Transient Analysis of a Single-Server Queue with Batch Arrivals Using Modeling and Functions Akin to the Modified Bessel Functions

Vitalice K. Oduol

Abstract—The paper considers a single-server queue with fixed-size batch Poisson arrivals and exponential service times, a model that is useful for a buffer that accepts messages arriving as fixed size batches of packets and releases them one packet at time. Transient performance measures for queues have long been recognized as being complementary to the steady-state analysis. The focus of the paper is on the use of the functions that arise in the analysis of the transient behaviour of the queuing system. The paper exploits practical modelling to obtain a solution to the integral equation encountered in the analysis. Results obtained indicate that under heavy load conditions, there is significant disparity in the statistics between the transient and steady state values.

Keywords—batch arrivals, modelling, single-server queue, time-varying probabilities, transient analysis.

I. INTRODUCTION

THE paper considers a single-server queue with fixed-size Poisson arrivals and exponential service times. This model is useful for a buffer that accepts messages arriving in fixed size batches of packets and releases one packet at time [1]. Justifications for the transient analysis of traffic in telecommunication systems abound [2]. It has long been recognized that transient performance measures for queues are complementary to the steady-state results [3-6] because there often exists a need to understand the initial behaviour of a system. In general queuing systems, arrivals at a service point (e.g. a switch) may occur in batches of different sizes. Due to congestion these arrivals may be queued for later forwarding to the destination. This paper considers the case in which the arrivals are of fixed size, and occur in continuous time. There may be cases where the network traffic is diverted suddenly to cope with faults, as in automatic protection switching (APS) in which the transport system re-directs traffic when faults and failures occur in subcomponents of the network. In such cases a service point may experience a sudden increase in its load, and this may continue until the original fault has been cleared. After the fault is corrected, traffic reverts to the previous distribution, and this presents another perturbation in the network. There exists now a reduction in the load in parts of the network,

V. K. Oduol is with the Department of Electrical and Information Engineering, University of Nairobi, Nairobi, Kenya (+254-02-318262 ext.28327, vkoduol@uonbi.ac.ke)

The restored element experiences an increased load transient starting from an empty state. This scenario presents the network service points with transient conditions that require the kind of analysis attempted here. The paper has limited the analysis to the case of fixed batch size. Of the two main approaches that have been adopted to obtain the transient behaviour of queuing systems one relies on the numerical inversion of Laplace transforms or generating functions [7,8], and the other is based on recursive computations. What is presented here is a method that uses a family of functions arising in the analysis of the queuing system. It is evident that these functions are akin to the modified Bessel functions of first kind, with which they share several similarities. Many of these similarities are not explored in this paper, since attention is on their use to obtain the transient behaviour of the queuing system.

A key integral equation is obtained and in solved using practical expressions to produce the results required in the analysis. For a batch size of one (B=1), the integral equation is easily solved [9] to obtain the results in terms of the modified Bessel functions of the first kind. For B >1, the triplet (B, ρ, k) represents the set of independent variables considered as inputs to the analysis, where B is the size of a batch, ρ the system load, and k the number of packets in the system. The results section presents the dependence of the system parameters on (B, ρ, k) and proceeds to use these to provide numerical results for the assessment quantities using equations derived in earlier sections of the paper.

The paper is organized as follows. Section II presents the system model, and sets the main point of reference for the rest of the presentation. It presents a brief discussion of the steady state probabilities, obtained by considering that time derivatives vanish. It proceeds to consider the more general case of time-varying probabilities, which is actually a discussion of the time-dependent quantities that, in the limit of large t, eventually yield the steady state results. Section III presents the key integral equation whose solution is used to determine the occupancy probabilities. Section IV addresses the modeling of the convolution kernels. This is done for both the main kernel and the general kernel. Section V gives the parameters for the convolution kernels and also discusses how to obtain the quantities $Q_k(t)$ from which the occupancy probabilities are obtained. Finally Section VI gives the results and conclusion.

