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Using Fuzzy Numbers in Heavy Aggregation Operators

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***Abstract*—**We consider different types of aggregation operators such as the heavy ordered weighted averaging (HOWA) operator and the fuzzy ordered weighted averaging (FOWA) operator. We introduce a new extension of the OWA operator called the fuzzy heavy ordered weighted averaging (FHOWA) operator. The main characteristic of this aggregation operator is that it deals with uncertain information represented in the form of fuzzy numbers (FN) in the HOWA operator. We develop the basic concepts of this operator and study some of its properties. We also develop a wide range of families of FHOWA operators such as the fuzzy push up allocation, the fuzzy push down allocation, the fuzzy median allocation and the fuzzy uniform allocation.

***Keywords*—**Aggregation operators, Fuzzy numbers, Fuzzy OWA operator, Heavy OWA operator.

I. Introduction

THE ordered weighted averaging (OWA) operator is a very common method for aggregating the information. It was introduced in [1] and since its appearance it has been used in a wide range of applications [2] – [20]. One of its main characteristics is that it provides a parameterized family of aggregation operators that includes among others, the maximum, the minimum and the average criteria.

In [18], Yager introduced a new extension of the OWA operator called the heavy ordered weighted averaging (HOWA) operator. The main characteristic of this operator is that it provides a parameterized family of aggregation operators that includes among others, the minimum, the OWA operator and the total operator. As we can see, this operator allows the weighting vector to range between the OWA operator and the total operator. This extension has also been studied in [7], [19].

Sometimes, the available information is uncertain and cannot be assessed with exact numbers. Then, it is necessary to use another approach to represent the information. A very useful approach for representing the uncertain information is the use of fuzzy numbers (FN) in the problem. The FN were introduced in the works of Chang and Zadeh [21], [22]. Since then, the FN have been studied by different authors [23] – [29]. Among the wide range of FN existing in the literature, we could mention for example, the triangular FN, trapezoidal FN, L-R FN, intuitionistic FN, interval-valued FN, etc. With this background, we can see that sometimes it is better to use the OWA operator with FN. Then, we need to use the fuzzy ordered weighted averaging (FOWA) operator. This operator has been studied by different authors such as in [3] – [5], [8], [10].

Going a step further, we can see that sometimes, the HOWA operator can also be affected by uncertain situations that need to be assessed with FN. Due to this, in this paper we suggest the use of FN in the HOWA operator. For doing so, we develop a new extension of the OWA operator called the fuzzy heavy ordered weighted averaging (FHOWA) operator. The main characteristic of this operator is that it is able to deal with uncertain information in the HOWA operator. Then, it can provide a wider class of aggregation operators by allowing the weighting vector to range from the FOWA operator to the fuzzy total operator. We will study the main concepts of this new extension and we will develop a wide range of particular cases such as the fuzzy push up allocation, the fuzzy push down allocation, the fuzzy median allocation, the fuzzy uniform allocation, etc.

In order to do so, this paper is organized as follows. In Section II we review some aggregation operators such as the FOWA and the HOWA operator. In Section III we introduce the FHOWA operator and in Section IV we develop different families of FHOWA operators. Finally, in Section V we summarize the main conclusions found in the paper.

II. Preliminaries

A. Fuzzy OWA Operator

The FOWA operator has been studied in [3] – [5], [8], [10] and it represents an extension of the OWA operator. Essentially, its main difference is that it uses uncertain information in the arguments of the OWA operator represented in the form of FN. The reason for using this aggregation operator is that sometimes the available information cannot be assessed with exact numbers and it is necessary to use other techniques such as FN. The FOWA operator provides a parameterized family of aggregation operators that include the

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fuzzy maximum, the fuzzy minimum and the fuzzy average criteria, among others.

Definition 1. Let Ψ be the set of FN. A FOWA operator of dimension n is a mapping $FOWA: \Psi^n \to \Psi$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$FOWA(\tilde{a}_1, \tilde{a}_2 \ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j b_j \quad (1)$$

where b_j is the jth largest of the $\tilde{a}_i$, and the $\tilde{a}_i$ are FN. Among others, we could mention as examples of FN, the triangular FN, the trapezoidal FN, the L-R FN, the interval-valued FN, the intuitionistic FN, etc. For further information of FN, see for example [21] – [29].

