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The Splitting Upwind Schemes for Spectral Action Balance Equation

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Abstract—The spectral action balance equation is an equation that used to simulate short-crested wind-generated waves in shallow water areas such as coastal regions and inland waters. This equation consists of two spatial dimensions, wave direction, and wave frequency which can be solved by finite difference method. When this equation with dominating convection term are discretized using central differences, stability problems occur when the grid spacing is chosen too coarse. In this paper, we introduce the splitting upwind schemes for avoiding stability problems and prove that it is consistent to the upwind scheme with same accuracy. The splitting upwind schemes was adopted to split the wave spectral action balance equation into four one-dimensional problems, which for each small problem obtains the independently tridiagonal linear systems. For each smaller system can be solved by direct or iterative methods at the same time which is very fast when performed by a multi-processor computer.

Keywords—upwind scheme, parallel algorithm, spectral action balance equation, splitting method.

I. Introduction

A third-generation model is a number of advanced spectral wind-wave models. It has been developed such as WAM model of WAMDI Group [9], in which all processes of wave generation, dissipation and nonlinear wave-wave interactions are accounted for explicitly. WAM model considers problems on oceanic scales, and make used of explicit propagation schemes in geographical and spectral spaces. Tolman [8] developed model base on spectral action balance equation, WAVEWATCH model incorporates all relevant wave-current interaction mechanism, including changes of absolute frequencies due to unsteadiness of depth and currents. The model explicitly accounts for growth and decay of wave energy and for nonlinear resonant wave-wave interactions. Booij [1] and Ris [7] et. al. summarized the research attainment in the wave energy, dissipation and nonlinear wave-wave interactions, and developed the third generation for coastal region in shallow water, SWAN(Simulating WAve Nearshore) model, which can be applied in coastal zones, lake and estuaries. The model uses the spectral action balance equation to represents the process of wave shoaling, refraction, bottom friction, depth-induced wave breaking, whitcapping, wind input and nonlinear wave-wave interactions reasonably.

For the numerical treatment of the spectral action balance equation, the finite difference approximation with the first and second order upwind schemes are applied. The discretizations yield a banded-9 and band-17 linear systems, respectively that can be solved by direct or iterative methods.

In recent years computers' evolution is going dramatically fast. Computers have been improved a lot and become much more powerful. One of the new types of computers is a multi-processing computer. So, we should develop algorithms that support and could be suitable for this evolution. In this paper we introduce the splitting upwind methods for solving spectral action balance equation. This method splits the original four-dimensional spaces problem into a set of one-dimensional space problems. At each fractional step one has to solve m independent one-dimensional space systems of linear equations where m is the number of unknowns in one-dimensional space problems in appropriate direction. Therefore, we can solve each of these systems of linear equations at every step by m independent parallel processors. This method is preferable for multi-processing computers.

II. Upwind Schemes for Spectral Action Balance Equation

We consider the wave spectral action balance equation which described the wave characteristic:

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial x}(c_xN)+\frac{\partial}{\partial y}(c_yN)+\frac{\partial}{\partial \sigma}(c_\sigma N)+\frac{\partial}{\partial \theta}(c_\theta N)=\frac{S}{\sigma}, \quad (1)$$
$$\forall(x,y,\sigma,\theta)\in\Omega\times\Gamma, \quad t\in[0,T],$$
$$\left.\frac{\partial N}{\partial n}\right|_{\partial\Omega\times\partial\Gamma}=0, \quad t\in[0,T],$$
$$N|_{t=0}=N_0(x,y,\sigma,\theta), \quad \forall(x,y,\sigma,\theta)\in\Omega\times\Gamma,$$

where Ω and Γ are domain in geographical and spectral $\mathbb{R}^2$, $\partial\Omega$ its boundary of Ω, $\partial\Gamma$ its boundary of Γ, $N_0(x,y,\sigma,\theta)$ is an initial values, and n is a normal direction of each variable. Which $N(x,y,\sigma,\theta,t)$ is the action density as a function of relative frequency σ, direction θ, horizontal coordinate x, y, and time t. The coefficients c_x, c_y, c_σ and c_θ are propagation velocity in -x, -y, -σ and -θ direction respectively. The first term of the left-hand side of the equation (1) represents the local rate of change of action density in time, the second and the third term represent propagation of action density in geographical space with propagation velocities c_x and c_y in x and y respectively. The fourth term represents shifting of relative frequency due to variations in depths and currents with propagation velocity c_σ in σ. The fifth term represents depth-induced and current-induced refraction with propagation velocity c_θ in θ. The term S in right hand side of the equation (1) represents the source term. More details are given in Booij et al. [1] and Ris et al. [7]

