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Unique Positive Solution of Nonlinear Fractional Differential Equation Boundary Value Problem

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Abstract—By using two new fixed point theorems for mixed monotone operators, the positive solution of nonlinear fractional differential equation boundary value problem is studied. Its existence and uniqueness is proved, and an iterative scheme is constructed to approximate it.

Keywords—Fractional differential equation, boundary value problem, positive solution, existence and uniqueness, fixed point theorem, mixed monotone operator.

I. Introduction

FRACTIONAL differential equations are used in various fields,see [1]-[7]. In recent decades, people depth study a variety of boundary value problems for fractional differential equations, and have achieved important results, see [8]-[13].

In particular, by using contraction map principle and some Lipschitz-type conditions, Zhanbing Bai [9] investigated the existence and uniqueness of positive solutions for a nonlocal boundary value problem of fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, 0<t<1, \\ u(0)=0, \beta u(\eta)=u(1), \end{cases} \tag{1}$$

where $1<\alpha\leq 2, 0<\beta\eta^{\alpha-1}<1, 0<\eta<1, D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order α and the function f is continuous on $[0,1]\times[0,\infty)$.

Inspired by the above literature, we study the existence and uniqueness of positive solutions for the following problem:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),u(t))+g(t,u(t))=0, 0<t<1, \\ u(0)=0, \beta u(\eta)=u(1), \end{cases} \tag{2}$$

where $1<\alpha\leq 2, 0<\beta\eta^{\alpha-1}<1, 0<\eta<1, D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order α and $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous and $g:[0,1]\times[0,+\infty)\to[0,+\infty)$ is also continuous.

Reference [9] gived the Green function for the problem (2), this paper gets good properties of the Green function. By means of two new fixed point theorems for mixed monotone operators, we obtain the existence and uniqueness of positive solutions for the problem (2).

II. Preliminaries and previous results

In this section, we present some definitions, lemmas and basic results that will be used in the proofs of our main results.

Definition 1 [4] The integral

$$I_{0+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}\mathrm{d}t, x>0$$

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is called the Riemann-Liouville fractional integral of order α, where$\alpha>0$ and $\Gamma(\alpha)$denotes the gamma function.

Definition 2 [4] For a function $f(x)$ given in the interval $[0,\infty)$, the expression

$$D_{0+}^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}x})^n\int_0^x\frac{f(t)}{(x-t)^{\alpha-n+1}}\mathrm{d}t,$$

is called the Riemann-Liouville fractional derivative of order α, where $n=[\alpha]+1, [\alpha]$ denotes the integer part of number α.

Lemma 1 [9] Let $y\in C[0,1]$ and $1<\alpha\leq 2$, the unique solution of the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t)+y(t)=0, 0<t<1, \\ u(0)=0, \beta u(\eta)=u(1), \end{cases} \tag{3}$$

is

$$u(t)=\int_0^1 G(t,s)y(s)\mathrm{d}s, \ \ t\in[0,1]$$

where

$$q\Gamma(\alpha)G(t,s) = \begin{cases} [t(1-s)]^{\alpha-1}-\beta t^{\alpha-1}(\eta-s)^{\alpha-1}-q(t-s)^{\alpha-1}, \\ \qquad 0\leq s\leq t\leq 1, \ s\leq\eta; \\ [t(1-s)]^{\alpha-1}-q(t-s)^{\alpha-1}, \\ \qquad 0<\eta\leq s\leq t\leq 1; \\ [t(1-s)]^{\alpha-1}-\beta t^{\alpha-1}(\eta-s)^{\alpha-1}, \\ \qquad 0\leq t\leq s\leq\eta<1; \\ [t(1-s)]^{\alpha-1}, \\ \qquad 0\leq t\leq s\leq 1, \eta\leq s. \end{cases} \tag{4}$$

where $q=1-\beta\eta^{\alpha-1}$. Here $G(t,s)$ is called the Green function of boundary value problem (2) and $G(t,s)>0, \forall t,s\in(0,1)$.