Note that it is also possible to use FN in the weighting vector of the FOWA operator. The motivation for doing so is because sometimes it is not clear the attitudinal character of the decision maker and he prefers to use different degrees of optimism or pessimism in order to take the decision. Due to the fact that this problem has a lot of internal problems such as the problem that the sum of the weights is not exactly one, etc., we will not consider this situation here.

Note also that sometimes, it is not clear how to reorder the arguments. Then, it is necessary to establish a criterion for comparing FN. For simplicity, we recommend to follow the policy explained in [26], [27].

From a generalized perspective of the reordering step, we have to distinguish between the descending FOWA (DFOWA) operator and the ascending FOWA (AFOWA) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DFOWA and w^*_{n-j+1} the jth weight of the AFOWA operator.

The FOWA operator is commutative, monotonic, bounded and idempotent. Different families of FOWA operators can be obtained by choosing a different manifestation in the weighting vector such as the step-FOWA operator, the window-FOWA operator, the FOWA median operator, the S-FOWA, the centered-FOWA operator, etc. Further information on these families can be found in [5].

Another interesting issue to consider is the measures for characterizing the weighting vector of the FOWA operator and the type of aggregation it performs. Among others, we can consider the attitudinal character, the entropy of dispersion, the divergence of W and the balance operator. The first measure, the attitudinal character [1], is defined as:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} \right) \quad (2)$$

It can be shown that $\alpha \in [0, 1]$. The more of the weight located toward the bottom of W, the closer α to 0 and the more of the weight located near the top of W, the closer α to 1. Note that for the maximum $\alpha(W) = 1$, for the minimum $\alpha(W) = 0$, and for the average criteria $\alpha(W) = 0.5$.

The second measure introduced also in [1], is called the entropy of dispersion of W and it is used to provide a measure of the information being used. It is defined as:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \quad (3)$$

That is, if $w_j = 1/n$ for all j, then $H(W) = \ln n$, and the amount of information used is maximum. If $w_j = 1$ for some j, known as step-FOWA [5], [14], then $H(W) = 0$, and the least amount of information is used.

The third measure was introduced in [18], it is called the divergence of W and it is useful in some exceptional situations. It is defined as:

$$Div(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W) \right)^2 \quad (4)$$

Finally, a fourth measure that could be used for the analysis of the weighting vector W is the balance operator [16]. It is useful to analyse the balance between favouring the arguments with high values or the arguments with low values. It can be defined as follows.

$$BAL(W) = \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1} \right) w_j \quad (5)$$

It can be shown that $BAL(W) \in [-1, 1]$. Note that for the maximum we get $BAL(W) = 1$, for the minimum, $BAL(W) = -1$ and for the average criteria, $BAL(W) = 0$. Also note that for the median and the olympic average, $BAL(W) = 0$. For the Arrow-Hurwicz aggregation, assuming that the usual aggregation of this method is $\lambda \text{Max}\{a_i\} + (1 - \lambda)\text{Min}\{a_i\}$, $BAL(W) = 2\lambda - 1$. As it can be shown, for an optimistic situation, where $\lambda > 0.5$, the balance is positive and for a pessimistic situation, where $\lambda < 0.5$, the balance is negative.

Note that it is also possible to study these measures with the AFOWA operator. The measures are equal with the only different that now the reordering is ascendant. That is, the weights of both orderings are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DFOWA and w^*_{n-j+1} the jth weight of the AFOWA operator.

B. Heavy OWA Operator

The Heavy OWA operator was introduced in [18] and it represents an extension to the OWA operator. The motivation for using this operator is because there are situations where the available information is independent from each other and this aspect needs to be considered in the aggregation. In this case, the difference with the OWA operator is that the sum of the weights is allowed to be between 1 and n instead of being restricted to sum up to 1. With this, we get a wider class of aggregation operators that include mean operators and

totalling operators. In the following, we provide a definition of the HOWA operator as suggested by Yager [18].

Definition 2. A Heavy OWA operator of dimension n is a mapping $HOWA$: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and the sum of the weights is between [1, n], then:

$$HOWA(a_1, a_2,\ldots, a_n) = \sum_{j=1}^{n} w_j b_j \quad (6)$$

where b_j is the jth largest of the a_i.

From a generalized perspective of the reordering step, we have to distinguish between the Descending HOWA (DHOWA) operator and the Ascending HOWA (AHOWA) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DHOWA and w^*_{n-j+1} the jth weight of the AHOWA operator.

