International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:2, No:11, 2008

The Structure of Weakly Left C-wrpp Semigroups

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***Abstract*—**In this paper, the class of weakly left C-wrpp semigroups which includes the class of weakly left C-rpp semigroups as a subclass is introduced. To particularly show that the spined product of a left C-wrpp semigroup and a right normal band which is a weakly left C-wrpp semifroup by virtue of left C-full Ehremann cyber groups recently obtained by authors Li-Shum, results obtained by Tang and Du-Shum are extended and strengthened.

***Keywords*—**Left C-semigroup,left C-wrpp semigroup,left quasi-normal band,weakly left C-wrpp semigroup

I. Introduction

THROUGHOUT this paper, we adopt the notation and terminologies given by Howei[1] and Li-Shum[2].

By modifying Green's relations on rpp semigroups,Tang [3]has introduced a new set of Green'srelations on a semigroup S and by using these new Green's relations, he was able to give a description for a wider class of C-rpp semigroups,namely, the class of C-wrpp semigroups. Tang[3] considered a Green-like right congruence relation $\mathcal{L}^{**}$ on a semigroup S for $a,b \in S$ $a\mathcal{L}^{**}b$ if and only $ax\mathcal{R}\, ay \Leftrightarrow bx\mathcal{R}\, by$ for all $x,y \in S^1$. Moreover,Tang pointed out in [3] that a semigroup S is a wrpp semigroup if and only if each $\mathcal{L}^{**}$-class of S contains at least one idempotent.

Recall that a wrpp semigroup S is a C-wrpp semigroup if the idempotents of S are central. It is well known that a semigroup S is a C-wrpp semigroup if and only if S is a strong semilattice of left-$\mathcal{R}$ cancellative monoids(see[3]).Because a Clifford semigroup can be expressed as a strong semilattice of groups and a C-rpp semigroup can be expressed as a strong semilattice of left cancellative monoids(see[4]),we see immediately that the concept of C-wrpp semigroups is a common generalization of Clifford semigroups and C-rpp semigroups.

For wrpp semigroups, Du-Shum [5] first introduced the concept of left C-wrpp semigroups, that is,a left C-wrpp semigroup whose satisfy the following conditions: (i)for all $e \in E(L_a^{**}), a = ae$ where $E(L_a^{**})$ is the set of idempotents in L_a^{**}; (ii)for all $a \in S$, there exists a unique idempotent a^+ satisfying $a\mathcal{L}^{**}a^+$ and $a = a^+a$; (iii) for all $a \in S, aS \subseteq L^{**}(a)$, where $L^{**}(a)$ is the smallest left $**$-ideal of S generated by a . For left C-wrpp semigroups, Du-Shum[5] gave a method of construction.

Guo [6] has investigated weakly left C-semigroups,and he pointed out that a semigroup S is a weakly left C-semigroup if and only if S is a completely regular semigroup with idempotents set $E(S)$ forming a left quasi-normal band.

In this paper, we first define the concept of weakly left C-wrpp semigroups.A structure theorem for weakly C-wrpp semigroups is obtained, and we prove this theorem in view of the structure of left C-full Ehresmann cyber groups recently obtained by Li and Shum[2].

II. Preliminaries

We first recall that some known results used in the sequel. The following results due to [2] and [7].

Let S be a semigroup, and U a subset of the set $E(S)$) which is the set of all idempotents of S . For all $a \in S$, let $U_a^l = \{u \in U \mid ua = a\}, U_a^r = \{u \in U \mid au = a\}$, $U_a = U_a^l \cap U_a^r$ $=\{u \in U \mid ua = a = au\}$. According to Lawson[8] and He[7], we have the following relations on S :

$\tilde{\mathcal{L}}^U = \{(a,b) \in S\times S \mid U_a^r = U_b^r\}, \tilde{\mathcal{R}}^U = \{(a,b) \in S\times S \mid U_a^l = U_b^l\}$,

$\tilde{\mathcal{H}}^U = \tilde{\mathcal{L}}^U \cap \tilde{\mathcal{R}}^U$, $\tilde{\mathcal{Q}}^U = \{(a,b) \in S\times S \mid U_a = U_b\}$.