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Let us choose a rectangular grid with constant mesh sizes $\triangle x$ and $\triangle y$ in $x-$ and $y-$ direction, respectively. The spectral space is divided into elementary bins with a constant directional resolution $\triangle\theta$ and a constant relative frequency resolution $\triangle\sigma$. We denote that the grid counters as $1 \leq i \leq N_x$, $1 \leq j \leq N_y$, $1 \leq l \leq N_\sigma$ and $1 \leq m \leq N_\theta$ in $x-, y-, \sigma-$ and $\theta-$ spaces, respectively. All variables are located at points (i, j, l, m).

A. The First-Order Upwind Scheme

When we replace the finite difference approximation using the first-order upwind scheme into the spectral action balance equation, yields

$$\begin{array}{rcl} \left.\frac{N^n - N^{n-1}}{\triangle t}\right|_{i,j,l,m} & & \\ +c_x^+ \frac{N_i - N_{i-1}}{\triangle x}\big|_{j,l,m}^n + c_x^- \frac{N_{i+1} - N_i}{\triangle x}\big|_{j,l,m}^n & & \\ +c_y^+ \frac{N_j - N_{j-1}}{\triangle y}\big|_{i,l,m}^n + c_y^- \frac{N_{j+1} - N_j}{\triangle y}\big|_{i,l,m}^n & & \\ +c_\sigma^+ \frac{N_l - N_{l-1}}{\triangle \sigma}\big|_{i,j,m}^n + c_\sigma^- \frac{N_{l+1} - N_l}{\triangle \sigma}\big|_{i,j,m}^n & & \\ +c_\theta^+ \frac{N_m - N_{m-1}}{\triangle \theta}\big|_{i,j,l}^n + c_\theta^- \frac{N_{m+1} - N_m}{\triangle \theta}\big|_{i,j,l}^n & = & \left.\frac{S}{\sigma_l}\right|_{i,j,l,m}^{n-1} \end{array} \quad (2)$$

where n is a time step with $\triangle t$ and for each point (i, j, l, m)

$$\begin{aligned} c_x^+ &= \max\{\bar{c}_x, 0\}, \quad c_x^- = \min\{\bar{c}_x, 0\}, \\ c_y^+ &= \max\{\bar{c}_y, 0\}, \quad c_y^- = \min\{\bar{c}_y, 0\}, \\ c_\sigma^+ &= \max\{\bar{c}_\sigma, 0\}, \quad c_\sigma^- = \min\{\bar{c}_\sigma, 0\}, \\ c_\theta^+ &= \max\{\bar{c}_\theta, 0\}, \quad c_\theta^- = \min\{\bar{c}_\theta, 0\}, \\ \bar{c}_x|_{i,j,l,m} &= \tfrac{1}{4}(c_{x_{i-1}} + 2c_{x_i} + c_{x_{i+1}})_{j,l,m}, \\ \bar{c}_y|_{i,j,l,m} &= \tfrac{1}{4}(c_{y_{j-1}} + 2c_{y_j} + c_{y_{j+1}})_{i,l,m}, \\ \bar{c}_\sigma|_{i,j,l,m} &= \tfrac{1}{4}(c_{\sigma_{l-1}} + 2c_{\sigma_l} + c_{\sigma_{l+1}})_{i,j,m}, \\ \bar{c}_\theta|_{i,j,l,m} &= \tfrac{1}{4}(c_{\theta_{m-1}} + 2c_{\theta_m} + c_{\theta_{m+1}})_{i,j,l}. \end{aligned}$$