II. SYSTEM MODEL

Messages arrive at a service point in fixed size batches of B packets according to a Poisson process of mean rate λ arrivals per second. The single server completes the service at the rate of μ packets per second. The probability flow rates are as shown in Fig.1.
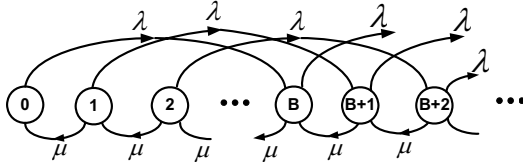

Fig.1 Probability Flow For the Queuing System

Denote by $P_k(t)$ the probability that there are k packets in the system at time t. The probability flow balance equations are given by

$$\frac{dP_k(t)}{dt} + (\lambda + \mu)P_k(t) = \begin{cases} \mu P_1(t) + \mu P_0(t) & k = 0 \\ \mu P_{k+1}(t) & 1 \le k < B \\ \mu P_{k+1}(t) + \lambda P_{k-B}(t) & k \ge B \end{cases} \quad (1)$$

A. Steady State Probabilities

For steady state analysis (1) can be solved for the number of packets in the system by setting the time derivatives to zero. This yields the set of equations

$$(\lambda + \mu)P_k = \begin{cases} \mu P_1 + \mu P_0 & k = 0 \\ \mu P_{k+1} & 1 \le k < B \\ \mu P_{k+1} + \lambda P_{k-B} & k \ge B \end{cases} \quad (2)$$

Let $P(z)$ be the moment generating function defined by

$$P(z) = \sum_{k=0}^{\infty} z^k P_k \quad (3)$$

Substituting (2) in (3) yields

$$P(z) = \frac{\mu(z-1)P_0}{(\lambda + \mu)z - \lambda z^{B+1} - \mu} \quad (4)$$

Since P(z) is analytic in the unit disc {z: |z| ≤ 1 }, there must be a pole-zero cancellation to remove the pole at z=1. This gives

$$P(z) = \frac{P_0}{1 - \frac{\lambda}{\mu} \sum_{m=1}^{B} z^m} \quad (5)$$

Since P(z) is a moment generating function, it is necessary to have $P(1) = 1$. Accordingly, $P_0 = 1 - B\lambda/\mu = 1 - \rho$, where $\rho = \lambda/\mu$ is the offered load. The steady state solution is obtained by solving (4) and (5) to give

$$P_k = \left(\frac{\lambda}{\mu}\right)\left(\sum_{m=1}^{\min(k, B)} P_{k-m}\right) \qquad k > 0 \quad (6)$$

B. Time-Varying Probabilities

This subsection considers the more general situation of the probabilities before they reach the steady state. The quantity $P_k(t)$ will eventually reach the steady state value P_k in the limit of large t. The equations in (1) can be simpler by letting

$$P_k(t) = Q_k(t)\exp(-(\lambda + \mu)t) \quad (7)$$

Substituting (7) in (1) results in

$$\frac{dQ_k(t)}{dt} = \begin{cases} \mu Q_1(t) + \mu Q_0(t) & k = 0 \\ \mu Q_{k+1}(t) & 1 \le k < B \\ \mu Q_{k+1}(t) + \lambda Q_{k-B}(t) & k \ge B \end{cases} \quad (8)$$

From (7) it is observed that he initial condition $P_k(0)=\delta_{i,k}$ is equivalent to $Q_k(0)=\delta_{i,k}$, where $\delta_{i,k}$ is the Kronecker delta, used here to signify the fact the system starts with i packets. Defining the moment generating function Q(z,t) as

$$Q(z,t) = \sum_{k=0}^{\infty} z^k Q_k(t) \quad (9)$$

and using (8) gives the partial differential equation.

$$\frac{\partial Q(z,t)}{\partial t} - (\lambda z^B + \mu z^{-1})Q(z,t) = \mu(1 - z^{-1})Q_0(t) \quad (10)$$

whose solution is

$$Q(z,t) = Q(z,0)\exp[(\lambda z^B + \mu z^{-1})t] + \mu(1 - \mu z^{-1})\int_0^t \exp[(\lambda z^B + \mu z^{-1})(t-\tau)]Q_0(\tau)d\tau \quad (11)$$

