial navigation system, or the relative wind information in the air data system. Rather than ignore any of these outputs, a Kalman filter could be built to combine all of this data and knowledge of the various systems' dynamics to generate an overall best estimate of velocity and sideslip angle.

**Keywords**—Aircraft motion, Kalman filter, LQG control, Lateral stability, State estimator.

## I. INTRODUCTION

THE Feedback control systems are widely used in manufacturing, mining, automobile and military hardware applications. In response to demands for increased efficiency and reliability, these control systems are being required to deliver more accurate and better overall performance in the face of difficult and changing operating conditions. In order to design control systems to meet the demands of improved performance and robustness when controlling complicated processes, control engineers will require new design tools and better underlying theory. In particular, a standard method of improving the performance of a control system is to add extra sensors and actuators. This necessarily leads to a multi-input multi-output control system. Thus, it is a requirement for any

modern feedback control system design methodology that it be able to handle the case of multiple actuators and sensors.

Linear Quadratic Gaussian optimal control theory (LQG) is one of the major achievements of the modern control area. This controller design methodology enables a controller to be synthesized which is optimal with respect to a specified quadratic performance index. Furthermore, this theory takes into account the presence of Gaussian white noise disturbances acting on the system. Indeed, in many practical control problems, it is straightforward to translate the required performance objective into a problem of minimizing a quadratic cost functional. Also, in many practical control problems, the system is subject to disturbances and measurement noise which are most naturally modeled as stochastic white noise processes.

The LQG controller design methodology based on the Kalman filter who in 1960 published his famous paper describing a recursive solution to the discrete-data linear filtering problem. A more complete introductory discussion can be found in [1] which also contains some interesting historical narrative. More extensive references include [2], [3] and [4]. It has also been used for motion prediction [7] and it is used for multi-sensor [10]. In practice, although it is possible to obtain process models either from first principles or from experimental measurements, these models will always be subject to errors. Thus, the control system needs to be designed to be robust against these modeling errors.

## II. LINEAR QUADRATIC GAUSSIAN REGULATOR

Linear Quadratic Gaussian (LQG) control is a modern state-space technique for designing optimal dynamic regulators. It enables you to trade off regulation performance and control effort, and to take into account process and measurement noise. Like pole placement, LQG design requires a state-space model of the plant. This section focuses on the discrete-time case. To form the LQG regulator, simply connect the Kalman filter and LQ-optimal gain  $K$  as shown below:

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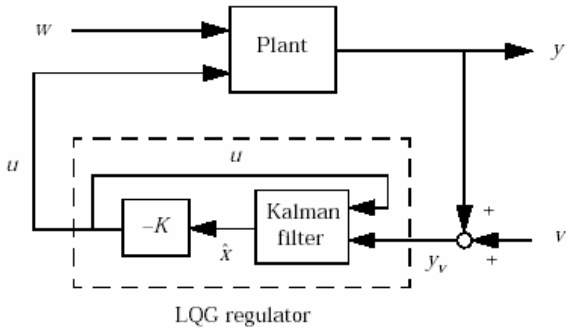


Fig. 1 Block diagram of LQG Controller

The goal is to regulate the output  $y$  around zero. The plant is driven by the process noise and the controls  $u$ , and the regulator relies on the noisy measurements  $y_v = y + v$  to generate these controls. The plant state and measurement equations are of the form

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) \end{aligned} \quad (1)$$

The LQG regulator consists of an optimal state-feedback gain and a Kalman state estimator. You can design these two components independently as shown next.

#### A. Optimal State-Feedback Gain

In LQG control, the regulation performance is measured by a quadratic performance criterion of the form

$$J(u) = \sum_{n=1}^{\infty} [x^T(n)Qx(n) + u^T(n)Ru(n) + 2x^T(n)Nu(n)] \quad (2)$$

The weighting matrices are user specified and define the trade-off between regulation performance (how fast goes to zero) and control effort. The first design step seeks a state-feedback law that minimizes the cost function. This gain is called the *LQ-optimal* gain.

