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Thermal and Mechanical Buckling of Short and Long Functionally Graded Cylindrical Shells Using First Order Shear Deformation Theory

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***Abstract*—**This paper presents the buckling analysis of short and long functionally graded cylindrical shells under thermal and mechanical loads. The shell properties are assumed to vary continuously from the inner surface to the outer surface of the shell. The equilibrium and stability equations are derived using the total potential energy equations, Euler equations and first order shear deformation theory assumptions. The resulting equations are solved for simply supported boundary conditions. The critical temperature and pressure loads are calculated for both short and long cylindrical shells. Comparison studies show the effects of functionally graded index, loading type and shell geometry on critical buckling loads of short and long functionally graded cylindrical shells.

***Keywords*—**Buckling, Functionally graded materials, Short and long cylindrical shell, Thermal and mechanical loads.

I. Introduction

THEORETICAL formulation on buckling of elastic shells have been reported by Donnell [1]. He obtained the critical buckling loads for short cylindrical shells under torsion. Koiter [2], Flügge [3] and Brush and Almorth [4] have derived other equations for cylindrical shells under different loads. Shahsiah and Eslami [5] employed the improved Donnell theory to study the instability of functionally graded cylindrical shells subjected to thermal loading. Woo and Meguid [6] investigated the postbuckling behaviour of functionally graded shallow cylindrical shells and plates under thermal and mechanical loads. Khazaeinejad and Najafizadeh [7] studied the critical buckling loads for functionally graded cylindrical shells under different mechanical loads. They employed the first order shear deformation theory to obtain the buckling equation.

The stability of short and long functionally graded cylindrical shells under thermal and mechanical loads are studied in the present paper. The properties are assumed to vary continuously from the inner surface to the outer surface of the shell. Employing the first order shear deformation theory, the kinematic relations are obtained. The equilibrium and stability equations are derived using the total potential energy equations and Euler equations. The resulting equations are solved for simply supported boundary conditions. The critical temperature and pressure loads are calculated for both short and long cylindrical shells.

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II. Fundamental Relations

A circular cylindrical shell made of functionally graded materials with thickness h, radius R and length L is considered. The material properties are assumed to be graded through the thickness direction. The constituent materials are assumed to be ceramic and metal. The volume fractions of the ceramic V_c and metal V_m corresponding to the power law are expressed as [8]

$$V_c = \left(\frac{2z+h}{2h}\right)^k, \qquad V_m = 1 - V_c \tag{1}$$

Here, subscripts m and c are the metal and ceramic constituents, respectively, z is the thickness coordinate $(-h/2 \le z \le h/2)$, and k is the power law index that takes values greater than or equal to zero. The variation of the composition of ceramic and metal is linear for k=1. The value of k equal to zero represents a fully ceramic shell. The properties of functionally graded cylindrical shell are determined from the volume fraction of the material constituents. The Young's modulus, E, and coefficient of thermal expansion, α, are assumed to change in the thickness direction [8]

$$\begin{aligned} E(z) &= E_c V_c + E_m V_m \\ \rho(z) &= \rho_c V_c + \rho_m V_m \end{aligned} \tag{2}$$

The Poisson' ratio, v, is assumed to be constant across the plate thickness. Substituting Eq. (1) into (2), the material properties of the FG plate are determined, which are the same as the equations proposed by Praveen and Reddy [8]

$$\begin{aligned} E(z) &= E_m + (E_c - E_m)\left(\frac{2z+h}{2h}\right)^k \\ \rho(z) &= \rho_m + (\rho_c - \rho_m)\left(\frac{2z+h}{2h}\right)^k \end{aligned} \tag{3}$$

For both short and long cylindrical shells, the displacement

field based on the first order shear deformation theory is expressed as

$$\begin{aligned} u(x,\theta,z) &= u_0(x,\theta) + z\beta_x(x,\theta) \\ v(x,\theta,z) &= v_0(x,\theta) + z\beta_\theta(x,\theta) \\ w(x,\theta,z) &= w(x,\theta) \end{aligned} \tag{4}$$

where u, v, w are the displacement of the shell in the x, θ, z directions, respectively. The displacement-strain relations for long cylindrical shells are written as

$$\begin{aligned} \varepsilon_{xx} &= u_{0,x} + 0.5w^2_{,x} + z\beta_{xx} \\ \varepsilon_{\theta\theta} &= \frac{v_{0,\theta} + w}{a} + \frac{1}{2}\left(\frac{v_0 - w_{,\theta}}{a}\right)^2 + \frac{z}{a}\beta_{\theta\theta} \\ \gamma_{x\theta} &= \frac{u_{0,\theta}}{a} + v_{0,x} + \left(\frac{-v_0 w_{,x} + w_{,x} w_{,\theta}}{a}\right) + 2z k_{x\theta} \\ \gamma_{xz} &= w_{,x} + \beta_x, \qquad \gamma_{\theta z} = \frac{1}{a} w,\theta + \beta_\theta \end{aligned} \tag{5}$$