Lemma 2 [9] Green function $G(t,s)$ in Lemma 1 has the following property:

$$\begin{aligned} &t^{\alpha-1}[(1-q)(1-s)^{\alpha-1}-\beta(\eta-s)^{\alpha-1}] \\ &\leq q\Gamma(\alpha)G(t,s) \\ &\leq t^{\alpha-1}(1-s)^{\alpha-1}, \forall t,s\in(0,1) \end{aligned} \tag{5}$$

Proof: Evidently, From(4), the right inequality holds. So, we only need to prove the left inequality. Classifications are discussed below:

If

$$0\leq s\leq t\leq 1, s\leq\eta,$$

then we have

$$0\leq t-s\leq t-ts=t(1-s),$$

and thus

$$(t-s)^{\alpha-1} \leq t^{\alpha-1}(1-s)^{\alpha-1}.$$

Hence,

$$\begin{aligned} &q\Gamma(\alpha)G(t,s) \\ &= t^{\alpha-1}(1-s)^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1} - q(t-s)^{\alpha-1} \\ &\geq t^{\alpha-1}(1-s)^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1} - qt^{\alpha-1}(1-s)^{\alpha-1} \\ &= t^{\alpha-1}[(1-q)(1-s)^{\alpha-1} - \beta(\eta-s)^{\alpha-1}]. \end{aligned}$$

when $0 < \eta \leq s \leq t \leq 1$,

$$\begin{aligned} &q\Gamma(\alpha)G(t,s) \\ &= t^{\alpha-1}(1-s)^{\alpha-1} - q(t-s)^{\alpha-1} \\ &\geq t^{\alpha-1}(1-s)^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1} - q(t-s)^{\alpha-1} \\ &\geq t^{\alpha-1}[(1-q)(1-s)^{\alpha-1} - \beta(\eta-s)^{\alpha-1}]. \end{aligned}$$

when $0 \leq t \leq s \leq \eta < 1$ and $0 \leq t \leq s \leq 1, \eta \leq s$,it can be proved similarly that above inequality is also true. So, the proof is complete.

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and a fixed point theorem which will be used later.

Suppose $(E, ||\cdot||)$ is a real Banach space which is partially ordered by a cone $P \subset E$, i.e. $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i)$x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii)$x \in P, -x \in P \Rightarrow x = \theta$.

Putting $P^0 = \{x \in P | x \text{ is an interior point of } P\}$, a cone P is said to be solid if P^0 is non-empty. Moreover, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E, \theta \leq x \leq y$ implies $||x|| \leq N||y||$; in this case N is called the normality constant of P. We say that an operator $A : E \to E$ is increasing if $x \leq y$ implies $Ax \leq Ay$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly $\sim$ is an equivalence relation. Given $w > \theta$ (i.e. $w \geq \theta$ and $w \neq \theta$), we denote the set $P_w = \{x \in E | x \sim w\}$ by P_w. It is easy to see that $P_w \subset P$ for $w \in P$.

Definition 3 [14] $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., $u_i, v_i (i = 1, 2) \in P,\ u_1 \leq u_2,\ v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$.Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 4 [15] An operator $A : P \to P$ is said to be sub-homogeneous if it is satisfies

$$A(tx) \geq tA(x),\ \ \forall t \in (0, 1),\ x \in P. \tag{6}$$

Definition 5 [15] Let $D = P$ and β be a real number with $0 \leq \beta < 1$. An operator $A : D \to D$ is said to be β-concave if it satisfies

$$A(tx) \geq t^{\beta} A(x),\ \ \forall t \in (0, 1),\ x \in D. \tag{7}$$

Lemma 3 (Theorem 2.1 in [14]) Let $w > \theta,\ \beta \in (0, 1)$. $A : P \times P \to P$ is a mixed monotone operator and satifies

$$A(tx, t^{-1}y) \geq t^{\beta} A(x, y),\quad \forall t \in (0, 1),\ x, y \in P. \tag{8}$$