#### B. Kalman State Estimator

As for pole placement, the LQ-optimal state feedback  $u(n) = -kx(n)$  is not implementable without full state measurement. However, we can derive a state estimate  $\hat{x}$  such that  $u(n) = -k\hat{x}(n)$  remains optimal for the output-feedback problem. This state estimate is generated by the Kalman filter.

Optimal estimation provides an alternative rationale for the choice of observer gains in the current estimator. Instead of arguments based on the pole placement, the optimal estimator is based on observer performance in the presence of process noise and measurement errors.

Suppose the discrete plant model of Eq.1 is extended

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) + \Gamma_1 w(k) \\ y(k) &= Cx(k) + v(k) \end{aligned} \quad (3)$$

where the process noise  $w(k)$  and measurement noise  $v(k)$  are white Gaussian random sequences with zero mean, that is

$$E\{w(k)\} = 0 \quad \text{and} \quad E\{v(k)\} = 0$$

where  $E\{\cdot\}$  denotes the statistical expectation, and zero correlation (white)

$$E\{w(i)w^T(j)\} = 0 \quad \text{and} \quad E\{v(i)v^T(j)\} = 0 \quad \text{for } i \neq j$$

and have covariances defined by

$$E\{w(k)w^T(k)\} = R_w \quad \text{and} \quad E\{v(k)v^T(k)\} = R_v$$

The optimization task is to determine a set of observer gains  $K_e$  to minimize the variance of the estimation error, which is denoted  $P(k)$ :

$$P(k) = E\{(x(k) - \hat{x}(k))(x(k) - \hat{x}(k))^T\} \quad (4)$$

The derivation is beyond the scope of this handout, we simply state that the solution is a structure the same as the full-state current observer with set of time-varying gains  $K_0$

$$\hat{x}(k) = \Phi \bar{x}(k) + K_0(k)[y(k) - C\bar{x}(k)] \quad (5)$$

where

$$K_0(k) = P(k)C^T R_v^{-1} \quad (6)$$

and where  $P(k)$  is found from

$$P(k) = M(k) - M(k)C^T [CM(k)C^T + R_v]^{-1} CM(k) \quad (7)$$

The matrix  $M(k)$  is the covariance of the state estimates  $\bar{x}(k)$  before the measurement. The estimate  $\bar{x}(k)$  is found from  $\hat{x}(k-1)$  using Eq.3 with  $w(k-1) = 0$  because we know that this is the expected value since the expected value of the plant noise is zero. Thus

$$\bar{x}(k) = \Phi \hat{x}(k-1) + \Gamma u(k-1) \quad (8)$$

This equation is known as the "time update", whereas the change in the estimate from  $\bar{x}(k)$  to  $\hat{x}(k)$  after measuring  $y(k)$  in Eq.5 is known as the "measurement update". The matrix  $M(k)$  is updated as

$$M(k+1) = \Phi P(k) \Phi^T + \Gamma_1 R_w \Gamma_1^T \quad (9)$$

The specific equations for the time and measurement updates [12] are presented below in Table I.

A block diagram for the closed-loop control of a continuous

TABLE I DISCRETE KALMAN FILTER TIME AND MEASUREMENT UPDATE EQUATIONS	
$\bar{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k)$ $M(k+1) = \Phi P(k) \Phi^T + \Gamma_1 R_w \Gamma_1^T$	Time update equations
$\hat{x}(k) = \Phi \bar{x}(k) + K_0(k)[y(k) - C\bar{x}(k)]$ $K_0(k+1) = P(k+1)C^T R_v^{-1}$ $P(k+1) = M(k+1) - M(k+1)C^T \times [CM(k+1)C^T + R_v]^{-1} CM(k+1)$	Measurement update equations

plant using a Kalman filter estimator is shown in Fig. 2.

