The study of the discrete risk model with random income

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Abstract—In this paper, we extend the compound binomial model to the case where the premium income process, based on a binomial process, is no longer a linear function. First, a mathematically recursive formula is derived for non ruin probability, and then, we examine the expected discounted penalty function, satisfy a defect renewal equation. Third, the asymptotic estimate for the expected discounted penalty function is then given. Finally, we give two examples of ruin quantities to illustrate applications of the recursive formula and the asymptotic estimate for penalty function.

Keywords—Discounted penalty function, Compound binomial process, Recursive formula, Discrete renewal equation, Asymptotic estimate.

I. INTRODUCTION

IN the classical compound binomial risk model, the number of claims is assumed to follow a binomial process $\{N(t), t = 0, 1, 2, \cdots\}$. The premium received in each period is one. In any period the probability of claim is $p(0 < p < 1)$, and the probability of no claim is $q = 1 - p$. We assume that claims occur at the end of the period and denote by $\xi_t = 1$ the event where a claim occurs in period $(t - 1, t]$ and $\xi_t = 0$ the event where no claim occurs in period $(t-1, t]$. The individual claim amounts $\{X_1, X_2, \cdots\}$ are independent and identically distributed(i.i.d.) positive integer valued random variables with distribution function(d.f.) $F(x) = 1 - \overline{F}(x) = Pr(X \leq x)$ and probability function(p.f.) $f(x)$ and finite mean μ, where the X is an arbitrary X_i and X_i is the size of the ith claim and X is independent of the binomial process $\{N(t)\}$. The aggregate claim amount up to time t is $S(t) = \sum_{i=1}^{N(t)} X_i$. For $t = 0, 1, 2, \cdots$, the surplus of the insurer at time t is

$$U(t) = u + t - \sum_{i=1}^{t} X_i \xi_i, \tag{1}$$

where $u = U(0)$, which is a non-negative integer, is the initial surplus. This model has been studied by many researchers in recent years. See, for example, Gerber[1], Shiu[2], Dickson[3], Cheng and Zhu[4]and Cheng et al.[5], and references therein.

In this paper, we suppose that the premium income is no longer a linear function of time but another binomial process $\{M(t), t = 0, 1, 2, \cdots\}$ with parameter p_1, independent of $\{N(t), t = 0, 1, 2, \cdots\}$ and $\{X_1, X_2, \cdots\}$. Where $M(t)$ is corresponding to the number of the customers up to time t. For simplicity, we assume that the size of premium payment is 1 for each one. The insurer's surplus process at the time of t is

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$$U(t) = u + M(t) - \sum_{i=1}^{N(t)} X_i = u - V(t), t = 0, 1, 2, \cdots, \tag{2}$$

the readers can see Temnov[6] for more details of related model. In detail, we denote by $\eta_t = 1$ the event where a payment occurs in period $(t - 1, t]$ and $\eta_t = 0$ the event where no payment occurs in period $(t-1, t]$. If the event occurs in period $(t-1, t]$, we suppose that the event happen at the beginning of the period. Let $Pr(\eta_t = 1) = p_1, Pr(\eta_t = 0) = q_1 = 1 - p_1$. Then the model(2) can also be expressed as

$$U(t) = u + \sum_{i=1}^{t} \eta_i - \sum_{i=1}^{t} X_i \xi_i, t = 0, 1, 2, \cdots.$$

We suppose that the positive security loading condition holds, that is, if we denote by θ the relative security loading then,

$$\theta = \frac{p_1}{p\mu} - 1 > 0, \tag{3}$$

Let $p(k) = Pr(X = k), k = 1.2.\cdots$ be the (p.f.) of the claim amounts. Let

$$p(0) = 0, \quad P(n) = \sum_{k=0}^{n} p(k) = 1 - \overline{P}(n), \quad P(0) = 0,$$

$$\mu = E(X) = \sum_{k=0}^{\infty} kp(k) = \sum_{k=0}^{\infty} \overline{P}(k),$$