The function $\exp[(\lambda z^B + \mu z^{-1})t]$ embedded in the solution for the moment generating function (11) can then be written as

$$\exp[(\lambda z^B + \mu z^{-1})t] = \sum_{k=-\infty}^{\infty} z^k \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} V_k^{(B)}(\alpha t) \quad (12)$$

with α defined as

$$\alpha = (\mu^B \lambda)^{\frac{1}{B+1}} = \left(\frac{\lambda}{\mu}\right)^{\frac{1}{B+1}}\mu \quad (13)$$

and for $k \ge 0$ the functions $V_{-k}^{(B)}(x)$ and $V_k^{(B)}(x)$ are defined as

$$V_{-k}^{(B)}(x) = \sum_{l=0}^{\infty} \frac{x^{l(B+1)+k}}{l!(lB+k)!} \quad (14)$$

$$V_k^{(B)}(x) = \sum_{l=\sigma_k}^{\infty} \frac{x^{l(B+1)-k}}{l!(lB-k)!} \quad (15)$$

where for k > 0 the variable σ_k is defined as $\sigma_k = \lceil k/B \rceil$ and the notation $\lceil k/B \rceil$ designates *the smallest integer not less than* k/B, otherwise known as the roof (or ceiling) of k/B. When *B* = 1 it is seen that $\sigma_k = k$, and it is easily shown by direct substitution that (14) and (15) coalesce and both become $I_k(2x)$, the modified Bessel function of the first kind of order k, evaluated at *2x*. There are other aspects of correspondence between the functions $V_k^{(B)}(x)$ and the modified Bessel functions, which are not explored here. The main purpose here is to use the functions as defined in order to obtain results for the transient behaviour of the queuing system. Continuing

these proceedings, the functions $V_k^{(B)}(x)$ can now be used in (11) to give

$$Q(z,t) = \sum_{k=-\infty}^{\infty} z^k \left(\frac{\lambda}{\mu}\right)^{\frac{k-i}{B+1}} V_{k-i}^{(B)}(\alpha t) \tag{16}$$

$$+ \sum_{k=-\infty}^{\infty} z^k \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} \int_0^t \mu Q_0(\tau) h_k(t-\tau) d\tau$$

where

$$h_k(t) = V_k^{(B)}(\alpha t) - \left(\frac{\lambda}{\mu}\right)^{\frac{1}{B+1}} V_{k+1}^{(B)}(\alpha t) \tag{17}$$

Defining the functions $q_{i-k}(t)$ according to

$$q_{i-k}(t) = \left(\frac{\lambda}{\mu}\right)^{\frac{k-i}{B+1}} V_{k-i}^{(B)}(\alpha t) \tag{18}$$

and subsequently equating coefficients of equal powers of z on both sides in (16) gives

$$Q_k(t) = q_{i-k}(t) + \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} \int_0^t \mu Q_0(\tau) h_k(t-\tau) d\tau \tag{19}$$

from which the dependence of $Q_k(t)$ on $Q_0(t)$ is evident. At this point it becomes necessary to seek some simplifying practical considerations that can simplify the analysis. These equations can be represented by the signal processing diagram of Fig.2, in which the factor μ in (19) has been omitted, as it amounts to a scaling of the time axis.
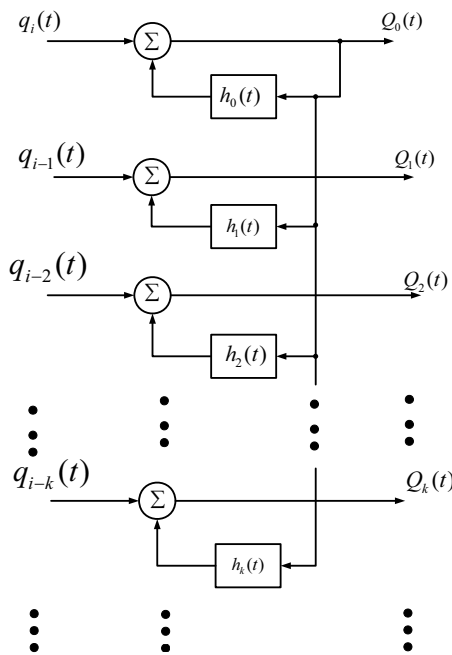