$B : P \to P$ is an increasing sub-homogeneous operator. Assume that
(i) there is $w_0 \in P_w$ such that $A(w_0, w_0) \in P_w$ and $Bw_0 \in P_w$;
(ii) there exists a constant $\delta_0 > 0$ such that $A(x, y) \geq \delta_0 Bx, \forall x, y \in P$.
Then:
(1) $A : P_w \times P_w \to P$ and $B : P_w \to P_w$;
(2) there exist $u_0, v_0 \in P_w$ and $\gamma \in (0, 1)$ such that

$$rv_0 \leq u_0 < v_0,\ u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0.$$

(3) the operator equation $A(x, x) + Bx = x$ has a unique solution x^* in P_w;
(4) for any initial values $x_0, y_0 \in P_w$,constructing successively the sequences

$$\begin{aligned} x_n &= A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \\ y_n &= A(y_{n-1}, x_{n-1}) + By_{n-1},\quad n = 1, 2, ... \end{aligned}$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Lemma 4 (Theorem 2.4 in [14]) Let$w > \theta,\ \beta \in (0, 1)$. $A : P \times P \to P$ is a mixed monotone operator and satisfies

$$A(tx, t^{-1}y) \geq tA(x, y),\quad \forall t \in (0, 1),\ x, y \in P. \tag{9}$$

$B : P \to P$ is an increasing β-concave operator. Assume that
(i) there is $w_0 \in P_w$ such that $A(w_0, w_0) \in P_w$ and $Bw_0 \in P_w$;
(ii) there exists a constant $\delta_0 > 0$ such that $A(x, y) \leq \delta_0 Bx, \forall x, y \in P$.
Then:
(1) $A : P_w \times P_w \to P$ and $B : P_w \to P_w$;
(2) there exist $u_0, v_0 \in P_w$ and $\gamma \in (0, 1)$ such that

$$rv_0 \leq u_0 < v_0,\ u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0.$$

(3) the operator equation $A(x, x) + Bx = x$ has a unique solution x^* in P_w;
(4) for any initial values $x_0, y_0 \in P_w$,constructing successively the sequences

$$\begin{aligned} x_n &= A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \\ y_n &= A(y_{n-1}, x_{n-1}) + By_{n-1},\quad n = 1, 2, ... \end{aligned}$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Remark 1 (i) If we take $B = \theta$ in Lemma 3, then the corresponding conclusion is still true (Corollary 2.2 in [14]);(ii) If we take $A = \theta$ in Lemma 4, then the corresponding conclusion is also true (Corollary 2.7 in [16]).

III. Main results

In this section, we apply Lemma 3 and Lemma 4 to investigate the problem (2), and we obtain some new results on the existence and uniqueness of positive solutions.

In this paper, we will work in the Banach space $C[0, 1] = \{x : [0, 1] \to R \text{ is continuous}\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$. Notice that this space can be endowed with a partial order given by $x, y \in C[0, 1],\ \ x \leq y \Leftrightarrow x(t) \leq y(t)$ for $t \in [0, 1]$.

Let $P = \{x \in C[0, 1] | x(t) \geq 0, t \in [0, 1]\}$ be the standard cone. Evidently, P is a normal cone in $C[0, 1]$ and the normality constant is 1.