let $T = \inf\{t \geq 0, U(t) < 0\}$ be the time of ruin, and $\Psi(u) = Pr\{T < \infty\}$ be the probability of ultimate ruin from initial surplus u, $\Phi(u) = 1 - \Psi(u)$ denotes the non ruin probability. If ruin occurs, $|U(T)|$ is the deficit at ruin and U(T-) is the surplus immediately prior to ruin. Denote by

$$m_v(u) = E[v^T w(U(T-), |U(T)|)I(T < \infty)|U(0) = u],$$

the (Gerber-Shiu) expected discounted penalty function, which was first introduced by Gerber and Shiu[7]. Here, $w(x_1, x_2), x_1 \geq 0, x_2 \geq 0$, is a non-negative bounded function, $0 < v \leq 1$ is the discount factor and $I(.)$ is the indicator function. The expected discounted penalty function provides a unified means of studying the joint distribution of the surplus immediately prior to ruin and the deficit at ruin. For further discussion of it, see Bao[8], K. P. Pavlova and G.E. Willmot[9],J. Cai and D.C.M. Dickson [10], G. E. Willmot and D.C.M. Dickson[11], and references therein.

In this paper we study the expected discounted penalty function of model (2) when the discount factor $v = 1$, for simplicity, we write $m(u) = m_1(u)$. In Section 2, We first derive a mathematically recursive formula for non ruin probability $\Phi(u)$. In Section 3, We derive a defective renewal equation for the penalty function $m(u)$. in Section 4, we give the asymptotic estimate for $m(u)$. Finally, we give two

examples of ruin quantities to illustrate applications of the recursive formula and the asymptotic estimate for $m(u)$.

II. NON RUIN PROBABILITY

In this section, we derive the recursive formula for $\Phi(u)$.

Theorem2.1. *The non ruin probability $\Phi(u)$ satisfies the following recursive formula*

$$\Phi(u+1) = \Phi(0) + \frac{p}{p_1 q} \sum_{k=0}^{u} \Phi(k)[p_1 \overline{P}(u+1-k) + q_1 \overline{P}(u-k)]$$

(4)

where

$$\Phi(0) = \frac{p_1 - p\mu}{p_1 q}, \tag{5}$$

Proof. We consider $U(t)$ in the first period $(0, 1]$ and separate the five possible cases as following:

(1) no premium arrives in $(0, 1]$ and no claim occurs in $(0, 1]$;

(2) a premium arrives in $(0, 1]$ and no claim occurs in $(0, 1]$;

(3) no premium arrives in $(0, 1]$ and a claim occurs in $(0, 1]$, but no ruin;

(4) a premium arrives in $(0, 1]$ and a claim occurs in $(0, 1]$, but no ruin;

(5) except the four cases of above.

According to the laws of conditional probability, the non ruin probability is equal to

$$\Phi(u) = q_1 q\Phi(u) + p_1 q\Phi(u+1) + q_1 p \sum_{k=1}^{u} \Phi(u-k)p(k) +$$

$$p_1 p \sum_{k=1}^{u+1} \Phi(u+1-k)p(k). \tag{6}$$

Using $p(0) = 0$, then (6) is equivalent to

$$(1 - q_1 q)\Phi(u) =$$

$$p_1 q\Phi(u+1) + p \sum_{k=0}^{u} \Phi(k)[p_1 p(u+1-k) + q_1 p(u-k)], \tag{7}$$

summing (7) over u from 0 to t, we obtain

$$(1 - q_1 q) \sum_{u=0}^{t} \Phi(u) =$$

$$p_1 q \sum_{u=0}^{t} \Phi(u+1) + p \sum_{u=0}^{t} \sum_{k=0}^{u} \Phi(k)[p_1 p(u+1-k) + q_1 p(u-k)]. \tag{8}$$