Fig. 2. Linear System Representing $Q_k(t)$

III. THE KEY INTEGRAL EQUATION

This section determines the integral equation for $Q_0(t)$, from whose solution the other quantities $Q_k(t)$ will be obtained. Setting k=0 in (19) results in the integral equation

$$Q_0(t) = q_i(t) + \int_0^t \mu Q_0(\tau) h_0(t-\tau) d\tau \tag{20}$$

This is the key integral equation that must be solved to obtain statistics for the behaviour of the queuing system. Indeed it can be shown that once $Q_0(t)$ is known, the average number in the system can be determined, as well as the delay experienced by packets. This is because the Laplace-Stieltjes transform $Q^*(z,s)$ of the moment generating function $Q(z,t)$ for $Q_k(t)$ is found to depend only on $Q_0(t)$, its initial conditions, and the parameters of the system, namely the arrival rate, batch size, and the service rate. This indicates that all the statistics of the system are derivable from the knowledge of $Q_0(t)$.

An equivalent integral equation for $Q_0(t)$ can be obtained by imposing the condition that for negative values of k, it is necessary to have for $Q_k(t) = 0$. In particular, setting k=-1 in (19) yields

$$-\left(\frac{\lambda}{\mu}\right)^{\frac{1}{B+1}} q_{i+1}(t) = \int_0^t \mu Q_0(\tau) h_{-1}(t-\tau) d\tau \tag{21}$$

By exploiting properties of the functions $V_k^{(B)}(x)$ it can be shown that the two integral equations are equivalent, in the sense that either one can be obtained from the other. In the sequel (20) is used.

A. Solution of the Key Integral Equation

The integral equations given here can be solved by a variety of methods. Some authors have suggested expressing both the kernel and the unknown function as series of derivatives of the known function in the method of differential inversion [10,11]. One special case considers an integral equation whose kernel is a Bessel function [12]. In this paper, the practical considerations given above are used to obtain numerical results that can be used in the assessment of the transient behaviour of the queuing systems in question. The integral equation in (20) can be represented by the linear system in the upper part of Fig. 3. The lower part of Fig.3 is obtained as follows. The Laplace transforms of $h_0(t)$ and $h(t)$ are respectively $H_0(s)$ and $H(s)$. In order to keep the expressions simple, the definition

$$\beta = (m_B - 1/B)\rho - 1 \tag{22}$$

yields

$$H_0(s) = \frac{1}{s+\beta+1} \tag{23}$$

$$H(s) = \frac{1}{1 - H_0(s)} = 1 + \frac{1}{s+\beta} \tag{24}$$
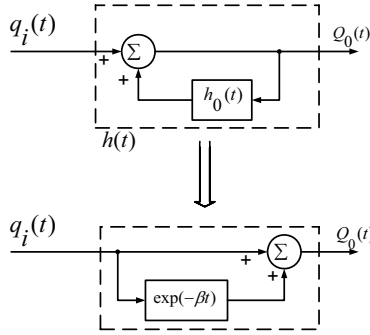
The time domain expression then becomes

$$h(t) = \delta(t) + \exp(-\beta t) \tag{25}$$

With $q_i(t)$ as input (25) immediately yields the lower part of Fig.3, where for the same reason as in Fig.2 the factor μ in (20) has once again been omitted.

Using the linear system in Fig.3 gives $Q_0(t)$ as

$$Q_0(t) = q_i(t) + q_i(t) * \exp(-\beta t) \tag{26}$$

where the symbol * is used to denote the convolution operation.

Fig. 3. Linear System Representing The Equation For $Q_0(t)$

The convolution term on the right hand side of (28) can be expressed as

$$q_i(t) * \exp(-\beta t) = \left(\frac{\mu}{\lambda}\right)^{\frac{i}{B+1}} \sum_{l=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^{\frac{l(B+1)+i}{B+1}} \frac{\left(t^{l(B+1)+i}\right) * \exp(-\beta t)}{l!(lB+i)!}$$

$$= \frac{1}{(-\beta)^{i+1}} \sum_{l=0}^{\infty} \binom{l(B+1)+i}{l} \left(\frac{\lambda}{\mu} \frac{1}{(-\beta)^{B+1}}\right)^l \left(\exp(-\beta t) - \sum_{q=0}^{l(B+1)+i} \frac{(-\beta t)^q}{q!}\right)$$
(30)

With this result every needed quantity is now known. Accordingly (28) can be taken as a working solution for $Q_0(t)$, which when substituted in (3) gives the probability $P_0(t)$ that the system is empty.