Rearranging the equation (2), we have the following equation

$$\begin{array}{rl} a_{i,j,l,m} N_{i,j,l,m}^n & - \frac{\triangle t c_x^+}{\triangle x} N_{i-1,j,l,m}^n + \frac{\triangle t c_x^-}{\triangle x} N_{i+1,j,l,m}^n \\ & - \frac{\triangle t c_y^+}{\triangle y} N_{i,j-1,l,m}^n + \frac{\triangle t c_y^-}{\triangle y} N_{i,j+1,l,m}^n \\ & - \frac{\triangle t c_\sigma^+}{\triangle \sigma} N_{i,j,l-1,m}^n + \frac{\triangle t c_\sigma^-}{\triangle \sigma} N_{i,j,l+1,m}^n \\ & - \frac{\triangle t c_\theta^+}{\triangle \theta} N_{i,j,l,m-1}^n + \frac{\triangle t c_\theta^-}{\triangle \theta} N_{i,j,l,m+1}^n \\ & = \frac{\triangle t}{\sigma_l} S_{i,j,l,m} + N_{i,j,l,m}^{n-1} \end{array} \quad (3)$$

where $i = 1, \ldots, N_x;\ j = 1, \ldots, N_y;\ l = 1, \ldots, N_\sigma;$ $m = 1, \ldots, N_\theta$. and

$$\begin{aligned} a_{i,j,l,m} = 1 + \triangle t \Bigg(& \frac{c_x^+ - c_x^-}{\triangle x} + \frac{c_y^+ - c_y^-}{\triangle y} \\ & + \frac{c_\sigma^+ - c_\sigma^-}{\triangle \sigma} + \frac{c_\theta^+ - c_\theta^-}{\triangle \theta} \Bigg)_{i,j,l,m}. \end{aligned} \quad (4)$$

We can see that the structure of the coefficient matrix of the linear system (3) is in the form of banded-9 matrix. This linear system can be solved by any direct or iterative methods under the diagonal dominant condition, that is sum of off diagonal entry must be less than the main diagonal of the coefficient matrix.

Now, we are analyzing the criteria of $\triangle t$, $\triangle x$, $\triangle y$, $\triangle\sigma$ and $\triangle\theta$ for existent and uniqueness solution of this linear system. Let us consider the diagonal dominant condition

$$\begin{array}{rcl} a_{i,j,l,m} & > & \left|\frac{\triangle t}{\triangle x} c_x^+\right|_{i-1,j,l,m} + \left|\frac{\triangle t}{\triangle x} c_x^-\right|_{i+1,j,l,m} \\ & & + \left|\frac{\triangle t}{\triangle y} c_y^+\right|_{i,j-1,l,m} + \left|\frac{\triangle t}{\triangle y} c_y^-\right|_{i,j+1,l,m} \\ & & + \left|\frac{\triangle t}{\triangle \sigma} c_\sigma^+\right|_{i,j,l-1,m} + \left|\frac{\triangle t}{\triangle \sigma} c_\sigma^-\right|_{i,j,l+1,m} \\ & & + \left|\frac{\triangle t}{\triangle \theta} c_\theta^+\right|_{i,j,l,m-1} + \left|\frac{\triangle t}{\triangle \theta} c_\theta^-\right|_{i,j,l,m+1} \\ & & \equiv \mathit{cond} \end{array} \quad (5)$$

where $i = 1, \ldots, N_x;\ j = 1, \ldots, N_y;\ l = 1, \ldots, N_\sigma;$ $m = 1, \ldots, N_\theta$.