Owning to

$$\sum_{u=0}^{t} \sum_{k=0}^{u} \Phi(k)[p_1 p(u+1-k) + q_1 p(u-k)]$$

$$= \sum_{k=0}^{t} \sum_{u=k}^{t} \Phi(k)[p_1 p(u+1-k) + q_1 p(u-k)]$$

$$= \sum_{k=0}^{t} \Phi(k)[p_1 P(t+1-k) + q_1 P(t-k)]. \tag{9}$$

Substitution (9) into (8) and rearranging terms, then (8) is equivalent to

$$p_1 q[\Phi(t+1) - \Phi(0)] =$$

$$p \sum_{k=0}^{t} \Phi(k) - p \sum_{k=0}^{t} \Phi(k)[p_1 P(t+1-k) + q_1 P(t-k)]$$

$$= p \sum_{k=0}^{t} \Phi(k)[p_1 \overline{P}(t+1-k) + q_1 \overline{P}(t-k)],$$

from which we get

$$p_1 q\Phi(t+1) =$$

$$p_1 q\Phi(0) + p \sum_{k=0}^{t} \Phi(k)[p_1 \overline{P}(t+1-k) + q_1 \overline{P}(t-k)]. \tag{10}$$

Owning to $p_1 > p\mu$, then we get

$$\lim_{u \to \infty} \Phi(u) = 1. \tag{11}$$

By the *Dominated Convergence Theorem* and (11), we have

$$\lim_{t \to \infty} \sum_{k=0}^{t} \Phi(k)[p_1 \overline{P}(t+1-k) + q_1 \overline{P}(t-k)]$$

$$= \lim_{t \to \infty} \sum_{k=0}^{t} \Phi(t-k)[p_1 \overline{P}(k+1) + q_1 \overline{P}(k)]$$

$$= \sum_{k=0}^{\infty} [p_1 \overline{P}(k+1) + q_1 \overline{P}(k)] = \mu - p_1 \tag{12}$$

Then we take $t \to \infty$ in (10), by the *Dominated Convergence Theorem* and (11)(12), yields

$$p_1 qm = p_1 q\Phi(0) + p(\mu - p_1). \tag{13}$$

Obviously Eq.(10) and Eq.(13) lead to (4) and (5).

Remark. Actually, we can get another recursive formula from (7), but we can easily get $\Phi(0)$ from (4).

III. DEFECTIVE RENEWAL EQUATION FOR THE PENALTY FUNCTION

In this section, we derive a defective renewal equation for the penalty function $m(u)$. Throughout this paper we will use the curly capital letters to denote the corresponding generating functions.

Theorem3.1. *The penalty function $m(u)$ satisfies the following defective renewal equation*

$$m(u) = \frac{p}{1-qq_1}\{\sum_{k=0}^{u} m(u-k)[p_1 \overline{P}(k) + q_1 \overline{P}(k-1)] +$$

$$\sum_{k=u}^{\infty} w(k) - p_1 w(u)\}, \quad u \in N, \tag{14}$$

where $w(u) = \sum_{k=u+1}^{\infty} w(u, k-u)p(k)$, and

$$m(0) = \frac{p}{qp_1}[\sum_{k=0}^{\infty} w(k) - p_1 w(0)]. \tag{15}$$

Proof. We consider $U(t)$ in the first period $(0, 1]$ and separate the four possible cases as following:

(1) no premium arrives in $(0, 1]$ and no claim occurs in $(0, 1]$;

(2) a premium arrives in $(0, 1]$ and no claim occurs in $(0, 1]$;

(3) no premium arrives in $(0, 1]$ and a claim occurs in $(0, 1]$;

(4) a premium arrives in $(0, 1]$ and a claim occurs in $(0, 1]$. According to the laws of conditional probability, the penalty function is equal to

$$m(u) = q_1qm(u) + p_1qm(u+1) + q_1p\sum_{k=1}^{u}m(u-k)p(k) + p_1p\sum_{k=1}^{u+1}m(u+1-k)p(k) + q_1p\sum_{k=u+1}^{\infty}w(u,k-u)p(k) + p_1p\sum_{k=u+2}^{\infty}w(u+1,k-u-1)p(k). \quad (16)$$