IV. MODELLING THE CONVOLUTION KERNELS

The convolution kernel $h_k(t)$ of (18) and (19) is found to satisfy the representation

$$h_k(t) = \begin{cases} \exp[-(m_B - 1/B)\rho t] & k = 0 \\ a_k t^{b_k} & k > 0 \end{cases}$$
(31)

where the parameters m_B, a_k and b_k are determined from the values for $h_k(t)$.

A. Modelling the Main Convolution Kernel $h_0(t)$

The convolution kernel $h_0(t)$, corresponding to k=0, is referred to as the main convolution kernel since it is used to determine $Q_0(t)$, which is used in (19) to obtain all the other quantities $Q_k(t$. Indeed once $Q_0(t)$ is known, it is possible to obtain all the main statistics, such as the mean number of packets in the system, the variance of this number, and even the delay seen by a packet in the system, all as functions of time. An examination of the values of $h_0(t)\exp(-(\lambda/\mu)t)$ reveals that the following relation holds

$$h_0(t)\exp(-(\lambda/\mu)t) = \exp(-m_B \rho t)$$
(32)

the parameter m_B being given by the values in Table I.

B. Modelling the General Convolution Kernel $h_k(t)$

The kernels $h_k(t)$ are used jointly with $Q_0(t$ in (19) to determine $Q_k(t)$. By examining the values of $h_k(t)$ it is found that the parameter b_k in (31) is independent of the load $\rho = B\lambda/\mu$, but depends on the batch size B and the index k, according to the relation

$$b_k = (B+1)\left[\frac{k+1}{B}\right]$$
(33)

where $[(k+1)/B]$ denotes the integer part of $(k+1)/B$. For a_k it is found that the following relation applies

$$a_k = g_k \rho^{r_k}$$
(34)

The values of g_k and r_k are given in Table II.

C. Determination of $Q_k(t)$

$Q_k(t)$ is determined by first substituting the expression for $Q_0(t)$ in (19) to write

$$Q_k(t) = q_{i-k}(t) + \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} h_k(t) * q_i(t) + \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} q_i(t) * h_k(t) * \exp(-\beta t)$$
(35)

The second term on the right hand side of (30) gives

$$\left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} h_k(t) * q_i(t) = \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} \left(\frac{\mu}{\lambda}\right)^{\frac{i}{B+1}} a_k \sum_{l=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^{\frac{l(B+1)+i}{B+1}} \frac{\left(t^{l(B+1)+i}\right) * \left(t^{b_k}\right)}{l!(lB+i)!}$$

$$= \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} a_k \sum_{l=0}^{\infty} \binom{l(B+1)+i}{l} \left(\frac{\lambda}{\mu}\right)^l \frac{b_k! t^{l(B+1)+i+b_k+1}}{(l(B+1)+i+b_k+1)!}$$
(31)

For the third (last) term in (30) it is reasonable to first consider

$$h_k(t) * \exp(-\beta t) = \frac{a_k b_k!}{(-\beta)^{b_k+1}} \left(\exp(-\beta t) - \sum_{q=0}^{b_k} \frac{(-\beta t)^q}{q!}\right)$$
(32)

which when substituted in (30), gives

$$\left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} q_i(t) * h_k(t) * \exp(-\beta t)$$
(33)

$$= \frac{a_k b_k!}{(-\beta)^{b_k+1}} \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} \left(q_i(t) * \exp(-\beta t) - q_i(t) * \sum_{q=0}^{b_k} \frac{(-\beta t)^q}{q!}\right)$$

The first term on the right hand side can be obtained by using (29), and so only the last term need be evaluated.

$$q_i(t) * \sum_{q=0}^{b_k} \frac{(\beta t)^q}{q!} =$$

$$\frac{1}{(-\beta)^{i+1}} \sum_{l=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^l \binom{l(B+1)+i}{l} \frac{1}{(-\beta)^{l(B+1)}} \sum_{q=0}^{b_k} \frac{(-\beta t)^{l(B+1)+q+i+1}}{(l(B+1)+q+i+1)!}$$
(34)

V. CONVOLUTION KERNEL PARAMETERS

The first set of results are a list of parameter values obtained by modelling the convolution kernels. Table I lists the values for the main kernel parameter m_B referred to in (31) and (32) for k=0.