Theorem 1 Assume that

(A1) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ is also continuous;

(A2) $f(t,u,v)$ is increasing in $u \in [0,+\infty)$ for fixed $t \in [0,1]$ and $v \in [0,+\infty)$,decreasing in $v \in [0,+\infty)$ for fixed $t \in [0,1]$ and $u \in [0,+\infty)$,and $g(t,u)$ is increasing in $u \in [0,+\infty)$ for fixed $t \in [0,1]$;

(A3) $g(t,0) \not\equiv 0$ and $g(t,\mu u) \geq \mu g(t,u), \forall t \in [0,1]$, $\mu \in (0,1), u \in [0,\infty)$,and there exists a constant $\beta \in (0,1)$ such that

$$f(t,\lambda u,\lambda^{-1}v) \geq \lambda^{\beta} f(t,u,v),\ \forall \lambda \in (0,1),\ u,v \in [0,\infty);$$

(A4) there exists a constant $\delta_0 > 0$ such that $f(t,u,v) \geq \delta_0 g(t,u)$,$t \in [0,1], u,v \geq 0$.

Then:

(a) there exist $u_0, v_0 \in P_w$ and $\gamma \in (0,1)$ such that $r v_o \leq u_0 < v_0$ and

$$\begin{array}{l} u_0(t) \leq \int_0^1 G(t,s)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]\mathrm{d}s, \\ v_0(t) \geq \int_0^1 G(t,s)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]\mathrm{d}s, \end{array}$$

where $w(t) = t^{\alpha-1}(1-t),\ t \in [0,1]$ and $G(t,s)$ is given as in (4);

(b)The problem (2) has a unique positive solution $u*$ in P_w;

(c) for any $x_0, y_0 \in P_w$,constructing successively the sequences

$$\begin{array}{rl} x_n(t) = & \int_0^1 G(t,s)f(s,x_{n-1}(s),y_{n-1}(s))\mathrm{d}s \\ & + \int_0^1 G(t,s)g(s,x_{n-1}(s))\mathrm{d}s,\ n = 1,2,\cdots, \\ y_n(t) = & \int_0^1 G(t,s)f(s,y_{n-1}(s),x_{n-1}(s))\mathrm{d}s \\ & + \int_0^1 G(t,s)g(s,y_{n-1}(s))\mathrm{d}s, \quad n = 1,2,\cdots \end{array}$$

We have $x_n(t) \to u^*(t)$ and $y_n(t) \to u^*(t)$as $n \to \infty$.

Proof: To begin with, from Lemma 1, the problem (2) has an integral formulation given by

$$u(t) = \int_0^1 G(t,s)[f(s,u(s),u(s)) + g(s,u(s))]\mathrm{d}s,$$

where is given as in (4).

Define two operators $A : P \times P \to E$ and$B : P \to E$ by

$$\begin{array}{l} A(u,v)(t) = \int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s, \\ Bu(t) = \int_0^1 G(t,s)g(s,u(s))\mathrm{d}s. \end{array}$$

It is easy to prove that is the solution of the problem (2) if and only if $u = A(u,u) + Bu$. From (A1), we know that $A : P \times P \to E$ and$B : P \to E$. In the sequel we check that A, B satisfy all assumptions of Lemma 3.

Firstly, we prove that A is a mixed monotone operator. In fact, for $u_i, v_i (i = 1,2) \in P$ with $u_1 \geq u_2,\ v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t),\ v_1(t) \leq v_2(t), t \in [0,1]$, and by (A2) and Lemma1,

$$\begin{array}{rl} A(u_1,v_1)(t) = & \int_0^1 G(t,s)f(s,u_1(s),v_1(s))\mathrm{d}s \\ \geq & \int_0^1 G(t,s)f(s,u_2(s),v_2(s))\mathrm{d}s \ . \\ = & A(u_2,v_2)(t) \end{array}$$

That is,$A(u_1,v_1) \geq A(u_2,v_2)$.