Let $w(u) = \sum_{k=u+1}^{\infty}w(u,k-u-)p(k)$, and after rearranging them in (16), we have

$$qp_1m(u+1) = (1-qq_1)m(u) - pq_1\sum_{k=1}^{u}m(u-k)p(k)$$

$$-pp_1\sum_{k=1}^{u+1}m(u+1-k)p(k) - pp_1w(u+1) - pq_1w(u). \quad (17)$$

Owing to

$$\mathscr{M}(z) = \sum_{u=0}^{\infty}z^um(u), \mathscr{W}(z) = \sum_{u=0}^{\infty}z^uw(u), \mathscr{P}(z) = \sum_{k=0}^{\infty}z^kp(k).$$

Multiplying by z^u and summing over u from 0 to ∞ Eq.(17) yields

$$qp_1[\mathscr{M}(z)-m(0)] = (1-qq_1)z\mathscr{M}(z) - pp_1\mathscr{M}(z)\mathscr{P}(z) - pq_1z\mathscr{M}(z)\mathscr{P}(z) - pp_1[\mathscr{W}(z)-w(0)] - pq_1z\mathscr{W}(z),$$

or equivalently,

$$[qp_1 - (1-qq_1)z + pp_1\mathscr{P}(z) - pq_1z\mathscr{P}(z)]\mathscr{M}(z) = qp_1m(0) + pp_1\mathscr{W}(z) - pq_1z\mathscr{W}(z) + pp_1w(0). \quad (18)$$

Owing to $\mathscr{P}(1) = \sum_{k=0}^{\infty}p(k) = 1$ and $pq_1+pp_1+qp_1+qq_1 = 1$, thus $z = 1$ is the root to the equation

$$qp_1 - (1-qq_1)z + pp_1\mathscr{P}(z) + pq_1z\mathscr{P}(z) = 0,$$

we set $z = 1$ in Eq(18), then we have

$$qp_1m(0) + pp_1w(0) = pp_1\mathscr{W}(1) + pq_1z\mathscr{W}(1) = p\mathscr{W}(1). \quad (19)$$

Substituting (19) into (18) results in

$$[qp_1 - (1-qq_1)z + pp_1\mathscr{P}(z) - pq_1z\mathscr{P}(z)]\mathscr{M}(z) = pp_1[\mathscr{W}(1) - \mathscr{W}(z)] + pq_1[\mathscr{W}(1) - z\mathscr{W}(z)]. \quad (20)$$

We subtract

$$qp_1 - (1-qq_1) + pp_1\mathscr{P}(1) + pq_1\mathscr{P}(1) = 0$$

from the first term on the left hand side of (20) to obtain

$$[(1-qq_1)(1-z) - pp_1(\mathscr{P}(1) - \mathscr{P}(z)) - pq_1(\mathscr{P}(1) - z\mathscr{P}(z))]\mathscr{M}(z)$$

$$= pp_1[\mathscr{W}(1) - \mathscr{W}(z)] + pq_1[\mathscr{W}(1) - z\mathscr{W}(z)].$$

that is

$$\mathscr{M}(z) = \frac{p}{1-qq_1}\{p_1\frac{\mathscr{P}(1)-\mathscr{P}(z)}{1-z}\mathscr{M}(z) + q_1\frac{\mathscr{P}(1)-z\mathscr{P}(z)}{1-z}\mathscr{M}(z) + p_1\frac{\mathscr{W}(1)-\mathscr{W}(z)}{1-z} + q_1\frac{\mathscr{W}(1)-z\mathscr{W}(z)}{1-z}\}. \quad (21)$$

Now for any function $a(x), x \in N$, with generating function $\mathscr{A}(z)$, one has (see Pavolva and Willot [9])

$$\frac{\mathscr{A}(t)-\mathscr{A}(z)}{t-z} = \sum_{u=0}^{\infty}z^u\sum_{i=u+1}^{\infty}t^{i-u-1}a(i).$$

Also, one has easily that

$$\frac{t\mathscr{A}(t)-z\mathscr{A}(z)}{t-z} = \sum_{u=0}^{\infty}z^u\sum_{i=u}^{\infty}t^{i-u}a(i).$$