The HOWA operator is monotonic and commutative both for the DHOWA and the AHOWA operator. It is monotonic because if $a_i \geq d_i$, for all i, then, $HOWA(a_1,\ldots, a_n) \geq HOWA(d_1,\ldots, d_n)$. It is commutative because any permutation of the arguments has the same evaluation. Note that the HOWA operator is not bounded by the minimum and the maximum. In this case, it is bounded by the minimum and the total operator which represents the sum of all the arguments.

By choosing a different manifestation of the weighting vector, we are able to obtain different types of aggregation operators. For example [18], the OWA operator is found when the sum of the weights is one. The total operator is found when the sum of the weights is n. The minimum is found when $w_n = 1$, $w_j = 0$ for all $j \neq n$ and the sum of the weights is one. For obtaining the maximum and the average criteria, we could find it from different aggregations as the weighting vector can be higher than 1. We should note that these results could also be obtained for the AHOWA operators.

Another interesting issue to comment is the sum of the elements of the weighting vector W that is denoted by Yager [18] as $|W|$ and it is called the magnitude of W. In order to normalize this feature, Yager introduced a characterizing parameter called the beta value of the vector W. It was defined as $\beta(W) = (|W| - 1) / (n - 1)$. Since $|W| \in [1, n]$, then, $\beta \in [0, 1]$. As it can be seen, if $\beta = 1$, we get the total operator and if $\beta = 0$, we get the usual OWA operator. Note that it is possible to look to the negation of β [18]. Then, $\rho = 1 - \beta$. In this case, if $\rho = 0$, we get the total operator and if $\rho = 1$, we get the usual OWA operator.

Once analysed the magnitude of W, it is possible to study the measures used for characterizing the weighting vector of the HOWA operator. The first measure, the attitudinal character, can be defined as:

$$\alpha(W) = \frac{1}{|W|}\sum_{j=1}^{n}\left(\frac{n-j}{n-1}\right) w_j \quad (7)$$

As it can be seen, $\alpha(W) \in [0, 1]$. Note that the total operator has $\alpha(W) = 0.5$.

The second measure, the entropy of dispersion, is defined as:

$$H(W) = -\frac{1}{|W|}\sum_{j=1}^{n} w_j \ln\left(\frac{w_j}{|W|}\right) \quad (8)$$

Note that for the total operator, $H(W) = -\ln n$.

The third measure that could be introduced for characterizing the weighting vector of the HOWA operator, is the divergence of W. It can be defined as:

$$Div(W) = \frac{1}{|W|}\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2 \quad (9)$$

Note that if $|W| = n$, we get the divergence for the total operator and it is the same divergence than the average. That is, $Div(W) = (1/12)[(n + 1)/(n - 1)]$.

Finally, the fourth measure, the balance operator, could be defined in this case as follows:

$$BAL(W) = \frac{1}{|W|}\sum_{j=1}^{n}\left(\frac{n+1-2j}{n-1}\right) w_j \quad (10)$$

It can be shown that $BAL(W) \in [-1, 1]$. In this case, if $|W| = n$, we get the balance for the total operator.

Note also that these four measures are reduced to the usual definitions of the OWA and the FOWA operator when $|W| = 1$.

By using the AHOWA operator, it is also possible to obtain these four measures. It is straightforward to obtain the ascending version of these measures by looking to the descending one and assuming that the weights of the DHOWA and the AHOWA are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DHOWA and w^*_{n-j+1} the jth weight of the AHOWA operator.

III. The Fuzzy Heavy OWA Operator

The fuzzy heavy OWA (FHOWA) operator represents an extension to the OWA operator. It consists in using in the same operator the characteristics of the FOWA operator with the characteristics of the HOWA operator. Then, the same operator will use uncertain information represented in the form of FN with a weighting vector that ranges from the FOWA operator to the fuzzy total operator. It can be defined as follows.

Definition 3. Let Ψ be the set of FN. A FHOWA operator of dimension n is a mapping $FHOWA: \Psi^n \rightarrow \Psi$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and the sum of the weights is between [1, n], then:

$$FHOWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j b_j \quad (11)$$

where b_j is the jth largest of the $\tilde{a}_i$, and the $\tilde{a}_i$ are FN. Among others, we could mention as examples of FN to be used in the FHOWA aggregation, the triangular FN, the trapezoidal FN, the L-R FN, the interval-valued FN, the intuitionistic FN, etc. For further information on FN, see for example [24] – [25].