Further, it follows from (A2) and Lemma 1 that B is increasing. Next we show that A satisfies the condition (8).For any $\lambda \in (0,1)$ and $u,v \in P$, from (A3) we know that

$$\begin{array}{rl} A(\lambda u,\lambda^{-1}v)(t) = & \int_0^1 G(t,s)f(s,\lambda u(s),\lambda^{-1}v(s))\mathrm{d}s \\ \geq & \lambda^{\beta} \int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s \\ = & \lambda^{\beta} A(u,v)(t) \end{array}$$

That is $A(\lambda u,\lambda^{-1}v) \geq \lambda^{\beta} A(u,v)$, for $\lambda \in (0,1)$ and $u,v \in P$.So, the operator A satisfies (8).Also, for any $\mu \in (0,1)$ and $u \in P$, from (A3) we have

$$\begin{array}{rl} B(\mu u)(t) = & \int_0^1 G(t,s)g(s,\mu u(s))\mathrm{d}s \\ \geq & \mu \int_0^1 G(t,s)g(s,u(s))\mathrm{d}s \ , \\ = & \mu Bu(t) \end{array}$$

That is $B(\mu u) \geq \mu Bu$, for $\mu \in (0,1)$ and $u \in P$. So the operator B is a sub-homogeneous operator. Now we show that $A(w,w) \in P_w$ and $Bw \in P_w$, where $w(t) = t^{\alpha-1}, t \in [0,1]$. By (A1), (A2) and Lemma 2,

$$\begin{array}{l} w(t) \int_0^1 [(1-q)(1-s)^{\alpha-1} - \beta(\eta-s)^{\alpha-1}]f(s,0,1)\mathrm{d}s \\ \leq q\Gamma(\alpha)A(w,w)(t) \\ = q\Gamma(\alpha) \int_0^1 G(t,s)f(s,w(s),w(s))\mathrm{d}s \\ \leq w(t) \int_0^1 (1-s)^{\alpha-1} f(s,1,0)\mathrm{d}s \end{array} ,$$

From (A2) and (A4), we have
$f(s,1,0) \geq f(s,0,1) \geq \delta_0 g(s,0) \geq 0$.

Since $g(t,0) \not\equiv 0$, we get

$$\int_0^1 f(s,1,0)\mathrm{d}s \geq \int_0^1 f(s,0,1)\mathrm{d}s \geq \delta_0 \int_0^1 g(s,0)\mathrm{d}s > 0,$$

and in consequence,

$$\begin{array}{l} l_1 := \frac{1}{q\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,1,0)\mathrm{d}s > 0 \\ l_2 := \frac{1}{q\Gamma(\alpha)} \int_0^1 (1-q)(1-s)^{\alpha-1} f(s,0,1)\mathrm{d}s \\ \quad -\frac{1}{q\Gamma(\alpha)} \int_0^1 \beta(\eta-s)^{\alpha-1} f(s,0,1)\mathrm{d}s > 0 \end{array}$$

So $l_2 w(t) \leq A(w,w)(t) \leq l_1 w(t), t \in [0,1]$; and hence we have $A(w,w) \in P_w$.Similarly,

$$\begin{array}{l} w(t) \int_0^1 [(1-q)(1-s)^{\alpha-1} - \beta(\eta-s)^{\alpha-1}]g(s,0)\mathrm{d}s \\ \leq q\Gamma(\alpha)Bw(t) \\ = q\Gamma(\alpha) \int_0^1 G(t,s)f(s,w(s),w(s))\mathrm{d}s \\ \leq w(t) \int_0^1 (1-s)^{\alpha-1} g(s,1)\mathrm{d}s \end{array} ,$$

from $g(t,0) \not\equiv 0$, we easily prove $Bw \in P_w$.Hence the condition (i) of Lemma 3 is satisfied.

In the following, we show that the condition (ii) of Lemma 3 is also satisfied. For $u,v \in P$ and any $t \in [0,1]$ by (A4),

$$\begin{array}{rl} A(u,v)(t) = & \int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s \\ \geq & \delta_0 \int_0^1 G(t,s)g(s,u(s))\mathrm{d}s \ . \\ = & \delta_0 Bu(t) \end{array}$$

Then we get $A(u,v) \geq \delta_0 Bu,\ u,v \in P$.