It is noted that for values of B larger than 5, this parameter is given by the reciprocal of the batch size. The physical reasons giving rise to this observation is not explored in this paper.

TABLE I
THE MAIN KERNEL PARAMETER m_B

B	m_B
1	1.042367
2	0.513729
3	0.336996
4	0.250781
5	0.200138
B > 5	1 / B

For $k > 0$, the general kernel (23) requires the values for a_k, which in turn are expressed in as a function of several variables whose values are obtained from the data and are given in Tables II and III. The notation Mod (k, B) represents the remainder when k is divided by B.

TABLE II
GENERAL KERNEL PARAMETER $Ü_k$

	B = 2	B = 3	B = 4
Mod(k, B) = 0	1.39916	1.1016	1.09821
Mod(k, B) = 1	1.83551	1.7010	1.95435
Mod(k, B) = 2	-	1.36146	1.60417
Mod(k, B) = 3	-	-	1.29080

TABLE III
GENERAL KERNEL PARAMETER $ø_k$

	B = 2	B = 3	B = 4
Mod(k, B) = 0	0.33413	0.3306	0.24646
Mod(k, B) = 1	0.33813	0.34660	0.25621
Mod(k, B) = 2	-	0.33685	0.25215
Mod(k, B) = 3	-	-	0.25046

The values in Table II and Table III were obtained from values of a_{kk} after observing that the log-log plots of a_{kk} versus ρ are are straight lines whose slopes and intercepts depend on k and B, a few of which are given in Tables II and III.

VI. RESULTS AND CONCLUSION

The paper has attempted to analyze the single-server queue with fixed size batch Poisson arrivals and exponential service times. Various parameters needed in the analysis are identified and their values determined by modelling. The values are obtained from observations made on the functions that arise in the solution of the differential equation describing the moment generating function for the occupancy probabilities. The differential equations have solutions which involve convolutions.

By modelling the convolution kernels, the integral equation for the empty probability is solved, and equations are provided showing how to obtain the other non-empty probabilities.

Fig. 4 shows how the state probabilities $P_k(t)$ vary with time for a queuing system with a batch size of 3 and starting with 5 packets (i=5) initially. The labels for the probabilities $P_k(t)$ are given on the right of the figure. The curves are such that the probabilities P_1, P_2, and P_3 appear to merge on the right of the figure. The actual values of these probabilities are different, but close. Since they appear to be merged, a single triple label P_1, P_2, P_3 is used. For the next set of probabilities P_4, P_5, and P_6, the label P_4,P_5,P_6 is used to mean that P_4, represents the curve immediately to the left of this label, P_5 represents the

curve below that an P_6 refers to the second curve below. This method was used to avoid cluttering the figure. The same convention is applied to P_7,P_8,P_9 and P_{10},P_{11},P_{12}. For the curves at the top (P_0) and at the bottom (P_{13}) single labels are used. It is observed that $P_5(t)$ starts from a value of unity (at t=0), and settles at the steady state value of 5.545×10^{-4}.

The other probabilities (for $k \neq i$) start from a value of zero and eventually settle at there steady state levels. Also observed is the fact that as k increases, the probabilities take longer to reach steady state.
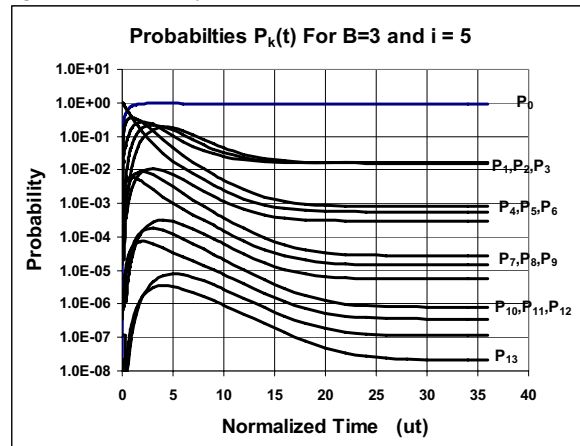