Thus, equating the coefficients of z^u in (21) we obtain

$$m(u) = \frac{p}{1-qq_1}\{\sum_{k=0}^{u}m(u-k)[p_1\overline{P}(k) + q_1\overline{P}(k-1)] + p_1\sum_{k=u+1}^{\infty}w(k) + q_1\sum_{k=u+1}^{\infty}w(k)\}$$

$$= \frac{p}{1-qq_1}\{\sum_{k=0}^{u}m(u-k)[p_1\overline{P}(k) + q_1\overline{P}(k-1)] + \sum_{k=u}^{\infty}w(k) - p_1w(u)\}. \quad (22)$$

We now demonstrate the renewal equation (22) is defective. In fact, by the positive relative security condition, we have

$$\frac{p}{1-qq_1}\sum_{k=0}^{\infty}[p_1\overline{P}(k) + q_1\overline{P}(k-1)] = \frac{p[p_1\mu+q_1\mu+q_1]}{1-qq_1},$$

$$= \frac{p\mu+pq_1}{1-qq_1} < \frac{p_1+pq_1}{1-qq_1} = 1,$$

which means renewal equation (22) is defective renewal equation. Let $u = 0$ in (22), then we get (15).

IV. ASYMPTOTIC ESTIMATE

In this section, we derive the asymptotic estimate for $m(u)$. Let

$$G_X(r) = \mathscr{P}(r) = E[r^X] = \sum_{n=1}^{\infty}p(n)r^n$$

We suppose that: there is a $r_\infty > 1$ such that $G_X(r) \to \infty$ as $r \to r_\infty$ (r_∞ is possibly $+\infty$).

Definition. If the equation $E[r^{V(1)}] = 1$ has a root $R > 1$, then R is called the adjustment coefficient and $E[r^{V(1)}] = 1$ is called the adjustment coefficient equation.

According to the definition $G_{V(1)}(r) = E[r^{S(1)-\eta_1}] = (pG_X(r) + q)(p_1\frac{1}{r} + q_1) = 1$, from which we get

$$(pG_X(r) + q)(q_1r + p_1) = r \quad (23)$$

Write $H(r) = (pG_X(r) + q)(q_1r + p_1), G(r) = r$, then $H(r)$ denotes the left side of Eq.(23), $G(r)$ denotes the right side of Eq.(23). It is easy to check out that $G(r)$ is a straight line with slope 1, $H(r)$ is a monotonously increasing convex function in $[0, r_\infty)$ and $H(0) = p_1q, H(1) = 1$. That is to say, there exist at most two real roots of Eq.(23) in $[0, r_\infty)$ and one of them is 1. According to (3) we have $H'(1) = p\mu + q_1 = \frac{p_1}{1+\theta} = \frac{1+q_1\theta}{1+\theta} < 1$, the slope of the straight line, Hence, if there exist two real roots of Eq.(23), the other is greater than 1, we denote by R the root greater than 1, R is the adjustment coefficient. According to the assumption $G_X(r) \to \infty$ as $r \to r_\infty$ and $H''(r) > 0$, R necessarily exists. In this paper we always assume that R exists.

Lemma. Let $\{a_k, k = 0, 1\cdots\}$ and $\{b_k, k = 0, 1\cdots\}$ be two nonnegative sequences. Suppose that $\sum_{k=0}^{\infty}a_k = 1, \sum_{k=1}^{\infty}ka_k <$

∞, $\sum\limits_{k=0}^{\infty} b_k < \infty$, *and that the greatest common divisor of the integers k for which* $a(k) > 0$ *is 1. if the renewal equation*

$$u_n = \sum_{k=0}^{n} a_{n-k} u_k + b_n, n = 0, 1 \cdots$$

is satisfied by a bounded sequence $\{u_n\}$ *of real numbers, then* $\lim\limits_{n\to\infty} u_n$ *exist and*

$$\lim_{n\to\infty} u_n = \frac{\sum\limits_{k=0}^{\infty} b_k}{\sum\limits_{k=1}^{\infty} k a_k}.$$

If $\sum\limits_{k=1}^{\infty} k a_k = \infty$, *let* $\lim\limits_{n\to\infty} u_n = 0$.