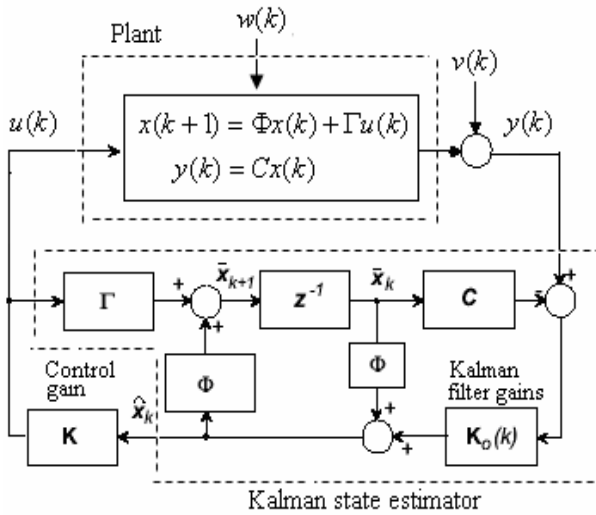


Fig. 2 Control of a plant with a Kalman filter based observer

In an actual design problem, meaningful values can be assigned to  $R_v$ , which is based on sensor noise which can often be found from the specifications. The same cannot be said of the process noise, which is often a mathematical artifice that is used to expedite the optimization. Physically,  $R_w$  is used to account for unknown disturbances and uncertainties in the plant model. The disturbance noise model should be chosen to approximate any known disturbances, but the designer is often forced to use "acceptable" values based on simulation studies.

### III. EQUATIONS OF AIRCRAFT MOTION

Fig. 3 shows the origin of the axes is at the aircraft's center

of gravity. The  $x$  axis is along the fuselage, the  $y$  axis is along the wingspan, and the  $z$  axis points downward.

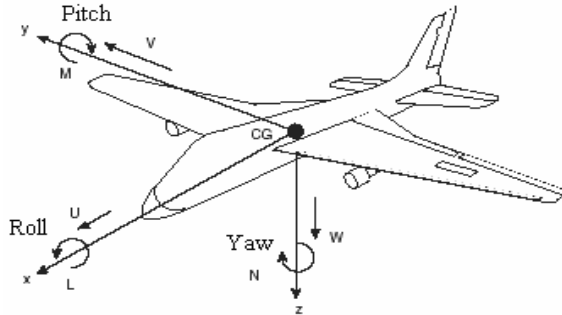


Fig. 3 Definition of the aircraft's axis system

The rigid body equations of motion are the differential equations that describe the evolution of the basic states of an aircraft. These equations of motion are all nonlinear first order ordinary differential equations. In addition they are highly coupled, i.e., each differential equation depends upon variables. However, we may gain some insight into the equations of motion by examining in steady state solutions, which then are in the matrix form:  $\dot{X} = A X + B U$

Where

$$X^T = [\beta \ p \ r \ \phi] : \text{state vector}$$

$$U^T = [\delta_a \ \delta_r] : \text{control vector}$$

$\delta_a, \delta_r$ : aileron and rudder deflection

$\beta, \phi$ : sideslip and roll angle

$p, r$ : roll and yaw rate

$$A = \begin{pmatrix} \frac{Y_\beta}{V_0} & \left( \frac{Y_p}{V_0} + \sin(\theta_0) \right) & \left( \frac{Y_r}{V_0} - \cos(\theta_0) \right) & \frac{g}{V_0} \cos(\theta_0) \\ \frac{L_\beta + aN_\beta}{\Delta} & \frac{L_p + aN_p}{\Delta} & \frac{L_r + aN_r}{\Delta} & 0 \\ \frac{N_\beta + aL_\beta}{\Delta} & \frac{N_p + aL_p}{\Delta} & \frac{N_r + aL_r}{\Delta} & 0 \\ 0 & 1 & \text{tg}(\theta_0) & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{Y_{\delta_r}}{V_0} & \frac{Y_{\delta_a}}{V_0} \\ \frac{L_{\delta_r} + aN_{\delta_r}}{\Delta} & \frac{L_{\delta_a} + aN_{\delta_a}}{\Delta} \\ \frac{N_{\delta_r} + bL_{\delta_r}}{\Delta} & \frac{N_{\delta_a} + bL_{\delta_a}}{\Delta} \\ 0 & 0 \end{pmatrix}$$

