Transfer Function of Piezoelectric Material

C. Worakitjaroenphon and A. Oonsivilai

Abstract—The study of piezoelectric material in the past was in T-Domain form; however, no one has studied piezoelectric material in the S-Domain form. This paper will present the piezoelectric material in the transfer function or S-Domain model. S-Domain is a well known mathematical model, used for analyzing the stability of the material and determining the stability limits. By using S-Domain in testing stability of piezoelectric material, it will provide a new tool for the scientific world to study this material in various forms.

Keywords—Piezoelectric, Stability, S-Domain, Transfer function

I. INTRODUCTION

PIEZOELECTRIC materials are widely used in electromechanical sensors and actuators, such as telephone handsets, transmitter and receiver insets, robotic sensors, ultrasonic transducers for medical imaging and nondestructive evaluation NDE, as well as transducers used in the upper MHz range. [1]

As mentioned previously, the poling process is the critical element in being able to utilize the piezoelectric effect in a ferroelectric ceramic. Without poling, the ceramic is inactive, even though each one of the individual crystallites is piezoelectric in itself. With poling, however, the ceramic becomes extremely useful, provided that it is not heated above its Curie temperature (TC). If overheated, it loses its polarization and all of the orientation of the polarization produced by the poling process. Both two effects are operative in piezoelectric crystals in general, and particularly in ferroelectric ceramics. The direct effect (designated as a generator) is identified with the phenomenon where electrical charge (polarization) is generated from a mechanical stress, and the converse effect (designated as a motor) is associated with mechanical movement generated by the application of an electrical field. Both of these effects are illustrated. [8]

Tiersten [6] presents Hamilton principle for linear piezoelectric media, in which Lagrangian can create a dynamic equation of piezoelectric continuum. Afterwards Reinhard Lerch [3], Allik, H. and their group [7], studied Piezoelectric by Finite Element Method, using data derived from T-Domain model. Vincent and his research team presented a General Finite Element Formulation for Piezoelectrically Coupled Systems. Piezoelectric finite elements were developed based on Mindlin shell elements and

integrated in the FE package Samcef. Volume elements have also been derived and integrated. [5] Vincent and his team used state space model for attaining analytical dynamic equations of the system. J Ajitsaria and his research team [9] focused on an analytical approach for voltage and power generation, based on Euler-Bernoulli beam theory and Timoshenko beam equations, which compared with two previously described models in literature; Electrical equivalent circuit and Energy method, the transfer function between input acceleration and output displacement can be obtained in the Laplace plane.

So, this paper will present the piezoelectric material in the transfer function or S-Domain model, by utilizing a model of Piezoelectric Material in state space model. In the end, the S-Domain model will produce characteristic polynomial and output/input ratio of piezoelectric material.

II. PIEZOELECTRIC EQUATION

A. Piezoelectricity Beam theory

Piezoelectricity in Ferroelectric Ceramics, as mentioned previously, has the poling process as the critical element in being able to utilize piezoelectric effect in a ferroelectric ceramic. Without poling, the ceramic is inactive, even though each one of the individual crystallites is piezoelectric itself. With poling, however, the ceramic becomes extremely useful, provided that it is not heated above its Curie temperature (TC), where it loses its polarization and all of the orientation of the polarization produced by the poling process. [8]

Two effects are operative in piezoelectric crystals, in general, and in ferroelectric ceramics, in particular. The direct effect (designated as a generator) is identified with the phenomenon where an electrical charge (polarization) is generated from a mechanical stress and a converse effect (designated as a motor) is associated with a mechanical movement generated by the application of an electrical field. Both of these effects are illustrated in Fig. 1 The easy grasp of the principles. [8] In longitudinal effect, deformations are produced parallel to the electric axis as in Fig. 1(a) and in transverse effect, deformations occur at right angles to the electric axis as in 1(b). [10]

The static analysis of a piezoelectric cantilever sensor is typically performed by the use of calculations employed for deflection of a thermal bimorph. The principle is based on the strain compatibility between three cantilever beams joined along the bending axis. Due to forces applied by one or all of the layers, deflection of the three-layer structure is derived from a static equilibrium state. The structure considered is a