Note that in this case, it is also possible to use FN in the weighting vector of the FHOWA operator. The motivation for doing so is the same as in the FOWA operator. That is, because sometimes it is not clear the attitudinal character of the decision maker and he prefers to use different degrees of optimism or pessimism in order to take the decision.

Note also that sometimes, it is not clear how to reorder the arguments. Then, it is necessary to establish a criterion for comparing FN. For simplicity, we recommend to follow the policy explained in [26] – [27]. Note that other methods could be used for comparing FN.

Analysing the reordering step, we see that we have to distinguish between the Descending FHOWA (DFHOWA) operator and the Ascending FHOWA (AFHOWA) operator. The DFHOWA operator is defined as in definition 3.

Definition 4. Let Ψ be the set of FN. An AFHOWA operator of dimension n is a mapping $FHOWA: \Psi^n \rightarrow \Psi$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and the sum of the weights is between [1, n], then:

$$AFHOWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j b_j \quad (12)$$

where b_j is the jth lowest of the $\tilde{a}_i$, and the $\tilde{a}_i$ are FN. As it can be seen, the only difference against the DFHOWA operator is that the elements b_j (j = 1, 2, …, n) are ordered in an increasing way: $b_1 \leq b_2 \leq \ldots \leq b_n$. Then, it is easy to see that the weights of these two operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DFHOWA and w^*_{n-j+1} the jth weight of the AFHOWA operator.

If we want to use a vector notation in the FHOWA operator, we could assume that B is a vector corresponding to the ordered arguments b_j and we should call it the ordered argument vector. We should also assume that W^T is the transpose of the weighting vector, then, we could express the FHOWA operator both for the DFHOWA and the AFHOWA operator as:

$$FHOWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = W^T B \quad (13)$$

The FHOWA operator is monotonic and commutative both for the DFHOWA and the AFHOWA operator. It is monotonic because if $\tilde{a}_i \geq \tilde{e}_i$ for all i, then, $FHOWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \geq FHOWA(\tilde{e}_1, \ldots, \tilde{e}_n)$. It is commutative because any permutation of the arguments has the same evaluation. That is, $FHOWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = FHOWA(\tilde{e}_1, \ldots, \tilde{e}_n)$, where $(\tilde{e}_1, \ldots, \tilde{e}_n)$ is any permutation of the arguments $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$. Note that this operator is also bounded by the minimum and the total operator.

In this case, it is also interesting to analyse the magnitude of the weighting vector $|W|$. Following the same methodology than the HOWA operator, we can define the magnitude $|W|$ of the FHOWA operator as $\beta(W) = (|W| - 1) / (n - 1)$. Since $|W| \in [1, n]$, then, $\beta \in [0, 1]$. As it can be seen, if $\beta = 1$, we get the fuzzy total operator and if $\beta = 0$, we get the usual FOWA operator. In the FHOWA operator, it is also possible to look to the negation of β. Then, $\rho = 1 - \beta$. If $\rho = 0$, we get the fuzzy total operator and if $\rho = 1$, we get the usual FOWA operator.

As it has been explained in the HOWA operator, once analysed the magnitude of $|W|$, it is possible to study the measures used for characterizing the weighting vector. For the FHOWA operator, we get the following. The first measure, the attitudinal character, can be defined as:

$$\alpha(W) = \frac{1}{|W|} \sum_{j=1}^{n} \left(\frac{n-j}{n-1} \right) w_j \quad (14)$$

As it can be seen, $\alpha(W) \in [0, 1]$.

Note that the formulation is the same than the HOWA operator because the fuzzy arguments do not affect the result. Also note that the fuzzy total operator has $\alpha(W) = 0.5$.

The second measure, the entropy of dispersion, can be defined as:

$$H(W) = -\frac{1}{|W|} \sum_{j=1}^{n} w_j \ln\left(\frac{w_j}{|W|} \right) \quad (15)$$

Note that for the fuzzy total operator, $H(W) = -\ln n$.

For the third measure, the divergence of W, we will use:

$$Div(W) = \frac{1}{|W|} \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W) \right)^2 \quad (16)$$

If $|W| = n$, we get the divergence of the fuzzy total operator and it is the same divergence than the fuzzy average. That is, $Div(W) = (1/12)[(n + 1)/(n - 1)]$.

Finally, the fourth measure, the balance operator, could be defined in this case as:

$$BAL(W) = \frac{1}{|W|} \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1} \right) w_j \quad (17)$$

It can be shown that $BAL(W) \in [-1, 1]$. In this case, if $|W| = n$, we get the balance for the fuzzy total operator.