Where  $a = \frac{I_{xz}}{I_{xx}}$ ,  $b = \frac{I_{xz}}{I_{zz}}$  and  $\Delta = 1 - a \times b$

## IV. APPLICATION TO CESSNA-182 AIRCRAFT

The CESSNA-182 was introduced in 1956, as a tricycle gear variant of the 180. In 1957, the name was changed to the 182A and the name Skylane was introduced. The characteristics of CESSNA-182 are presented [9], [11] on Table II and having the following lateral factors of stability represented on Table III.

TABLE II  
GEOMETRIC CHARACTERISTICS

Description	Values for CESSNA-182
Wing area	174.00 sq. ft
Wight	2645.00 Ibs
Wing span	35.80 ft
Mean aerody. chord	4.90 ft
Air speed	219.00 ft/sec
Air density	0.00205 slugs/ cu. ft
Initial theta	0.00 rad
High	5000 ft
$x_{cg}$	0.25
$I_{yy}$	1346 slugs. sq. ft
$I_{xx}$	948 slugs. sq. ft
$I_{zz}$	1967 slugs. sq. ft
$I_{xz}$	0.00 slugs. sq. ft

TABLE III  
LATERAL FACTORS OF STABILITY

Quantity	Values for CESSNA-182
$L_{\beta}$	-28.7492 / rad. sec <sup>2</sup>
$L_p$	-12.4092 / rad. sec
$L_r$	2.5346 / rad. sec
$L_{\delta a}$	57.4984 / rad. sec <sup>2</sup>
$L_{\delta r}$	4.7485 / rad. sec <sup>2</sup>
$N_{\beta}$	10.1194 / rad. sec <sup>2</sup>
$N_p$	-0.3817 / rad. sec
$N_r$	-1.2597 / rad. sec
$N_{\delta a}$	-8.2512 / rad. sec <sup>2</sup>
$N_{\delta r}$	-10.2284 / rad. sec <sup>2</sup>
$Y_{\beta}$	-32.2554 ft / rad. sec <sup>2</sup>
$Y_p$	-0.3147 ft / rad. sec
$Y_r$	1.7859 ft / rad. sec
$Y_{\delta a}$	0.0000 ft / rad. sec <sup>2</sup>
$Y_{\delta r}$	19.4730 ft / rad. sec <sup>2</sup>

## V. SIMULATION AND RESULTS

If we assume that the measurable outputs are the sideslip angle  $\beta$  and roll angle  $\phi$ , the matrixes  $A$ ,  $B$  and  $C$  are:

$$A = \begin{bmatrix} -0.1473 & -0.0014 & -0.9918 & 0.1498 \\ -28.749 & -12.409 & 2.5346 & 0 \\ 10.119 & -0.3817 & -1.2597 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0886 & 0 \\ 4.7485 & 57.498 \\ 10.228 & -8.2512 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Can solve for the eigenvalues of the matrix  $A$  to find the modes of the system:

$$\lambda_1 = -0.0112 \quad \Rightarrow \quad \text{Spiral Mode}$$

$$\lambda_2 = -12.4341 \quad \Rightarrow \quad \text{Roll Damping}$$

$$\lambda_{3,4} = -0.6855 \pm 3.3073i \quad \Rightarrow \quad \text{Dutch Roll}$$

Stable with 3 modes, and after converting to a discrete model, we have:

$$\Phi = \begin{bmatrix} 0.32598 & 0.0092028 & -0.21875 & 0.044601 \\ -0.72234 & -0.012649 & 0.52021 & -0.075613 \\ 2.4234 & -0.0033903 & 0.077197 & 0.093014 \\ -0.50476 & 0.076212 & 0.13779 & 0.98496 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} -0.54925 & 0.57559 \\ 1.7265 & 3.2451 \\ 2.3998 & -2.2492 \\ 0.32196 & 1.2859 \end{bmatrix}$$

The following MATLAB script takes the plant model, discretizes it using a ZOH discrete model, chooses LQR gains, forms a Kalman filter observer for state estimation, combines the plant, controller, and observer models and plots some initial condition responses of the closed-loop system.