C. Worakitjaroenphon is currently a graduate student of School of Electrical Engineering, Suranaree University of Technology, Thailand, pursuing a degree in Electrical Engineering.(e-mail: chalida_w@live.com)
*A. Oonsivilai is an Assistant Professor at School of Electrical Engineering, Suranaree University of Technology, Nakhon Ratchasima, Thailand. He serves as an advisor for the Smart Materials, Health Sciences and Energy Research Center, and the Alternative and Sustainable Energy Research Unit (corresponding author: e-mail: anant@sut.ac.th).
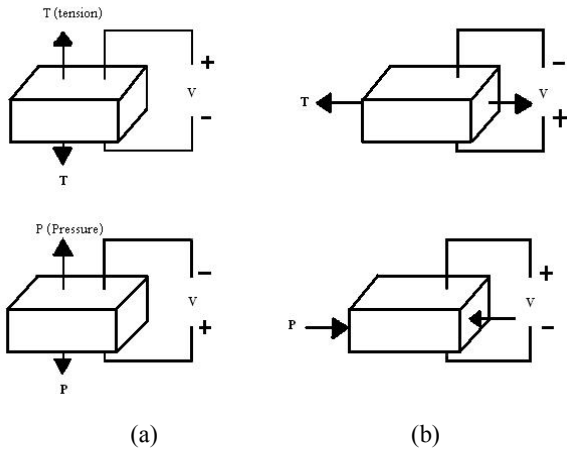
(a)   (b)

Fig. 1 Piezoelectric effects: (a) longitudinal effects; (b) transverse effects [10]
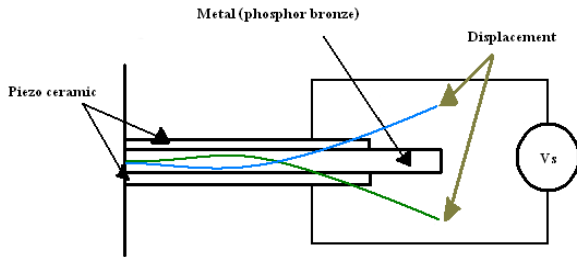

Fig. 2 Principles of producing flexion on a lever, using the transverse effect [10]

piezoelectric heterogeneous bimorph, where two piezoelectric layers are bonded on both sides of a purely elastic layer, i.e., brass. Fig. 2 shows a basic geometry of the three-layer multi-morph. A brass with a pure elasticity is sandwiched between the upper and lower layers of the PZT material. The modeling of this structure neglects shear effects and ignores residual stress-induced curvature. In addition, the beam thickness is much less than the piezoelectric-induced curvature, so the second order effects such as electrostriction can be ignored. [9]

Moreover, the radius of curvature for all the layers is assumed to be approximately the same as those of the structure, because of the assumption that the thickness is much less than the overall beam curvature. The total strain at the surface of each layer is the sum of the strains caused by the piezoelectric effect, the axial force, and the bending. It is noted that the sign of the surface strain depends on the bending of either the top or bottom surface of the layer. [9]

This theory is the foundation of the equations under the next section, which will provide the basis to the Space State Model.

B. Basic Piezoelectric Equation

The matrix equations in (1)-(2) are the basis of piezoelectric, which relate mechanical and electrical quantities in piezoelectric media [1]-[4]. It expresses the relationship between the stress, strain, electric field, and electric displacement field in a stress-charge form.

$$\{T\} = \left[c^E\right]\{S\} - \left[e\right]^t\{E\} \tag{1}$$

$$\{D\} = \left[e\right]\{S\} + \left[\varepsilon^S\right]\{E\} \tag{2}$$

Where

$\{T\} = \{T_{11}, T_{22}, T_{33}, T_{23}, T_{13}, T_{12}\}^t$ is the stress vector,

$\{S\} = \{S_{11}, S_{22}, S_{33}, 2S_{23}, 2S_{13}, 2S_{12}\}^t$ is the deformation vector,

$\{E\} = \{E_1 \quad E_2 \quad E_3\}$ is the electric field,

$\{D\} = \{D_1 \quad D_2 \quad D_3\}$ is the electric displacement,

$\left[c^E\right]$ is the elasticity matrix for constant electric,

$\left[\varepsilon^S\right]$ is the dielectric matrix for constant mechanical strain,

$\left[e\right]$ is the piezoelectric coupling coefficients matrix. [5]

The electric field $\{E\}$ is related to the electrical potential ϕ by :

$$\{E\} = -\nabla\phi = -\left\{\frac{\partial\phi}{\partial x}u_x + \frac{\partial\phi}{\partial y}u_y + \frac{\partial\phi}{\partial z}u_z\right\} \tag{3}$$

And the mechanical strain $\{S\}$ to the mechanical displacement $\{u\}$ by:

$$\{S\} = [B]\{u_{x,y,z}\} \tag{4}$$

Where $[B]$ is the Cartesian coordinates

$$[B] = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \tag{5}$$

The elastic behavior of piezoelectric media is governed by Newton's law.

$$DIV\{T\} = \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{\partial T_z}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} = \rho \ddot{u} \qquad (6)$$

Where

DIV is divergence of a dyadic.