It is easy to verify that above relations are equivalent relations. For all $a \in S$, a $\tilde{\mathcal{L}}^U$-class, a $\tilde{\mathcal{R}}^U$-class, a $\tilde{\mathcal{H}}^U$-class and a $\tilde{\mathcal{Q}}^U$-class of S containing a , denoted by $\tilde{L}_a^U, \tilde{R}_a^U, \tilde{H}_a^U$ and $\tilde{Q}_a^U$, respectively. For the sake of convenience, we denote the semigroup S with a projective set U which is a subset of all idempotents $E(S)$ by $S(U)$.

Consider the special semigroup $S(U)$ with $U = E(S)$. Then the equivalent relations on $S = S(U)$, say $\tilde{\mathcal{L}}^{E(S)}, \tilde{\mathcal{R}}^{E(S)}$, $\tilde{\mathcal{H}}^{E(S)}$ and $\tilde{Q}^{E(S)}$, respectively. For brevity, we write $\tilde{\mathcal{L}}$, $\tilde{\mathcal{R}}$, $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{Q}}$, respectively.

Defintion2.1 A semigroup $S(U)$ is called a U-semi-lpp semigroup if each $\tilde{\mathcal{R}}^U$-class of S contains at least one element in U , that is, $\tilde{R}_a^U \cap U \neq \varnothing$ for all $a \in S$. A semigroup $S(U)$ is called a U-semi-rpp semigroup if each $\tilde{\mathcal{L}}^U$-class of S contains at least one element in U , that is, $\tilde{L}_a^U \cap U \neq \varnothing$ for all

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$a \in S$. A semigroup $S(U)$ is called a U -semiabundant semigroup if both each $\tilde{L}^U$ -class and each $\tilde{R}^U$ -class of S contain at least one element in U , that is, $\tilde{L}_a^U \cap U \neq \varnothing$ and $\tilde{R}_a^U \cap U \neq \varnothing$ for all $a \in S$. A semigroup $S(U)$ is called U -abundant semigroup if each $\tilde{Q}^U$ -class of S contains at least one element in U , that is, $\tilde{Q}_a^U \cap U \neq \varnothing$. Denoted the unique element in U by a_U° if $|\tilde{Q}_a^U \cap U| = 1$, In special case $U = E(S)$, a $E(S)$ -abundant semigroup is called a full abundant semigroup, in this case, a_E° is usually written as a° .

Lawson[8] point out that $\tilde{L}^U$ is not necessarily a riht congruence and $\tilde{R}^U$ is not necessarily a left congruence on $S(U)$. We have

Definition 2.2 A semigroup $S(U)$ is called satisfying U -right(left) congruence condition if $\tilde{L}^U \in \mathrm{RC}(S)$ ($\tilde{R}^U \in \mathrm{LC}(S)$, a semigroup $S(U)$ is called satisfying U - congruence condition if $\tilde{L}^U \in \mathrm{RC}(S)$ and $\tilde{R}^U \in \mathrm{LC}(S)$, where $\mathrm{RC}(S)$ is a lattice that all right congruences form, and $\mathrm{LC}(S)$is a lattice that all left congruences form.

Definition 2.3 A U -semiabundant semigroup $S(U)$ is called an Ehresmann semigroup if $S(U)$ satisfies U -congruence condition and U is a subsemilattice of U . In particular, an Ehresmann semigroup $S(U)$ is called a C-Ehresmann semigroup if U lies in center of $S(U)$.

Definition 2.4 A U -abundant semigroup $S(U)$ is called an orthodox U -abundant semigroup if U is a subsemigroup of $S(U)$ and $(ab)_U^\circ \, \mathrm{D}(U) a_U^\circ b_U^\circ$ for all $a, b \in S$, where $\mathrm{D} = \mathrm{L} \vee \mathrm{R}$ is a usual Green's D relation.