Finally, an application of Lemma 3 implies: there exist $u_0, v_0 \in P_w$ and $\gamma \in (0,1)$ such that

$$\begin{array}{l} r v_0 \leq u_0 < v_0, \\ u_0 \leq A(u_0,v_0) + Bu_0 \leq A(v_0,u_0) + Bv_0 \leq v_0. \end{array}$$

the operator equation $A(u,u)+Bu=u$ has a unique solution u^* inP_w ; for any initial values $x_0, y_0 \in P_w$, constructing successively the sequences

$$\begin{array}{l} x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \\ y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, ... \end{array},$$

We have $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$.That is, there exist $u_0, v_0 \in P_w$ and $\gamma \in (0,1)$ such that$rv_o \le u_0 < v_0$ and

$$\begin{array}{l} u_0(t) \le \int_0^1 G(t,s)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]\mathrm{d}s, \\ v_0(t) \ge \int_0^1 G(t,s)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]\mathrm{d}s, \end{array}$$

The problem (2) has a unique positive solution u^* in P_w; for any initial values $x_0, y_0 \in P_w$, constructing successively the sequences

$$\begin{array}{l} x_n(t) = \int_0^1 G(t,s)f(s,x_{n-1}(s),y_{n-1}(s))\mathrm{d}s \\ \qquad + \int_0^1 G(t,s)f(s,x_{n-1}(s),y_{n-1}(s))\mathrm{d}s, \quad n = 1,2,\cdots, \\ y_n(t) = \int_0^1 G(t,s)f(s,y_{n-1}(s),x_{n-1}(s))\mathrm{d}s \\ \qquad + \int_0^1 G(t,s)g(s,y_{n-1}(s))\mathrm{d}s, \quad n = 1,2,\cdots \end{array}$$

We have $x_n(t) \to u^*(t)$ and $y_n(t) \to u^*(t)$as $n \to \infty$.

Theorem 2 Assume that (A1), (A2) and

(A5) $f(t, \lambda u, \lambda^{-1}v) \ge \lambda f(t,u,v), \quad \forall t \in [0,1], \lambda \in (0,1), u,v \in [0,\infty)$,and there exists a constant $\beta \in (0,1)$ such that

$$g(t,\mu u) \ge \mu^\beta g(t,u), \quad \forall t \in [0,1], \mu \in (0,1),\ u \in [0,\infty);$$

(A6) $f(t,0,1) \not\equiv 0$ for $t \in [0,1]$ and there exists a constant $\delta_0 > 0$ such that

$$f(t,u,v) \le \delta_0 g(t,u), t \in [0,1], u,v \ge 0.$$

Then:

(a) there exist $u_0, v_0 \in P_w$ and $\gamma \in (0,1)$ such that $rv_o \le u_0 < v_0$ and

$$\begin{array}{l} u_0(t) \le \int_0^1 G(t,s)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]\mathrm{d}s, \\ v_0(t) \ge \int_0^1 G(t,s)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]\mathrm{d}s, \end{array}$$

where $w(t) = t^{\alpha-1}(1-t),\ t \in [0,1]$ and $G(t,s)$ is given as in (3);

(b)The problem (1) has a unique positive solution u^* in P_w;

(c) for any $x_0, y_0 \in P_w$,constructing successively the sequences

$$\begin{array}{l} x_n(t) = \int_0^1 G(t,s)f(s,x_{n-1}(s),y_{n-1}(s))\mathrm{d}s \\ \qquad + \int_0^1 G(t,s)g(s,x_{n-1}(s))\mathrm{d}s,\ n = 1,2,\cdots, \\ y_n(t) = \int_0^1 G(t,s)f(s,y_{n-1}(s),x_{n-1}(s))\mathrm{d}s \\ \qquad + \int_0^1 G(t,s)g(s,y_{n-1}(s))\mathrm{d}s, \quad n = 1,2,\cdots \end{array}$$

We have $x_n(t) \to u^*(t)$ and $y_n(t) \to u^*(t)$as $n \to \infty$.