For cylindrical shells under thermal and mechanical loads, the total strain components may be expressed as

$$\begin{aligned} \varepsilon_{ij} &= \varepsilon^m{}_{ij} + \varepsilon^T{}_{ij}, \qquad \varepsilon^T{}_{ij} = \alpha T \\ \varepsilon^m_{xx} &= \frac{1}{E}(\sigma_{xx} - \upsilon\sigma_{\theta\theta}), \qquad \varepsilon^m_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \upsilon\sigma_{xx}) \\ \gamma^m_{x\theta} &= \frac{\tau_{x\theta}}{G}, \qquad \gamma^m_{xz} = \frac{\tau_{xz}}{G}, \qquad \gamma^m_{\theta z} = \frac{\tau_{\theta z}}{G}, \qquad G = \frac{E}{2(1+\upsilon)} \end{aligned} \tag{6}$$

The related stress components can be written as

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\upsilon_2}\left[\varepsilon_{xx} + \upsilon\varepsilon_{\theta\theta} - (1+\upsilon)\alpha T\right] \\ \sigma_{\theta\theta} &= \frac{E}{1-\upsilon^2}\left[\varepsilon_{\theta\theta} + \upsilon\varepsilon_{xx} - (1+\upsilon)\alpha T\right] \\ \tau_{x\theta} &= G\gamma_{x\theta} = \frac{E}{2(1+\upsilon)}\gamma_{x\theta} \\ \tau_{\theta z} &= G\gamma_{\theta z} = \frac{E}{2(1+\upsilon)}\gamma_{\theta z} \end{aligned} \tag{7}$$

The stress and moment resultants are defined by the following relations

$$\begin{aligned} (N_i.M_i,P_i) &= \int_{-h/2}^{h/2} \sigma_i(1,z,z^3)dz, \qquad i = x,\theta,x\theta \\ (Q_i,R_i) &= K\int_{-h/2}^{h/2} \sigma_{iz}(1,z^2)dz, \qquad i = x,\theta \end{aligned} \tag{8}$$

Substitution of Eq. (7) into Eq. (8), the stress and moment resultants can be obtained in terms of strain components

$$\begin{aligned} N_x &= \frac{A}{1-\upsilon^2}(\varepsilon_{xm} + \upsilon\varepsilon_{\theta m}) + \frac{B}{1-\upsilon^2}(k_x + \upsilon k_\theta) - \frac{T_0}{1-\upsilon} \\ N_\theta &= \frac{A}{1-\upsilon^2}(\varepsilon_{\theta m} + \upsilon\varepsilon_{xm}) + \frac{B}{1-\upsilon^2}(k_\theta + \upsilon k_x) - \frac{T_0}{1-\upsilon} \\ N_{x\theta} &= \frac{A}{2(1+\upsilon)}\gamma_{x\theta m} + \frac{B}{1+\upsilon}k_{x\theta} \\ M_x &= \frac{B}{1-\upsilon^2}(\varepsilon_{xm} + \upsilon\varepsilon_{\theta m}) + \frac{C}{1-\upsilon^2}(k_x + \upsilon k_\theta) - \frac{T_1}{1-\upsilon} \\ M_\theta &= \frac{B}{1-\upsilon^2}(\varepsilon_{\theta m} + \upsilon\varepsilon_{xm}) + \frac{C}{1-\upsilon^2}(k_\theta + \upsilon k_x) - \frac{T_1}{1-\upsilon} \\ M_{x\theta} &= \frac{B}{2(1+\upsilon)}\gamma_{x\theta m} + \frac{C}{1+\upsilon}k_{x\theta} \\ Q_x &= K\left(\frac{A}{2(1+\upsilon)}\gamma_{xzm}\right. \\ Q_\theta &= K\left(\frac{A}{2(1+\upsilon)}\right)\gamma_{\theta zm} \end{aligned} \tag{9}$$