Definition 2.5 An orthodox U -abundant semigroup is called a left C-Ehresmann semigroup if $uS \subseteq Su$ holds for all $u \in U$.

Definition 2.6 An orthodox U -abundant semigroup is called a left C-Ehresmann cyber group if the identity $uxuy = uxy$ holds for all $u \in U, x, y \in S$.

When $U = E(S)$, we call an orthodox $E(S)$ -abundant semigroup(left C-Ehresmann semigroup, left C-Ehresmann cyber group) an orthodox full abundant semigroup(left C-full Ehresmann semigroup, left C-full Ehresmann cyber group).

Recall that the direct product $I \times T$ of a left zero band I and a monoid T is called a left monoid,and the direct product $I \times T$ of a left zero band I and a unipotent semigroup T is called a left unipotent semigroup. It is well known that a right normal band Λ can be expressed as a strong semilattice of right zero bands, that is, $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ .By using the above results, He[7] have proved that the following results:

Lemma 2.7 The following statements are equivalent for a semigroup S :

(i) $S(U)$ is a left C-Ehresmann semigroup for some $U \subseteq E(S)$;

(ii) S is a semilattice Y of left monoids $S_\alpha = I_\alpha \times T_\alpha$ and $U = \{(i, 1_\alpha) \mid i \in I_\alpha\}$ is a subsemigroup of S where $\alpha \in Y$, 1_α is the identical element in T_α .

By virtue of above lemma, a left C-Ehresmann semigroup $S(U) = [Y; S_\alpha = I_\alpha \times T_\alpha]$ may be defined as a semilattice of left monoids $S_\alpha = I_\alpha \times T_\alpha$, and the set $U = \{(i, 1_\alpha) \mid i \in I_\alpha, \alpha \in Y\}$ is also a subsemigroup of $S(U)$.

The followinglemma have been recently proved by Li-Shum [2].

Lemma 2.8 Let S be a semigroup. Then $S(U)$ is a left C-Ehresmann cyber group for some $U \subseteq E(S)$ if only and if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C-Ehresmann semigroup $S_1 = [Y; I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to the semilattice Y .

By using Lemma 2.7, we can easily follow that S is a left C-full Ehresmann semigroup if and only if S is a semilattice of left unipotent semigroups. Hence we can denote a left C-full Ehresmann semigroup S by $S = [Y; S_\alpha = I_\alpha \times T_\alpha]$ (see[7]).

By using Lemma 2.8, we can easily imply that S is a left C-full Ehresmann cyber group if and only if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C-full Ehresmann semigroup $S_1 = [Y; I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to the semilattice Y (see [2]).

III. The structure of weakly left C-wrpp semigroups

In this section, the concept of weakly left C-wrpp semigroups is introduced. We shall prove that a structure theorem for weakly left C-wrpp semigroups. First, we introduce the cocept of weakly left C-wrpp semigroups.

Definition 3.1 A semigroup S is called a weakly left C-wrpp semigroup ,if S is a strong wrpp semigroup and satisfy identity $exey = exy$ for all $e \in E(S)$ and $x, y \in S$.

According to [5]. we know that the left C-wrpp semigroup is a special case of the weakly left C-wrpp semigroup.

Lemma 3.2 Let S be a strongly wrpp semigroup. Then S is a full abundant semigroup with $a^\circ = a^+$, for all $a \in S$.

Proof. Let S be a strongly wrpp. To prove S is a full abundant semigroup,we only need to prove $a^\circ = a^+$ for all $a \in S$. Let $I_a = \{e \mid ea = ae = a\}$. For all $e \in I_a$, since