Next, we try to simplify these stability criteria, by letting

$$\begin{aligned} M_x &\equiv \max\nolimits_{\forall i,j,l,m} |c_x|_{i,j,l,m}, \\ M_y &\equiv \max\nolimits_{\forall i,j,l,m} |c_y|_{i,j,l,m}, \\ M_\sigma &\equiv \max\nolimits_{\forall i,j,l,m} |c_\sigma|_{i,j,l,m}, \\ M_\theta &\equiv \max\nolimits_{\forall i,j,l,m} |c_\theta|_{i,j,l,m}. \end{aligned} \quad (6)$$

Substituting the notations (6) into the relation (5), yields

$$\begin{aligned} \mathit{cond} &\leq \frac{2\triangle t M_x}{\triangle x} + \frac{2\triangle t M_y}{\triangle y} + \frac{2\triangle t M_\sigma}{\triangle \sigma} + \frac{2\triangle t M_\theta}{\triangle \theta} \\ &\leq 8\triangle t \max\left\{\frac{M_x}{\triangle x}, \frac{M_y}{\triangle y}, \frac{M_\sigma}{\triangle \sigma}, \frac{M_\theta}{\triangle \theta}\right\}. \end{aligned}$$

Since $|a_{i,j,l,m}| \geq 1$ and $8\triangle t \max\left\{\frac{M_x}{\triangle x}, \frac{M_y}{\triangle y}, \frac{M_\sigma}{\triangle \sigma}, \frac{M_\theta}{\triangle \theta}\right\} \geq$ *cond*, we can choose

$$1 > 8\triangle t \max\left\{\frac{M_x}{\triangle x}, \frac{M_y}{\triangle y}, \frac{M_\sigma}{\triangle \sigma}, \frac{M_\theta}{\triangle \theta}\right\}.$$

Thus the condition of $\triangle t$ that satisfy the diagonal dominant of the linear system is following

$$\triangle t < \frac{1}{8 \max\left\{\frac{M_x}{\triangle x}, \frac{M_y}{\triangle y}, \frac{M_\sigma}{\triangle \sigma}, \frac{M_\theta}{\triangle \theta}\right\}}.$$

B. The Second-Order Upwind Scheme

When we replace the finite difference approximation using the second-order upwind scheme into the spectral action balance equation, yields

$$\begin{array}{rl} \left.\frac{N^n - N^{n-1}}{\triangle t}\right|_{i,j,l,m} & + \ c_x^+ \left.\frac{3N_i - 4N_{i-1} + N_{i-2}}{2\triangle x}\right|_{j,l,m}^n \\ & + \ c_x^- \left.\frac{-N_{i+2} + 4N_{i+1} - 3N_i}{2\triangle x}\right|_{j,l,m}^n \\ & + \ c_y^+ \left.\frac{3N_j - 4N_{j-1} + N_{j-2}}{2\triangle y}\right|_{i,l,m}^n \\ & + \ c_y^- \left.\frac{-N_{j+2} + 4N_{j+1} - 3N_j}{2\triangle y}\right|_{i,l,m}^n \\ & + \ c_\sigma^+ \left.\frac{3N_l - 4N_{l-1} + N_{l-2}}{2\triangle \sigma}\right|_{i,j,m}^n \\ & + \ c_\sigma^- \left.\frac{-N_{l+2} + 4N_{l+1} - 3N_l}{2\triangle \sigma}\right|_{i,j,m}^n \\ & + \ c_\theta^+ \left.\frac{3N_m - 4N_{m-1} + N_{m-2}}{2\triangle \theta}\right|_{i,j,l}^n \\ & + \ c_\theta^- \left.\frac{-N_{m+2} + 4N_{m+1} - 3N_m}{2\triangle \theta}\right|_{i,j,l}^n \\ & = \ \left.\frac{S}{\sigma_l}\right|_{i,j,l,m}^{n-1} \end{array} \quad (7)$$

where n is a time step with $\triangle t$ and for each point (i,j,l,m)

$$c_x^+ = \max\{\bar{c}_x, 0\}, \quad c_x^- = \min\{\bar{c}_x, 0\},$$
$$c_y^+ = \max\{\bar{c}_y, 0\}, \quad c_y^- = \min\{\bar{c}_y, 0\},$$
$$c_\sigma^+ = \max\{\bar{c}_\sigma, 0\}, \quad c_\sigma^- = \min\{\bar{c}_\sigma, 0\},$$
$$c_\theta^+ = \max\{\bar{c}_\theta, 0\}, \quad c_\theta^- = \min\{\bar{c}_\theta, 0\},$$