Proof: Consider two operators A, B defined in the proof of Theorem 1. Similarly, from (A1), (A2), we obtain that $A : P \times P \to E$ is a mixed monotone operator and $B : P \to E$ is increasing. From (A5), we have

$$\begin{array}{l} A(\lambda u, \lambda^{-1}v) \ge \lambda A(u,v), \lambda \in (0,1),\ u,v \in P \\ B(\mu u) \ge \mu^\beta Bu, \mu \in (0,1), u \in p. \end{array}$$

From (A2) and (A6), we have

$$\begin{array}{l} g(s,0) \ge \frac{1}{\delta_0} f(s,0,1), \\ f(s,1,0) \ge f(s,0,1), \quad s \in [0,1]. \end{array}$$

Since $f(t,0,1) \not\equiv 0$,we get

$$\begin{array}{l} \int_0^1 f(s,1,0)\mathrm{d}s \ge \int_0^1 f(s,0,1)\mathrm{d}s > 0, \\ \int_0^1 g(s,1)\mathrm{d}s \ge \int_0^1 g(s,0)\mathrm{d}s \ge \frac{1}{\delta_0}\int_0^1 f(s,0,1)\mathrm{d}s > 0, \end{array}$$

and in consequence,

$$\begin{array}{l} \frac{1}{q\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,1,0)\mathrm{d}s \\ \ge \frac{1}{q\Gamma(\alpha)}\int_0^1 [(1-q)(1-s)^{\alpha-1} - \beta(\eta-s)^{\alpha-1}] f(s,0,1)\mathrm{d}s \\ > 0, \\ \frac{1}{q\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} g(s,1)\mathrm{d}s \\ \ge \frac{1}{q\Gamma(\alpha)}\int_0^1 [(1-q)(1-s)^{\alpha-1} - \beta(\eta-s)^{\alpha-1}] g(s,0)\mathrm{d}s \\ > 0. \end{array}$$

So, we can easily prove that $A(w,w) \in P_w$ and $Bw \in P_w$. For $u,v \in P$, and any $t \in [0,1]$ by (A6),

$$\begin{array}{rl} A(u,v)(t) = & \int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s \\ \le & \delta_0 \int_0^1 G(t,s)g(s,u(s))\mathrm{d}s \\ = & \delta_0 Bu(t) \end{array}.$$

Then we get $A(u,v) \le \delta_0 Bu,\ u,v \in P$.

Finally, an application of Lemma 4 implies: there exist $u_0, v_0 \in P_w$ and $\gamma \in (0,1)$ such that

$$\begin{array}{l} rv_0 \le u_0 < v_0, \\ u_0 \le A(u_0,v_0) + Bu_0 \le A(v_0,u_0) + Bv_0 \le v_0. \end{array}$$

the operator equation $A(u,u)+Bu=u$ has a unique solution u^* inP_w ; for any initial values $x_0, y_0 \in P_w$, constructing successively the sequences

$$\begin{array}{l} x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \\ y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, ... \end{array},$$

We have $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$.That is, there exist $u_0, v_0 \in P_w$ and $\gamma \in (0,1)$ such that $rv_o \le u_0 < v_0$ and

$$\begin{array}{l} u_0(t) \le \int_0^1 G(t,s)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]\mathrm{d}s, \\ v_0(t) \ge \int_0^1 G(t,s)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]\mathrm{d}s, \end{array}$$