Employing the total potential energy associated with Euler equation, the equilibrium equations for long functionally graded cylindrical shells can be derived based on the first order shear deformation theory as follows

$$\begin{aligned} &aN_{x,x} + N_{x\theta,x} = 0 \\ &N_{\theta,\theta} + aN_{x\theta,x} - \frac{1}{a}(v_0 - w_{,\theta})N_\theta + w_{,x}N_{x\theta} = 0 \\ &-N_\theta + aN_x w_{,xx} + (-v_{0,x} + w_{,x\theta})N_{x\theta} + aQ_{x,x} \\ &\qquad + \frac{1}{a}(-v_{0,\theta} + w_{,\theta\theta})N_\theta + N_{x\theta}w_{,x\theta} + Q_{\theta,\theta} = 0 \\ &aM_{x,x} + M_{x\theta,\theta} - aQ_x = 0 \\ &aM_{x\theta,x} + M_{\theta,\theta} - aQ_\theta = 0 \end{aligned} \tag{10}$$

For short cylindrical shells the v_0 term in Eq. (5) should be set to zero, thus, the equilibrium equations for short functionally graded are reduced to

$$\begin{aligned} &aN_{x,x} + N_{x\theta,\theta} = 0 \\ &aN_{x\theta,x} + N_{\theta,\theta} = 0 \\ &-N_\theta + aN_x w_{,xx} + aQ_{x,x} + \frac{1}{a}(w_{,\theta\theta})N_\theta + 2N_{x\theta}w_{,x\theta} + Q_{\theta,\theta} = 0 \\ &aM_{x,x} + M_{x\theta,\theta} - aQ_x = 0 \\ &aM_{x\theta,x} + M_{\theta,\theta} - aQ_\theta = 0 \end{aligned} \tag{11}$$

If V is the total potential energy of the shell embedded in an elastic medium, its variation in equilibrium state using the Taylor series can be expressed as

$$\Delta V = \delta V + \frac{1}{2!}\delta^2 V + \frac{1}{3!}\delta^3 V + \ldots . \tag{12}$$

where the first term is associated with the state of equilibrium. To establish the stability equations, the condition $\delta^2 V = 0$ is used. The stability of original configuration of the shell in the neighborhood of the equilibrium state can be obtained by the

sign of the second variation $\delta^2 V$. For all virtual displacement, the equilibrium is stable for $\delta^2 V > 0$ and for at least one admissible set of virtual displacements, the equilibrium is unstable for $\delta^2 V < 0$. If we assume that the equilibrium state of a FG cylindrical shell under external pressure is defined in terms of displacement components u^0, v^0, and w^0, the displacement components of a neighboring stable state differ by u^1, v^1, and w^1 with respect to the equilibrium position. Thus, the total displacements of a neighboring state are [9]

$$\begin{aligned} u &= u^0 + u^1 \\ v &= v^0 + v^1 \\ w &= w^0 + w^1 \end{aligned} \tag{13}$$

In similar way, the resultants of a neighboring state may be related to the equilibrium state according to the following relations:

$$\begin{aligned} N_x &= N_x^0 + N_x^1 \\ M_x &= M_x^0 + M_x^1 \\ N_\theta &= N_\theta^0 + N_\theta^1 \\ M_\theta &= M_\theta^0 + M_\theta^1 \\ N_{x\theta} &= N_{x\theta}^0 + N_{x\theta}^1 \\ M_{x\theta} &= M_{x\theta}^0 + M_{x\theta}^1 \\ Q_x &= Q_x^0 + Q_x^1 \\ Q_\theta &= Q_\theta^0 + Q_\theta^1 \end{aligned} \tag{14}$$

where N_x^0, N_θ^0 and $N_{x\theta}^0$ are the prebuckling mechanical forces that describe the linear parts of the force increments corresponding to u^1, v^1, and w^1. The stability equations may be obtained by substituting Eqs. (13) and (14) into equilibrium equations (11) as follows
For long cylindrical shell:

$$\begin{aligned} &aN_{x1,x} + N_{x\theta 1,\theta} = 0 \\ &aN_{x\theta 1,x} + N_{\theta 1,\theta} = 0 \\ &-N_{\theta 1} + aQ_{x1,x} + aw_{1,xx}N_{x0} - (v_{1,x} - w_{1,x\theta})N_{x\theta 0} \\ &\qquad + Q_{\theta 1,\theta} - \frac{1}{a}(v_{1,\theta} - w_{1,\theta\theta})N_{\theta 0} + w_{1,x\theta}N_{x\theta 0} = 0 \\ &-aQ_{x1} + aM_{x1,x} + M_{x\theta 1,\theta} = 0 \\ &-aQ_{\theta 1} + aM_{x\theta 1,x} + M_{\theta 1,\theta} = 0 \end{aligned} \tag{15}$$