For the weighting matrices,

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Design the regulator by computing the LQR Gain matrix  $K$ ,

$$K = \begin{bmatrix} 0.6421 & 0.0071 & 0.0027 & 0.1166 \\ -0.1655 & 0.0121 & -0.0679 & 0.1171 \end{bmatrix}$$

Compute the Kalman filter gains  $K_0$ , the process noise  $w(k)$  and measurement noise  $v(k)$  are white Gaussian random sequences with zero mean and have the following covariances matrices as:

$$R_w = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \quad R_v = \begin{bmatrix} 0.0100 & 0 \\ 0 & 0.0100 \end{bmatrix}$$

The Kalman filter gain: 
$$K_0 = \begin{bmatrix} 1.2190 & 0.0513 \\ -2.8578 & -0.0629 \\ 2.1032 & 0.0918 \\ -1.1702 & 1.2330 \end{bmatrix}$$

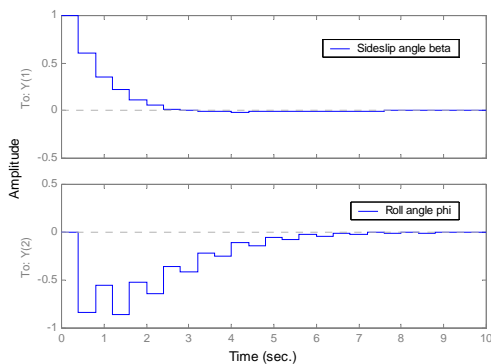


Fig. 4 Initial condition response of the aircraft

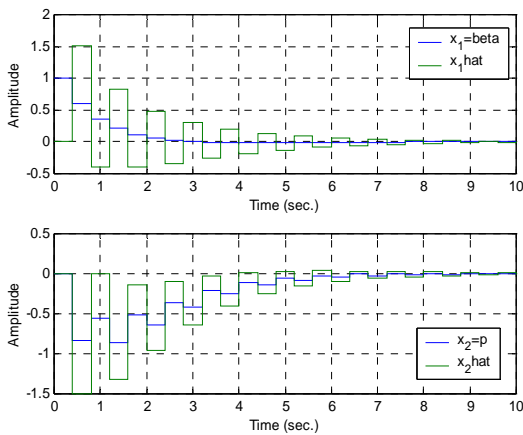


Fig. 5 Response of states and predictive estimates

VI. CONCLUSION

The Kalman filter estimates a process by using a form of feedback control: the filter estimates the process state at some time and then obtains feedback in the form of (noisy) measurements. As such, the equations for the Kalman filter fall into two groups: *time update* equations and *measurement*

*update* equations. The time update equations are responsible for projecting forward (in time) the current state and error covariance estimates to obtain the a priori estimates for the next time step. The measurement update equations are responsible for the feedback i.e. for incorporating a new measurement into the a priori estimate to obtain an improved a posteriori estimate.

An application for lateral motion of aircraft was presented to show that phenomena such as a limited steady shift error (SSE) could occur. To eliminate some of the undesirable phenomena, we suggested a good choosing of the noise covariance data that is dominantly rich to eliminate the SSE.

Finally, the LQG gives a very good following to the outputs of plant with a steady shift error limited and the Kalman filter is an optimal estimator when dealing with Gaussian white noise. Optimal estimation provides an alternative rationale for the choice of observer gains in the current estimator which is based on observer performance in the presence of process noise and measurement errors.

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