The problem (2) has a unique positive solution u^* in P_w; for any initial values $x_0, y_0 \in P_w$, constructing successively the sequences

$$\begin{array}{l} x_n(t) = \int_0^1 G(t,s)f(s,x_{n-1}(s),y_{n-1}(s))\mathrm{d}s \\ \qquad + \int_0^1 G(t,s)f(s,x_{n-1}(s),y_{n-1}(s))\mathrm{d}s, \quad n = 1,2,\cdots, \\ y_n(t) = \int_0^1 G(t,s)f(s,y_{n-1}(s),x_{n-1}(s))\mathrm{d}s \\ \qquad + \int_0^1 G(t,s)g(s,y_{n-1}(s))\mathrm{d}s, \quad n = 1,2,\cdots \end{array}$$

We have $x_n(t) \to u^*(t)$ and $y_n(t) \to u^*(t)$as $n \to \infty$.

From Remark 1 and similar to the proofs of Theorems 1-2, we can prove the following conclusions.

Corollary 1 Let $g \equiv 0$.Assume that f satisfies the conditions of Theorem 1 and $f(t,0,1) \not\equiv 0$.

Then:

(i) there exist $u_0, v_0 \in P_w$ and $\gamma \in (0,1)$ such that $rv_o \le u_0 < v_0$ and

$$\begin{array}{l} u_0(t) \le \int_0^1 G(t,s)f(s,u_0(s),v_0(s))\mathrm{d}s, \\ v_0(t) \ge \int_0^1 G(t,s)f(s,v_0(s),u_0(s))\mathrm{d}s, \end{array}$$

where $w(t) = t^{\alpha-1}(1-t),\ t \in [0,1]$ and $G(t,s)$ is given as in (3);
(ii)The problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t), u(t)) = 0, & 0 < t < 1, \ 1 < \alpha \le 2, \\ u(0) = 0, \quad \beta u(\eta) = u(1), \end{cases}$$

has a unique positive solution u^* in P_w;
(iii) for any $x_0, y_0 \in P_w$,constructing successively the sequences

$$\begin{aligned} x_n(t) &= \int_0^1 G(t,s) f(s, x_{n-1}(s), y_{n-1}(s)) \mathrm{d}s, \quad n = 1, 2, \cdots, \\ y_n(t) &= \int_0^1 G(t,s) f(s, y_{n-1}(s), x_{n-1}(s)) \mathrm{d}s, \quad n = 1, 2, \cdots \end{aligned}$$

We have $x_n(t) \to u^*(t)$ and $y_n(t) \to u^*(t)$as $n \to \infty$.
Corollary 2 Let $f \equiv 0$.Assume that g satisfies the conditions of Theorem 2 and $g(t,0) \not\equiv 0$,for $t \in [0,1]$.
Then:
(i) there exist $u_0, v_0 \in P_w$ and $\gamma \in (0,1)$ such that $rv_o \le u_0 < v_0$ and

$$\begin{aligned} u_0(t) &\le \int_0^1 G(t,s) g(s, u_0(s)) \mathrm{d}s, \\ v_0(t) &\ge \int_0^1 G(t,s) g(s, v_0(s)) \mathrm{d}s, \end{aligned}$$

where $w(t) = t^{\alpha-1}(1-t),\ t \in [0,1]$ and $G(t,s)$ is given as in (4);
(ii)The problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + g(t, u(t)) = 0, & 0 < t < 1, \ 1 < \alpha \le 2, \\ u(0) = 0, \quad \beta u(\eta) = u(1), \end{cases}$$

has a unique positive solution u^* in P_w;
(iii) for any $x_0, y_0 \in P_w$,constructing successively the sequences

$$\begin{aligned} x_n(t) &= \int_0^1 G(t,s) g(s, x_{n-1}(s)) \mathrm{d}s, \quad n = 1, 2, \cdots, \\ y_n(t) &= \int_0^1 G(t,s) g(s, y_{n-1}(s)) \mathrm{d}s, \quad n = 1, 2, \cdots \end{aligned}$$

We have $x_n(t) \to u^*(t)$ and $y_n(t) \to u^*(t)$as $n \to \infty$.

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