Proof. According to the Karlin and Taylor[12] (Chapter 3), it is easy to get the conclusion.

Theorem4.1. *The asymptotic estimate for the penalty function* $m(u)$ *is*

$$m(u) \sim C R^{-u} (u \to \infty), \tag{24}$$

where

$$C = \frac{1-R}{R} \frac{(p_1 + q_1 R)\mathscr{W}(R) - \sum_{i=0}^{\infty} w(i)}{(1-R)(p_1+q_1R)\mathscr{P}'(R)+\mathscr{P}(R)-1}. \tag{25}$$

Proof.

Denote by $m^*(u) = R^u m(u)$,

$$a(k) = R^k \frac{p}{1-qq_1}[p_1 \overline{P}(k) + q_1 \overline{P}(k-1)],,$$

$$b(u) = R^u \frac{p}{1-qq_1}[\sum_{k=u+1}^{\infty} w(k) + q_1 w(u))],$$

where R is the adjustment coefficient.
Thus, Eq.(22) becomes

$$m^*(u) = \sum_{k=0}^{u} m^*(k) a(u-k) + b(u), \quad u \in N.$$

Owing to R is the root of Eq.(23), then we have

$$\sum_{k=0}^{\infty} a(k)$$

$$= \frac{p}{1-qq_1}[p_1 \sum_{k=0}^{\infty} R^k \sum_{i=k+1}^{\infty} p(i) + q_1 \sum_{k=0}^{\infty} R^k \sum_{i=k}^{\infty} p(i)]$$

$$= \frac{p}{1-qq_1}[p_1 \sum_{i=1}^{\infty} p(i) \sum_{k=0}^{i-1} R^k + q_1 \sum_{i=0}^{\infty} p(i) \sum_{k=0}^{i} R^k]$$

$$= \frac{p}{1-qq_1}[p_1 \sum_{i=1}^{\infty} p(i) \frac{1-R^i}{1-R} + q_1 \sum_{i=0}^{\infty} p(i) \frac{1-R^{i+1}}{1-R}]$$

$$= \frac{p}{1-qq_1}[\frac{p_1}{1-R}(1 - \mathscr{P}(R)) + \frac{q_1}{1-R}(1 - R\mathscr{P}(R))]$$

$$= \frac{p[1 - \mathscr{P}(R)(p_1+q_1R)]}{(1-qq_1)(1-R)}$$

$$= 1$$

Note that

$$\sum_{k=0}^{\infty} k a(k)$$

$$= \frac{p}{1-qq_1}[p_1 \sum_{k=1}^{\infty} k R^k \overline{P}(k) + q_1 \sum_{k=1}^{\infty} k R^k \overline{P}(k-1)]$$

$$= \frac{p}{1-qq_1}[p_1 \sum_{k=1}^{\infty} k R^k (\sum_{i=k}^{\infty} p(i) - p(k)) + q_1 \sum_{k=1}^{\infty} k R^k \sum_{i=k}^{\infty} p(i)]$$

$$= \frac{p}{1-qq_1}[\sum_{k=1}^{\infty} k R^k \sum_{i=k}^{\infty} p(i) - p_1 \sum_{k=1}^{\infty} k R^k p(k)]$$

$$= \frac{p}{1-qq_1}[\sum_{i=1}^{\infty} p(i) \sum_{k=1}^{i} k R^k - p_1 R \mathscr{P}'(R)]$$

$$= \frac{p}{1-qq_1}[\frac{R}{(1-R)^2} \sum_{i=1}^{\infty} p(i)(1 - R^i) - \frac{R}{1-R} \sum_{i=1}^{\infty} i R^{i+1} p(i) - p_1 R \mathscr{P}'(R)]$$

$$= \frac{p}{1-qq_1}[\frac{R}{(1-R)^2}(1 - \mathscr{P}(R)) - \frac{R^2}{1-R}\mathscr{P}'(R) - p_1 R\mathscr{P}'(R)]$$

$$= \frac{p}{1-qq_1}[\frac{R}{(1-R)^2}(1 - \mathscr{P}(R)) - (\frac{R}{1-R} + p_1)R\mathscr{P}'(R)]$$