ρ is density of the piezoelectric medium.

Whereas the electric behavior is described by Maxwell's Equation considering that piezoelectric media are insulating (no free volume charge).

$$DIV\{D\} = 0 \qquad (7)$$

Equations (1)-(7) constitute a complete set of differential equations which can be solved with appropriate mechanical (displacements and forces) and electrical (potential and charge) boundary conditions. [1]

The dynamic equations of a piezoelectric continuum can be derived from the Hamilton principle, in which the Lagrangian and the virtual work are properly adapted to include the electrical contributions, as well as the mechanical ones. The potential energy density of a piezoelectric material includes contributions from the strain energy and from the electrostatic energy. [5], [6]

$$H = \frac{1}{2}\left[\{S\}^T\{T\} - \{E\}^T\{D\}\right] \qquad (8)$$

Similarly, the virtual work density

$$\delta W = \{\delta u\}^t\{F\} - \delta\phi\sigma \qquad (9)$$

Where $\{F\}$ is the external force and σ is the electric charge. From (8), (9), into the Hamilton principle. [7]

$$0 = -\int_V \begin{pmatrix} \rho\{\delta u\}^t\{\ddot{u}\} - \{\delta S\}^t\left[c^E\right]\{S\} \\ + \{\delta S\}^t\left[e\right]^t\{E\} + \{\delta E\}^t\left[e\right]\{S\} \\ + \{\delta E\}^t\left[\varepsilon^S\right]\{E\} + \{\delta u\}^t\{P_b\} \end{pmatrix} dV$$

$$+ \int_{S_1}\{\delta u\}^t\{P_S\}dS \qquad (10)$$

$$+ \{\delta u\}^t\{P_c\} - \int_{S_2}\delta\phi\sigma dS - \delta\phi Q$$

In the finite element formulation, the displacement field $\{u\}$ and the electric potential ϕ over an element are related to the corresponding node values $\{u_i\}$ and $\{\phi_i\}$ by the mean of the shape functions $\{N_u\}$, $\{N_\phi\}$.

$$\{u\} = \left[N_u\right]\{u_i\} \qquad (11)$$

$$\phi = \left[N_\phi\right]\{\phi_i\} \qquad (12)$$

And therefore, the strain field $\{S\}$ and the electric field $\{E\}$ are related to the nodal displacements and potential by the shape functions derivatives $\left[B_u\right]$ and $\left[B_\phi\right]$ defined by:

$$\{S\} = [D]\{u\} = [D][N_u]\{u_i\} = [B_u]\{u_i\} \qquad (13)$$

$$\{E\} = -\nabla\phi = -\nabla\left[N_\phi\right]\{\phi_i\} = -\left[B_\phi\right]\{\phi_i\} \qquad (14)$$

Substituting expressions (11) – (14), into the variation principle (10), yields:

$$\begin{aligned} 0 = &-\{\delta u_i\}^t\int_V \rho[N_u]^t[N_u]dV\{\ddot{u}\} \\ &-\{\delta u_i\}^t\int_V [B_u]^t\left[c^E\right][B_u]dV\{u_i\} \\ &-\{\delta u_i\}^t\int_V [B_u]^t[e]\left[B_\phi\right]dV\{\phi_i\} \\ &-\{\delta\phi_i\}^t\int_V \left[B_\phi\right]^t[e]^t[B_u]dV\{u_i\} \\ &+\{\delta\phi_i\}^t\int_V \left[B_\phi\right]^t\left[\varepsilon^S\right]\left[B_\phi\right]dV\{\phi_i\} \\ &+\{\delta u_i\}^t\int_V [N_u]^t\{P_b\}dV \\ &+\{\delta u_i\}^t\int_{S_1} [N_u]^t\{P_S\}dS + \{\delta u_i\}^t[N_u]^t\{P_c\} \\ &-\{\delta\phi_i\}^t\int_{S_2}\left[N_\phi\right]^t\sigma dS - \{\delta\phi_i\}^t\left[N_\phi\right]^t Q \end{aligned} \qquad (15)$$