For short cylindrical shell:

$$\begin{aligned} &aN_{x1,x} + N_{x\theta 1,\theta} = 0 \\ &aN_{x\theta 1,x} + N_{\theta 1,\theta} = 0 \\ &-N_{\theta 1} + aQ_{x1,x} + aw_{1,xx}N_{x0} + Q_{\theta 1,\theta} \\ &\qquad + \frac{1}{a}w_{1,\theta\theta}N_{\theta 0} + 2w_{1,x\theta}N_{x\theta 0} = 0 \\ &-aQ_{x1} + aM_{x1,x} + M_{x\theta 1,\theta} = 0 \\ &-aQ_{\theta 1} + aM_{x\theta 1,x} + M_{\theta 1,\theta} = 0 \end{aligned} \tag{16}$$

Here, the superscript 1 and 0 describe the states of stability and equilibrium conditions, respectively. The terms in the stability equations with superscript 0 satisfy the equilibrium condition and therefore drop out of the equations. The functionally graded cylindrical shell has simply supported boundary conditions and the following approximate solutions satisfy the resulting equations and the simply supported boundary conditions are assumed

$$\begin{aligned} &u_1 = u'_1 \cos(\lambda x)\ \sin(n\theta), \qquad 0 \le x \le L \\ &v_1 = v'_1 \sin(\lambda x)\ \cos(n\theta), \qquad 0 \le \theta \le 2\pi \\ &w_1 = w'_1 \sin(\lambda x)\ \sin(n\theta) \\ &\beta_{x1} = \beta'_{x1} \cos(\lambda x)\ \sin(n\theta) \\ &\beta_{\theta 1} = \beta'_{\theta 1} \sin(\lambda x)\ \cos(n\theta) \end{aligned} \tag{17}$$

where $\lambda = m\pi/L$. For functionally graded cylindrical shell under thermal loading, the temperature function and pre-buckling forces are expressed as

For Uniform temperature changes:

$$T(x) = \frac{\Delta T x}{L}, \qquad 0 \le x \le L$$

For radial temperature changes:

$$T(z) = T_m + \frac{\Delta T}{[S]} I(z)$$

$$\begin{aligned} I(z) = &\left(\frac{2z+h}{2h}\right) - \frac{K_{cm}}{(k+1)K_m}\left(\frac{2z+h}{2h}\right)^{k+1} + \frac{K_{cm}^2}{(2k+1)K_m^2}\left(\frac{2z+h}{2h}\right)^{2k+1} \\ &- \frac{K_{cm}^3}{(3k+1)K_m^3}\left(\frac{2z+h}{2h}\right)^{3k+1} + \frac{K_{cm}^4}{(4k+1)K_m^4}\left(\frac{2z+h}{2h}\right)^{4k+1} \\ &- \frac{K_{cm}^5}{(5k+1)K_m^5}\left(\frac{2z+h}{2h}\right)^{5k+1} \end{aligned}$$

$$N_{\theta 0} = N_{x\theta 0} = 0, \qquad N_{x0} = -\frac{1}{1-\upsilon}\int_{\frac{-h}{2}}^{\frac{h}{2}} \alpha(z)E(z)T(z)dz \tag{18}$$

For mechanical loading, the pre-buckling forces are defined by

$$N_{\theta 0} = -P_e a,\ N_{x0} = N_{x\theta 0} = 0 \tag{19}$$

Substituting Eq. (17) into the Eqs. (15) and (16), leads to a five sets of differential equations with respect the unknown constants defined in Eq. (17). Solving these equations gives a

function for buckling pressure dependent on the half-wave numbers power law index of functionally graded material, and geometry parameters of the functionally graded shell.

III. Results And Discussion

The critical temperature and pressure loads of a functionally graded cylindrical shell under thermal and mechanical loads are obtained based on the first order shear deformation theory. The functionally graded cylindrical shell is composed of aluminum and alumina as metal and ceramic materials, respectively. The material properties are listed in Table 1. The numerical results are calculated for a=0.5 m and h/a=0.05. Figures 1-3 show the variations of critical temperature and pressure loads for short and long functionally graded cylindrical shells with different geometry.