Also note that these four measures are reduced to the usual definitions shown in Section II.A when $|W| = 1$.

These three measures could also be studied with the AUHOWA operator. Then, for the first measure we get:

$$\alpha(W) = \frac{1}{|W|}\sum_{j=1}^{n}\left(\frac{j-1}{n-1}\right) w_j \tag{18}$$

As it can be seen, $\alpha(W) \in [0, 1]$. Note that the fuzzy total operator has also $\alpha(W) = 0.5$.

For the second measure, the result is the same as with DFHOWA operators although the reordering step is different.

For the third measure, we get:

$$Div(W) = \frac{1}{|W|}\sum_{j=1}^{n} w_j\left(\frac{j-1}{n-1} - \alpha(W)\right)^2 \tag{19}$$

If $|W| = n$, we get the divergence for the uncertain total operator and it is the same divergence than the average. Also note that if $w_k = 1$ and $w_j = 0$ for all $j \neq k$, the $Div(W) = 0$.

And for the balance operator, we get:

$$BAL(W) = \frac{1}{|W|}\sum_{j=1}^{n}\left(\frac{2j-1-n}{n-1}\right) w_j \tag{20}$$

It can be shown that $BAL(W) \in [-1, 1]$. In this case, we also get the balance for the fuzzy total operator if $|W| = n$.

IV. Families of FHOWA Operators

By using a different manifestation of the weighting vector, we are able to obtain different families of FHOWA operators. For example, we can obtain the FOWA operator, the fuzzy total operator, the fuzzy weighted average and the fuzzy minimum. The FOWA operator is obtained when $\beta = 0$. The fuzzy total operator is found when $\beta = 1$. The fuzzy weighted average is obtained when the ordered position of the b_j is the same than the ordered position of the a_i and $\beta = 0$. Finally, the fuzzy minimum is found when $w_n = 1$, $w_j = 0$, for all $j \neq n$ and $\beta = 0$.

Following the same methodology that Yager [18] used for the HOWA operator, we can develop another group of particular cases of the FHOWA operator.

The first type of FHOWA operator we will study is the fuzzy push up allocation. In this case, we get $w_j = (1 \wedge (|W| - (j - 1))) \vee 0$. Note that if $\beta = 0$, $W_{pu} = W^*$, $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$, then, $\alpha(W_{pu}) = \alpha(W^*) = 1$. If $\beta = 1$, $W_{pu} = W_T$, $w_j = 1$ for all j and $\alpha(W_{pu}) = \alpha(W_T) = 0.5$.

The second type of FHOWA operator we will study is the dual allocation to the push up. This type is known as the fuzzy push down allocation and it is defined as $w_{n-j+1} = (1 \wedge (|W| - (j - 1))) \vee 0$. Note that if $\beta = 0$, $W_{pd} = W_*$, $w_n = 1$ and $w_j = 0$ for all $j \neq$ n, then, $\alpha(W_{pd}) = \alpha(W_*) = 0$. If $\beta = 1$, $W_{pd} = W_T$, $w_j = 1$ for all j and $\alpha(W_{pd}) = \alpha(W_T) = 0.5$.

Another special allocation that it is possible to use in the FHOWA operator is the fuzzy median type allocation. In this case, we have to distinguish between the case when n is even or odd. If n is even, we allocate the weights for $j = 1$ to a as $w_{a+j} = w_{a+1-j} = [1 \wedge ((|W| - 2(j - 1))/2)] \vee 0$. If n is odd, we allocate the weights for $j = 1$ to a as $w_{a+1} = 1$ and $w_{a+1-j} = w_{a+1+j} = [1 \wedge ([(|W| - 1) - 2(j - 1)]/2)] \vee 0$. As the weighting vector is symmetric, $\alpha(W) = 0.5$. Note that if $\beta = 0$, we get the FOWA median and if $\beta = 1$, we get the fuzzy total operator.