Which must be verified for any arbitrary variation of the displacements $\{\delta u_i\}$ and electrical potential $\{\delta\phi_i\}$ compatible with the essential boundary conditions. For an element, (15), can be written under the form:

$$[M]\{\ddot{u}_i\} + \left[K_{uu}\right]\{u_i\} + \left[K_{u\phi}\right]\{\phi_i\} = \{f_i\} \qquad (16)$$

$$\left[K_{\phi u}\right]\{u_i\} + \left[K_{\phi\phi}\right]\{\phi_i\} = \{g_i\} \qquad (17)$$

With

$$[M] = \int_V \rho[N_u]^t[N_u]dV \qquad (18)$$

$$\left[K_{uu}\right] = \int_V [B_u]^t\left[c^E\right][B_u]dV \qquad (19)$$

$$\left[K_{u\phi}\right] = \int_V [B_u]^t[e]^t\left[B_\phi\right]dV \qquad (20)$$

$$\left[K_{\phi\phi}\right] = -\int_V \left[B_\phi\right]^t\left[\varepsilon^S\right]\left[B_\phi\right]dV \qquad (21)$$

$$\left[K_{\phi u}\right] = \left[K_{u\phi}\right]^t \quad (22)$$

The matrixes above are element mass, stiffness, piezoelectric coupling and capacitance matrix respectively. The models below showed the external mechanical force and electric charge vectors.

$$\{f_i\} = \int_V [N_u]^t \{P_b\} dV + \int_{S_1} [N_u]^t \{P_S\} dS + [N_u]^t \{P_c\} \quad (23)$$

$$\{g_i\} = \int_{S_2} [N_\phi]^t \sigma dS - [N_\phi]^t Q \quad (24)$$

Each element k of the mesh is connected to its neighbouring elements at the global nodes and the displacement is continuous from one element to the next. Based on that formulation, piezoelectric finite elements of type multilayered Mindlin shell and volume has been derived. For shell elements, it is assumed that the electric field and displacement are uniform across the thickness and aligned on the normal to the mid-plane. The electrical degrees of freedom are the voltages ϕ_k across the piezoelectric layers; it is assumed that the voltage is constant over each element (this implies that the finite element mesh follows the shape of the electrodes). One electrical degree of freedom of type voltage per piezoelectric layer is defined. The assembly takes into account the equipotentiality condition of the electrodes; this reduces the number of electric variables to the number of electrodes. For volume elements, one additional degree of freedom of type electric potential is defined in each node of the piezoelectric volume element. [5]

C. State Space Model

The idea behind modeling structures embedding piezoelectric actuators and sensors using finite elements is indeed to gather the necessary information to design a good control strategy. It is therefore necessary to interface the structural analysis software (finite element package) with a control design software. The assembled system of equations can be complemented with a damping term $[C]\{\dot{U}\}$ to obtain the full equation of dynamics and the sensor equation:

$$\{0\} = [M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K_{UU}]\{U\} + \left[K_{U\Phi}^{(i)}\right]\{\Phi\} \quad (25)$$

$$\{G\} = \left[K_{\Phi U}^{(0)}\right]\{U\} + [K_{\Phi\Phi}]\{\Phi\} \quad (26)$$

Where

$\{U\}$ represents the mechanical degree of freedom,

$\{\Phi\}$ the electric potential degree of freedom,

$[M]$ the inertial matrix,

$[C]$ the damping matrix,

$[K_{UU}]$ the mechanical stiffness matrix,

$[K_{U\Phi}] = [K_{\Phi U}]^T$ the electromechanical coupling matrix,

$[K_{\Phi\Phi}]$ is the electric capacitance matrix.