TABLE I
Material Properties

Material	E (GPa)	$\alpha \times 10^{-6}$ (1/°C)	v
Stainless steel	200	11.7	0.3
Alumina	380	7.4	0.3

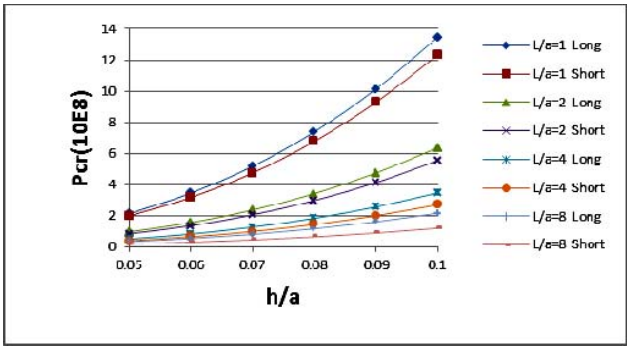

Fig. 1 Comparison of critical pressure loads (Pa) for short and long functionally graded cylindrical shells.

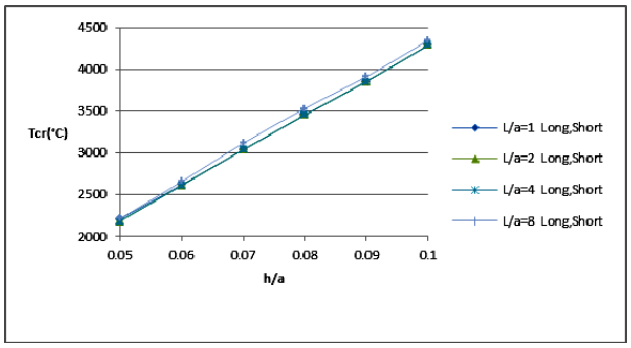

Fig. 2 Comparison of critical temperatures (°C) for short and long functionally graded cylindrical shells under uniform temperature.

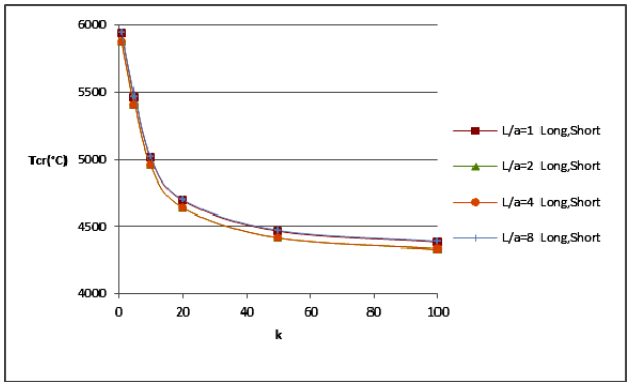

Fig. 3 Variations of critical temperatures (°C) for short and long functionally graded cylindrical shells under radial temperature with functionally graded index and L/a ratio.

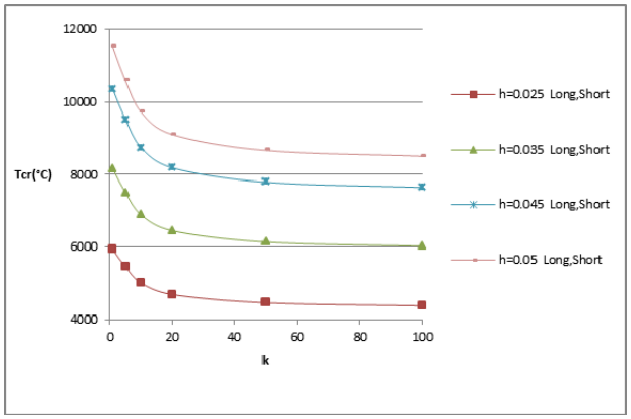

Fig. 4 Variations of critical temperatures (°C) for short and long functionally graded cylindrical shells under radial temperature with functionally graded index and L/a=1.

IV. Conclusions

The buckling analysis of functionally graded cylindrical shells under thermal and mechanical loads is presented in this paper. The following are concluded:

1. The difference between the short and long functionally graded cylindrical shells can be seen in their longitudinal strain. For long functionally graded cylindrical shells, this strain is equal to zero.
2. The critical pressure loads are increased for both short and long functionally graded cylindrical shells as the thickness-to-radius ratio h/a is increased.
3. The critical pressure loads are increased for both short and long functionally graded cylindrical shells as the length-to-radius ratio L/a is increased

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