The next type of allocation we will study is the step-FHOWA operator. In this case, the weighting vector is focused at the Kth largest element. That is, assuming $b = \text{Min}[(K - 1), (n - K)]$, we allocate the weights for j to b as $w_K = 1$ and $w_{K+j} = w_{K-j} = [1 \wedge ([(|W| - 1) - 2(j - 1)]/2)] \vee 0$. If $b = K - 1$, $K - 1 < n - K$, then, $w_{j+2K-1} = [1 \wedge ((|W| - (1 + 2b)) - (j - 1))] \vee 0$, for $j = 1$ to $n - 2K + 1$. If $b = n - K$, $K - 1 > n - K$, then, $w_{2K-n-j} = [1 \wedge ((|W| - (1 + 2b)) - (j - 1))] \vee 0$, for $j = 1$ to $n - 2K + 1$. Note that if $K = 1$, the step-FHOWA becomes the fuzzy push up allocation, for $K = n$, the step-FHOWA becomes the fuzzy push down allocation and for $K = (n + 1)/2$, it becomes the fuzzy median allocation.

Another possible allocation for the FHOWA operator is the fuzzy uniform allocation. In this case, we assign the weights as $w_j = |W|/n$ for all j. In this allocation we always find a neutral attitudinal character $\alpha(W) = 0.5$. Note that if $\beta = 0$, we get the fuzzy arithmetic mean.

A further special allocation for the FHOWA operator is the fuzzy olympic average allocation. In this type of allocation, we have to distinguish between two cases. In the first case $|W| < n - 2m$, we allocate the weight as $w_j = |W|/(n - 2m)$ for $j = m + 1$ to $n - m$, and $w_j = 0$ for $j = 1$ to m and for $j = n - m + 1$ to n. In the second case, where $|W| \geq n - 2m$, we allocate the weights as $w_j = 1$ for $j = m + 1$ to $n - m$ and $w_{m+1-j} = w_{n-m+j} = [1 \wedge ([(|W| - (n - 2m)) - 2(j - 1)]/2)] \vee 0$ for $j = 1$ to m.

Finally, the last type of allocation we will consider for the FHOWA operator is the Arrow-Hurwicz aggregation [30]. Assuming that $|W| = q$ and dimension n, we define the weights in two directions, push up and push down. First, we calculate $\omega_j = (1 \wedge (\lambda q - (j - 1))) \vee 0$ for $j = 1$ to n and $\hat{w}_{n-j+1} = (1 \wedge ((1 - \lambda)q - (j - 1))) \vee 0$ for $j = 1$ to n. Then, we define the weights as $w_i = \omega_i + \hat{w}_i$. Note that $\omega_j = 0$ for all $j \geq \lambda q + 1 \geq \lambda|W| + 1$ and $\hat{w}_j = 0$ for $j \leq n - (1 - \lambda)q \leq n - |W| + \lambda|W|$.

If we use a similar methodology for the AFHOWA operator as it has been shown above for the DFHOWA operator, we can obtain a wide range of special cases of AFHOWA operators. For example, we could analyse the AFOWA operator, the fuzzy total operator, the fuzzy weighted average and the fuzzy minimum criteria. The AFOWA operator is found when $\beta = 0$. The fuzzy total operator is found when $\beta = 1$. The fuzzy weighted average is obtained when the ordered position of the b_j is the same than the ordered position of the a_i and $\beta = 0$.

Finally, the fuzzy minimum is found when $w_1 = 1$, $w_j = 0$, for all $j \neq 1$ and $\beta = 0$.

Another group of AFHOWA operators that could be obtained are the fuzzy push up allocation, the fuzzy push down allocation, the fuzzy median type, the step-AFHOWA type, the fuzzy uniform allocation, the olympic AFHOWA average and the Arrow-Hurwicz AFHOWA aggregation. Note that the formulation of these families is very similar to their corresponding descending version with the only difference that now the reordering is ascendant. Therefore, the weights of the ascending families are related to the descending families by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DFHOWA and w^*_{n-j+1} the jth weight of the AFHOWA operator.

V. Conclusion

In this paper we have developed a new extension of the OWA operator. We have called it the fuzzy heavy OWA (FHOWA) operator. Basically, this operator consists in using uncertain information represented in the form of FN in situations where the weighting vector can range from the usual FOWA operator to the fuzzy total operator. Focusing on the HOWA operator, we could say that we have considered situations of the HOWA operator where the available information is uncertain and can be assessed with FN.

We have studied this new operator giving its definition and studying some of its main properties such as the distinction between descending and ascending orders. We have also developed a wide range of families of FHOWA operators such as the fuzzy push up allocation, the fuzzy push down allocation, the fuzzy median allocation, the fuzzy uniform allocation, etc.

In future research, we expect to develop new extensions to the FHOWA operator by considering other characteristics in the aggregation such as the possibility of using FN in the weighting vector of the FHOWA operator.

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