The voltage actuation and charge sensing are considered. Actuation is done by imposing a voltage $\{\Phi\}$ on the actuators and sensing by imposing a zero voltage $(\{\Phi\} = \{0\})$ and measuring the electric charges $\{G\}$ appearing on the sensors. Using a truncated modal decomposition (n decoupled modes) $\{U\} = [Z]\{x(t)\}$, where $[Z]$ represents the n modal shapes and $\{x(t)\}$ the n modal amplitudes, (20) and (21), become:

$$\{0\} = [M][Z]\{\ddot{x}\} + [C][Z]\{\dot{x}\} + [K_{UU}][Z]\{x\} + \left[K_{U\phi}^{(i)}\right]\{\Phi\} \quad (27)$$

$$\{G\} = \left[K_{U\Phi}^{(0)}\right]^t [Z]\{x\} + [K_{\phi\phi}]\{\Phi\} \quad (28)$$

From the property of $[Z]^t[Z] = unitvactor$ will be:

$$[Z]^t[M][Z] = diag(\mu_k) \quad (29)$$

$$[Z]^t[K][Z] = diag(\mu_k \omega_k^2) \quad (30)$$

$$[Z]^t[C][Z] = diag(2\xi_k \mu_k \omega_k) \quad (31)$$

The modal frequencies:

$$[\Omega] = diag(\omega_k) \quad (32)$$

The modal masses:

$$[\mu] = diag(\mu_k) \quad (33)$$

The modal classical damping ratios of the considered structure:

$$[\xi] = diag(\xi_k) \quad (34)$$

So, multiply $[Z]^t$ to both sides of (27), and use the quality of (29)-(34), will make :

$$\{\dot{x}\} = I\{\dot{x}\} \quad (35)$$

$$\{\ddot{x}\} = -2[\xi][\Omega]\{\dot{x}\} - [\Omega^2]\{x\} - [\mu^{-1}][Z]^t[K_{U\Phi}^{(i)}]\{\Phi\} \quad (36)$$

$$\{G\} = [K_{U\Phi}^{(0)}]^t[Z]x + [K_{\Phi\Phi}]\{\Phi\} \quad (37)$$

Equations (35-37) can reformulated as (38-39)

$$\begin{Bmatrix}\dot{x}\\\ddot{x}\end{Bmatrix} = \begin{bmatrix}0 & I\\-\Omega^2 & -2\xi\Omega\end{bmatrix}\begin{Bmatrix}x\\\dot{x}\end{Bmatrix} - \begin{bmatrix}0\\\mu^{-1}Z^tK_{U\Phi}^{(i)}\end{bmatrix}\{\Phi\} \quad (38)$$

$$\{G\} = [K_{U\Phi}^{(0)^t}Z \quad 0]\begin{Bmatrix}x\\\dot{x}\end{Bmatrix} + [K_{\Phi\Phi}]\{\Phi\} \quad (39)$$

Which is in the same format as :

$$\dot{x} = Ax + Bu \quad (40)$$

$$y = Dx + Hu \quad (41)$$

Where

$[Z]^t$ is the modal shapes, $[K_{\Phi U}^{(0)}][Z]$ is the modal electric charge on the sensor, $[Z]^t[K_{U\Phi}^{(i)}]$ is the modal electric charge on the actuators, transposed (by reciprocity), representing the participation factor of the actuators to each mode, are obtained from a dynamic finite element analysis.

III. TRANSFER FUNCTION MODEL

From (40) - (41) is converted to S-Domain will be:

$$sX(s) = AX(s) + B\Phi(s) \quad (42)$$

$$\Upsilon(s) = DX(s) + H\Phi(s) \quad (43)$$

When set (42)-(43) to the new form:

$$(sI - A)X(s) = B\Phi(s) \quad (44)$$

$$X(s) = (sI - A)^{-1}B\Phi(s) \quad (45)$$

$$\frac{\Upsilon(s)}{\Phi(s)} = D(sI - A)^{-1}B + H \quad (46)$$

where.

$$A = \begin{bmatrix}0 & I\\-\Omega^2 & -2\xi\Omega\end{bmatrix} \quad (47)$$

$$B = \begin{bmatrix}0\\\mu^{-1}Z^tK_{U\Phi}^{(i)}\end{bmatrix} \quad (48)$$

$$H = [K_{\Phi\Phi}] \quad (49)$$

$$D = [K_{U\Phi}^{(0)^t}Z \quad 0] \quad (50)$$

$$I = Unit \quad matrix$$

Of these values could be determined. Characteristic polynomial is.

$$P(s) = |sI - A| = \begin{vmatrix}s & I\\-\Omega^2 & s + 2\xi\Omega\end{vmatrix} = s(s + 2\xi\Omega) + \Omega^2I \quad (51)$$