Fig. 4. State Probabilities $P_k(t)$ with 5 Initial Packets and $\rho = 0.05$

Fig.5 and Fig.6 show how the empty state probability $P_0(t)$ varies with time for non-zero initial state (i=5) and for an empty state initial state (i=0), respectively.
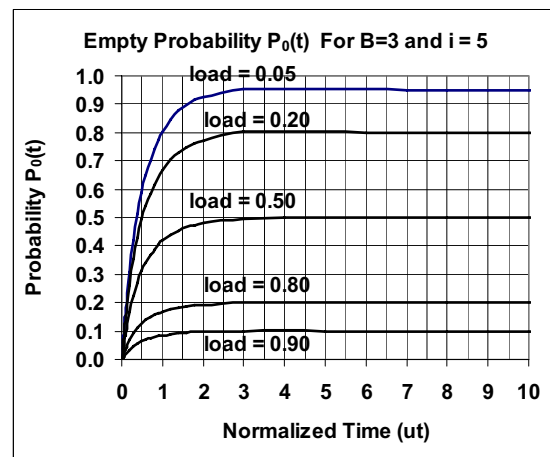

Fig. 5. Empty Probability $P_0(t)$ with 5 Initial Packets

When the system is initially empty (Fig.6), the empty state probability starts from a value of unity, and when the initial state is non-zero (as in Fig.5), it starts from a value of zero. In both cases $P_0(t)$ eventually settles at a steady state value determined by the load.

Fig.7 shows the variation of the mean number of packets in the system for some time instants. For load values below 0.4 (i.e. for $\rho < 0.4$), there is little difference between

the transient and steady state values for the number of packets. However, for heavier loads (e.g. for $\rho = 0.9$), there is much greater difference between the transient and steady state values. At $\mu t = 5$, the average number of packets is 2.14, while for $\mu t = 80$, this number is 18, which is the steady state value.
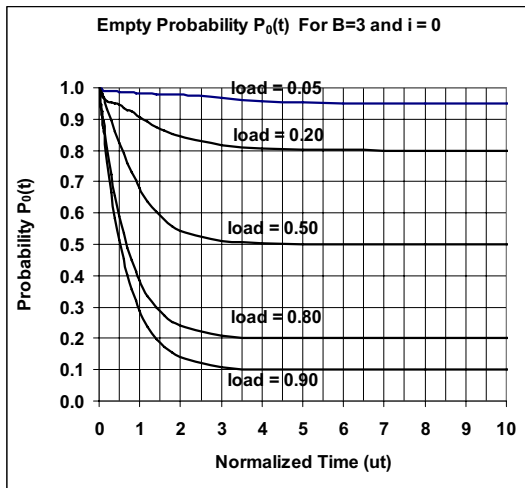

Fig. 6 Probability $P_0(t)$ With Queue Initially Empty (i=0)

Thus, the disparity between the transient and steady state values are more pronounced under heavy load conditions. Although at low loads, e.g. $\rho < 0.4$, the system *still takes some time to reach steady state,* the difference between the transient and steady state values is small. Accordingly, steady state results will be insufficient to determine the relevant statistics of the system following a perturbation. This conclusion is not surprising, the paper has only provided a means to arrive at it by using modeling and the functions described in the text.
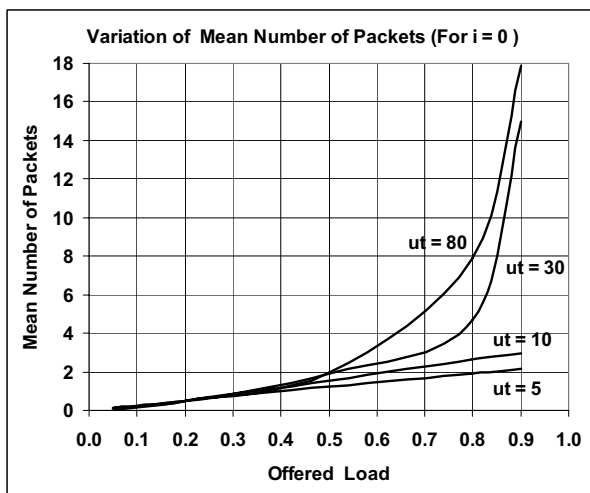