$(a^+e)^+ a = (a^+e)^+ (a^+e) a = (a^+e) a = a (a^+e)(a^+e)^+ = a (a^+e)^+$,

and $a = ae \mathrm{L}^{**} a^+ e \mathrm{L}^{**} (a^+e)^+$. So we have $(a^+e)^+ \in L_a^{**} \cap I_a$ .This implies that $(a^+e)^+ = a^+$. Thus, we have $a^+ e a^+ = a^+ e$ and whence $a^+ e \in E(S)$. Consequently, we obtain that $a^+ e = (a^+ e)^+ = a^+$. On the other hand, we can easily verify that $ea^+ \in L_a^{**} \cap I_a$. Therefore, $ea^+ = a^+$, and so $e \in I_{a^+}$. Thus, it follows that $I_a \subseteq I_{a^+}$. Clearly, $I_{a^+} \subseteq I_a$ and whence $I_a = I_{a^+}$. This means that $(a, a^+) \in \tilde{Q}$. Therefore, S is a full abundant semigroup with $a^\circ = a^+$. The proof is completed.

Lemma 3.3 Let S be a weakly left C-wrpp semigroup. Then

S is a left C-full Ehresmann cyber group.

Proof. Let S be a weakly left C-wrpp semigroup. Then we have $exey = exy$ for all $e \in E(S), x, y \in S$, and hence $E(S)$ is a left quasi-normal band. According to Lemma 3.2, we know that S is a full abundant semigroup with $a^\circ = a^+$. To prove that S is a left C-full Ehresmann cyber group, it suffices to prove $S(E(S))$ satisfying $(ab)^\circ \mathrm{D}(E(S))a^\circ b^\circ$ for all $a,b \in S$. Now, Let $a,b \in S$. For all $x, y \in S^1$, we can infer that

$(ab)^+ x \mathrm{R}\, (ab)^+ y \Leftrightarrow abx \mathrm{R}\, aby$ (R is a left congruence)

$\Leftrightarrow a^+ba^+b^+x \mathrm{R}\, a^+ba^+b^+y$ ($E(S)$ is a left quasi-normal band)

$\Leftrightarrow aba^+b^+x \mathrm{R}\, aba^+b^+y$ (R is a left congruence)

$\Leftrightarrow (ab)^+a^+b^+x \mathrm{R}\, (ab)^+a^+b^+y$.

This means that $(ab)^+ \mathrm{L}^{**} (ab)^+a^+b^+$. Since $(ab)^+, (ab)^+a^+b^+ \in E(S)$, we can verify that $(ab)^+ \mathrm{L} (ab)^+a^+b^+$, and whence $(ab)^+ = (ab)^+a^+b^+$. For all $x, y \in S$, we have

$(ab)^+a^+x \mathrm{R}\, (ab)^+a^+y \Leftrightarrow aba^+x \mathrm{R}\, aba^+y$

$\Leftrightarrow a^+ba^+x \mathrm{R}\, a^+ba^+y$

$\Leftrightarrow b^+a^+ba^+x \mathrm{R}\, b^+a^+ba^+y$ (R is a left congruence)

$\Leftrightarrow (ba)^+b^+a^+ba^+x \mathrm{R}\, (ba)^+b^+a^+ba^+y$ (R is a left congruence)

$\Leftrightarrow ba^+x \mathrm{R}\, ba^+y \Leftrightarrow b^+a^+x \mathrm{R}\, b^+a^+y$

$\Leftrightarrow a^+b^+a^+x \mathrm{R}\, a^+b^+a^+y$ (R is a left congruence)

$\Leftrightarrow (ab)^+a^+b^+a^+x \mathrm{R}\, (ab)^+a^+b^+a^+y$ (R is a left congruence)

$\Leftrightarrow (ab)^+a^+x \mathrm{R}\, (ab)^+a^+y$.

So we have $(ab)^+a^+ \mathrm{L}^{**} a^+b^+a^+$, and hence $(ab)^\circ = (ab)^+ \mathrm{R}\, (ab)^+a^+ \mathrm{L}\, a^+b^+a^+ \mathrm{R}\, a^+b^+ = a^\circ b^\circ$. The proof is completed.

We now characterize the weakly left C-wrpp semigroups.

Theorem 3.4 A semigroup S is a weakly left C-wrpp semigroup if and only if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C-wrpp semigroup $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to semilattice Y.