Characteristic polynomial:

$$s^2 + 2\xi\Omega s + \Omega^2I \quad (52)$$

From (46) - (52) can be found in the following output/input ratio

$$\frac{G(s)}{\Phi(s)} = D(sI - A)^{-1}B + H \quad (53)$$

$$\frac{G(s)}{\Phi(s)} = \frac{[K_{\Phi\Phi}]s^2 + 2[K_{\Phi\Phi}]\xi\Omega s + ([K_{\Phi\Phi}]\Omega^2 - K_{U\Phi}^{(0)^t}Z\mu^{-1}Z^tK_{U\Phi}^{(i)})I}{s^2 + 2\xi\Omega s + \Omega^2I} \quad (54)$$

IV. CONCLUSION

Model (52) is the characteristic polynomial and model (54) is the output/input ratio. However, these values are in the form of a matrix. We can also make the calculation easier by using Cartesian coordinates to estimate the value in form of one-dimensional system.

ACKNOWLEDGMENT

REFERENCES

[1] Reinhard Lerch, "Simulation of Piezoelectric Devices by Two- and Three Dimensional Finite Elements," IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control, vol. 37, pp. 233-247, May 1990.
[2] C.Kusakabe, Y.Tomikawa, T.Takano, H.Tarnura and H.Okamoto, "Thin Vibrator Alarm Using a Cantilever Piezoelectric Actuator," Journal of L. Appl. Phys. Vol. 35,pp. 5018-5022 1996.
[3] R. Lerch, H. Kaarmann, "Three-Dimensional Finite Element Analysis of Piezoelectric Media," Proceeding of IEEE Ultrasonics Symposiuni 1987.
[4] M. Sunar, S.S. Rao, "The piezoelectric Control Design and Actuator Placement," AIAA Juornal, Vol. 35, NO. 3, pp.534-539, March 1997
[5] Vincent PIEFORT, Andre' PREUMONT ,"Finite element modeling of piezoelectric structures," Active Structures Laboratory, ULB-CP 165/42, Av F.D. Roosevelt 50, B-1050 Brussels, Belgium.
[6] Tiersten, H.F., 1967, "Hamilton's Principle For Linear Piezoelectric Media," in "Proceedings of the IEEE", pages 1523-1524.
[7] Allik, H. and T. J. R. Hughes, 1970, "Finite Element Method for Piezoelectric Vibration," International Journal for Numerical Mathods in Engineering, 2:151-157.

[8] Gene H. Haertling, "Ferroelectric Ceramics: History and Technology," Department of Ceramic and Materials Engineering, Clemson University, Clemson, South Carolina 29634-0907

[9] J Ajitsaria, S Y Choe, D Shen and D J Kim, "Modeling and analysis of bimorph piezoelectric cantilever beam for voltage generation," Auburn University, Auburn, Albama, 36849, U.S.A.

[10] Chang-Hwan Lee, Hyun Kyo Jung, "Two-Dimensional Finite Element Analysis of Piezoelectric Cantilever for Silent Alarm System", IEEE Transl. Electrical machines and systems., Vol. 2, pp. 1159-1162, 2001.

Chalida Worakitjaroenphon, was born in Saraburi Province, Thailand, in 1986. She received a Bachelor's Degree in Electrical Engineering, from King Mongkut's Institute of Technology, Ladkrabang, Thailand, in 2008. She is currently pursuing a Master's Degree in Electrical Engineering, at School of Electrical Engineering, Suranaree University of Technology, Thailand.

Her Summer Internship was spent at Technical Department, Aircraft Maintenance Engineering Group, Suvarnabhumi Airport, with Thai Airways International Public Company Limited, Thailand, in 2007. She is currently the owner of Charoenpholkehapun Shop, General manager of Anankehapun Shop, Saraburi, Shareholder and General manager of Seejamnongkehakij.Co.,Ltd., Nakhon Ratchasima, and Electrical Engineerof Anawach Engineering Co., Ltd., Bangkok, Thailand. Her areas of interests are electrical power system, stability and piezoelectric.

Anant Oonsivilai, was born in Khon Kaen Province, Thailand, in 1963. He received a Bachelor's Degree from Khon Kaen University, and a Master's Degree from King Mongkut's Institute of Technology, North Bangkok, Thailand and a PhD. from Dalhousie University, Canada, all in field of Electrical Engineering, in 1986, 1992 and 2000, respectively.

He is currently an Assistant Professor in School of Electrical Engineering, Suranaree University of Technology, Thailand. His areas of interests are electrical power system, stability, control technology, advance alternative, and sustainable energy.