Fig. 7. Mean Number of Packets - Queue Initially Empty (i=0)

REFERENCES

[1] G. N. Higginbottom, *Performance Evaluation of Communication Networks*, Artech House 1998
[2] H. P. Schwefel, L. Lipsky, M. Jobmann, "On the Necessity of Transient Performance Analysis in Telecommunication Networks," *17th International Teletraffic Congress* (ITC17), Salvador da Bahia, Brazil, September 24-28 2001
[3] B. van Holt, C. Blondia, "Approximated Transient Queue Length and Waiting Time Distribution via Steady State Analysis", *Stochastic Models* **21**, pp.725-744, 2005
[4] T. Hofkens, K. Spacy, C. Blondia, "Transient Analysis of the D-BMAP/G/1 Queue with an Applications to the Dimensioning of Video Playout Buffer for VBR Traffic", *Proceedings of Networking*, Athens Greece, 2004
[5] D. M. Lucantoni, G. L. Choudhury, W. Witt, "The Transient BMAP/PH/1 Queue", *Stochastic Models* **10**, pp.461-478, 1994
[6] W. Böhm. S. G. Mohanty, "Transient Analysis of Queues with Heterogeneous Arrivals", *Queuing Systems,* Vol.18, pp.27-45, 1994
[7] G. L. Choudhury, D. M. Lucantoni, W. Witt, "Multidimensional Transform Inversion with Application to the Transient M/G/1 Queue", *Annals of Applied Probability*, **4**, 1994, pp.719-740.
[8] J. Abate, G. L. Choudhury, W. Whitt, "An Introduction to Numerical Transform Inversion and its Application to Probability Models" In: W. Grassman, (ed.) *Computational Probability*, pp. 257–323. Kluwer, Boston , 1999.
[9] L. Kelinrock, R. Gail, *Queuing Systems*: *Problems and Solutions*, John Wiley & Sons, 1996.
[10] R. G. Hohlfeld, J. I. F. King, T. W. Drueding, G. v. H. Sandri, "Solution of convolution integral equations by the method of differential inversion", *SIAM Journal on Applied Mathematics*, Vol. 53 , No.1 (February 1993), Pages: 154 - 167
[11] A. S. Vasudeva Murthy, **"**A note on the differential inversion method of Hohlfeld et al.",*SIAM Journal on Applied Mathematics*, Vol. 55 , No.3 (June 1995), pp. 719 - 722
[12] P. L. Bharatiya , "The Inversion of a Convolution Transform Whose Kernel is a Bessel Function", *The American Mathematical Monthly*, Vol. 72, No. 4. (Apr., 1965), pp. 393-397

Vitalice K. Oduol received his pre-university education at Alliance High School in Kenya. In 1981 he was awarded a CIDA scholarship to study electrical engineering at McGill University, Canada, where he received the B.Eng. (Hons.) and M.Eng. degrees in 1985 and 1987, respectively, both in electrical engineering. In June 1992, he received the Ph.D. degree in electrical engineering at McGill University.

He was a research associate and teaching assistant while a graduate student at McGill University. He joined MPB Technologies, Inc. in 1989, where he participated in a variety of projects, including meteor burst communication systems, satellite on-board processing, low probability of intercept radio, among others. In 1994 he joined INTELSAT where he initiated research and development work on the integration of terrestrial wireless and satellite systems. After working at COMSAT Labs. (1996-1997) on VSAT networks, and TranSwitch Corp.(1998-2002) on product definition and architecture, he returned to Kenya, where since 2003 he has been with Department of Electrical and Information Engineering, University of Nairobi.

Dr. Oduol was a two-time recipient of the Douglas tutorial scholarship at McGill University. He is currently chairman, Department of Electrical and Information Engineering, University of Nairobi. His research interests include performance analysis, modeling and simulation of telecommunication systems, adaptive error control, feedback communication.