Proof. Necessity. Let S be a weakly left C-wrpp semigroup. Then by Lemma 3.3, S is a left C-full Ehresmann cyber group with $a^\circ = a^+$ for all $a \in S$. According to Lemma 2.8, we know that S can express as $S_1 \times_Y \Lambda$, where $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ is a left C-full Ehresmann semigroup and $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ is a right normal band. We only need to show that S_1 is a strongly wrpp semigroup and T_α is a left-R cancellative monoid.

For all $(i,a) \in S_\alpha, (j,b) \in S_\beta$ and $(k,c) \in S_\gamma$, we have

$(i,a)(j,b) \mathrm{R}\, (i,a)(k,c) \Rightarrow (i,a)(i,1_\alpha)(j,b) \mathrm{R}\, (i,a)(i,1_\alpha)(k,c)$

$\Rightarrow ((i,a),\lambda)((i,1_\alpha)(j,b),\mu) \mathrm{R}\, ((i,a),\lambda)((i,1_\alpha)(k,c),\mu)$ ($\lambda \in \Lambda_\alpha$, $\mu \in \Lambda_{\alpha\beta} = \Lambda_{\alpha\gamma}$)

$\Rightarrow ((i,1_\alpha),\lambda)((i,1_\alpha)(j,b),\mu) \mathrm{R}\, ((i,1_\alpha),\lambda)((i,1_\alpha)(k,c),\mu)$ ($\lambda \in \Lambda_\alpha$, $\mu \in \Lambda_{\alpha\beta} = \Lambda_{\alpha\gamma}$) $\Rightarrow (i,1_\alpha)(j,b) \mathrm{R}\, (i,1_\alpha)(k,c)$

since $((i,a),\lambda)^+ = ((i,a),\lambda)^\circ = ((i,1_\alpha),\lambda)$, where 1_α is the identity in $T_\alpha (\alpha \in Y)$. Similarly, we can deduce that $(i,a)(j,b) \mathrm{R}\, (i,a) \Rightarrow (i,1_\alpha)(j,b) \mathrm{R}\, (i,1_\alpha)$. Thus $(i,a) \mathrm{L}^{**} (i,1_\alpha)$. By $(i,1_\alpha)$ being the unique element in $L_\alpha^{**} \cap I_{(i,a)}$, we observe that S_1 is a strongly wrpp semigroup with $(i,a)^+ = (i,1_\alpha)$. If $a,b,c \in T_\alpha$ such that $ab \mathrm{R}\, ac$, then $(i,a)(i,b) \mathrm{R}\, (i,a)(i,c)$ for all $i \in I_\alpha$. By (i,a) $\mathrm{L}^{**} (i,1_\alpha)$, we have $(i,1_\alpha)(i,b) \mathrm{R}\, (i,1_\alpha)(i,c)$, and whence $b \mathrm{R}\, c$, this means that T_α is a left-R cancellative monoid.

Sufficiency. Assume that $S = S_1 \times_Y \Lambda$, where $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ is a left C-wrpp semigroup and $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ is a right normal band, then $(i,a)^+ = (i,1_\alpha)$ for $(i,a) \in S_\alpha$, where 1_α is the identity in T_α. We easily verify that S is a strongly wrpp semigroup with $((i,a),\lambda)^+ = ((i,1_\alpha),\lambda)$, and whence we can also check that S is a weakly left C-wrpp semigroup.

Weakly left C-semigroups were first investigated by Guo[6] in 1996, and weakly left C-rpp semigroups were investigated by Cao [9] in 2000. It is clear that weakly left C-semigroups and weakly left C-rpp semigroups are special weakly left C-wrpp. As applications of Theorem 3.4, we have the following corollaries:

Corollary 3.5 A semigroup S is a weakly left C-rpp semigroup if and only if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C-rpp semigroup $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to semilattice Y.

Corollary 3.6 A semigroup S is a weakly left C- semigroup if and only if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C- semigroup $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to semilattice Y.

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