Similarly, we have

$$\sum_{k=0}^{\infty} b(k) = \frac{p}{1-qq_1}[\sum_{k=0}^{\infty} R^k \sum_{i=k+1}^{\infty} w(i) + q_1 \sum_{k=0}^{\infty} R^k w(k)]$$

$$= \frac{p}{1-qq_1}[\sum_{k=0}^{\infty} R^k \sum_{i=k}^{\infty} w(i) - p_1 \sum_{k=0}^{\infty} R^k w(k)]$$

$$= \frac{p}{1-qq_1}[\sum_{i=0}^{\infty} w(i) \sum_{k=o}^{i} R^k - p_1 \mathscr{W} R]$$

$$= \frac{p}{1-qq_1}[\frac{1}{1-R} \sum_{i=0}^{\infty} w(i)(1 - R^{i+1}) - p_1 \mathscr{W} R]$$

$$= \frac{p}{1-qq_1}[\frac{1}{1-R} \sum_{i=0}^{\infty} w(i) - (\frac{R}{1-R} + p_1)\mathscr{W} R]$$

According to the Lemma, we get

$$\lim_{u\to\infty} m^*(u) = \frac{1-R}{R} \frac{(p_1+q_1R)\mathscr{W}(R) - \sum_{i=0}^{\infty} w(i)}{(1-R)(p_1+q_1R)\mathscr{P}'(R)+\mathscr{P}(R)-1}. \tag{26}$$

Then (24) and (25) follow from (26).

V. SOME EXAMPLES OF RUIN QUANTITIES

In this section, we give two examples of ruin quantities to illustrate applications of the asymptotic estimate for $m(u)$.

Example 5.1. Letting $w(x_1, x_2) = 1$, we have $m(u) = E[I(T < \infty)] = \Psi(u)$. In this case

$$\mathscr{W}(R) = \frac{1}{1-R}[1 - \mathscr{P}(R)],$$

and

$$\sum_{i=0}^{\infty} w(i) = \mu.$$

Thus, By Theorem4.1 , the asymptotic estimate for the ultimate ruin probability $\Psi(u)$ is

$$\Psi(u) \sim C_1 R^{-u}(u \to \infty),$$

where

$$C_1 = \frac{(p_1+q_1R)(1-\mathscr{P}(R))-(1-R)\mu}{R\{(1-R)(p_1+q_1R)\mathscr{P}'(R)+\mathscr{P}(R)-1\}}.$$

Example 5.2. Letting $w(x_1, x_2) = I(x_2 \leq y), (y \in N)$, then $m(u) = Pr(|U(T)| \leq y, T < \infty) = G(u, y)$ is the distribution function of the deficit at ruin. In this case

$$\mathscr{W}(R) = \frac{1-R^{-y}}{1-R}[1 - \mathscr{P}(R)] + R^{-y} \sum_{i=0}^{y-1} R^i \overline{P}(i),$$

and

$$\sum_{i=0}^{\infty} w(i) = \sum_{i=0}^{y-1} \overline{P}(i).$$

Thus, By Theorem4.1 , the asymptotic estimate for $G(u,y)$ is

$$G(u,y) \sim C_2 R^{-u}(u \to \infty),$$

where

$$C_2 =$$

$$\frac{(p_1 + q_1 R)(1 - \mathscr{P}(R))(1 - R^{-y}) + (1-R)[R^{-y} \sum_{i=0}^{y-1} R^i \overline{P}(i) - \sum_{i=0}^{y-1} \overline{P}(i)]}{R\{(1-R)(p_1 + q_1 R)\mathscr{P}'(R) + \mathscr{P}(R) - 1\}} .$$

Remark. We introduce the new discrete time risk model, which is the generalization of the classical discrete time risk. However, the discrete time model with random income in this paper can be seen as analogous to the continuous time model in Temmov[6] in a way.

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