$$\begin{aligned}
\bar{c}_x|_{i,j,l,m} &= \tfrac{1}{6}(c_{x_{i-2}} + c_{x_{i-1}} + 2c_{x_i} + c_{x_{i+1}} + c_{x_{i+2}})_{j,l,m},\\
\bar{c}_y|_{i,j,l,m} &= \tfrac{1}{6}(c_{y_{j-2}} + c_{y_{j-1}} + 2c_{y_j} + c_{y_{j+1}} + c_{y_{j+2}})_{i,l,m},\\
\bar{c}_\sigma|_{i,j,l,m} &= \tfrac{1}{6}(c_{\sigma_{l-2}} + c_{\sigma_{l-1}} + 2c_{\sigma_l} + c_{\sigma_{l+1}} + c_{\sigma_{l+2}})_{i,j,m},\\
\bar{c}_\theta|_{i,j,l,m} &= \tfrac{1}{6}(c_{\theta_{m-2}} + c_{\theta_{m-1}} + 2c_{\theta_m} + c_{\theta_{m+1}} + c_{\theta_{m+2}})_{i,j,l}.
\end{aligned}$$

When rearranging the equation (7), then we have the following equation

$$\begin{aligned}
a_{i,j,l,m}N^n_{i,j,l,m} &- \tfrac{4\triangle tc_x^+}{2\triangle x}N^n_{i-1,j,l,m} + \tfrac{\triangle tc_x^+}{2\triangle x}N^n_{i-2,j,l,m}\\
&- \tfrac{\triangle tc_x^-}{2\triangle x}N^n_{i+2,j,l,m} + \tfrac{4\triangle tc_x^-}{2\triangle x}N^n_{i+1,j,l,m}\\
&- \tfrac{4\triangle tc_y^+}{2\triangle y}N^n_{i,j-1,l,m} + \tfrac{\triangle tc_y^+}{2\triangle y}N^n_{i,j-2,l,m}\\
&- \tfrac{\triangle tc_y^-}{2\triangle y}N^n_{i,j+2,l,m} + \tfrac{4\triangle tc_y^-}{2\triangle y}N^n_{i,j+1,l,m}\\
&- \tfrac{4\triangle tc_\sigma^+}{2\triangle \sigma}N^n_{i,j,l-1,m} + \tfrac{4\triangle tc_\sigma^+}{2\triangle \sigma}N^n_{i,j,l-2,m}\\
&- \tfrac{\triangle tc_\sigma^-}{2\triangle \sigma}N^n_{i,j,l+2,m} + \tfrac{4\triangle tc_\sigma^-}{\triangle \sigma}N^n_{i,j,l+1,m}\\
&- \tfrac{4\triangle tc_\theta^+}{2\triangle \theta}N^n_{i,j,l,m-1} + \tfrac{\triangle tc_\theta^+}{2\triangle \theta}N^n_{i,j,l,m-2}\\
&- \tfrac{\triangle tc_\theta^-}{2\triangle \theta}N^n_{i,j,l,m+2} + \tfrac{4\triangle tc_\theta^-}{2\triangle \theta}N^n_{i,j,l,m+1}\\
&= \tfrac{\triangle t}{\sigma_l}S^{n-1}_{i,j,l,m} + N^{n-1}_{i,j,l,m}
\end{aligned}$$

where $i = 1,\dots,N_x;\ j = 1,\dots,N_y;\ l = 1,\dots,N_\sigma;$ $m = 1,\dots,N_\theta$, and

$$\begin{aligned}
a_{i,j,l,m} &= 1 + \frac{3\triangle t}{2}\left(\frac{c_x^+ - c_x^-}{\triangle x} + \frac{c_y^+ - c_y^-}{\triangle y}\right.\\
&\qquad \left. + \frac{c_\sigma^+ - c_\sigma^-}{\triangle \sigma} + \frac{c_\theta^+ - c_\theta^-}{\triangle \theta}\right)_{i,j,l,m}. \qquad (8)
\end{aligned}$$

We can see that the structure of the coefficient matrix of the linear system (3) is in the form of banded-17 matrix. This linear system can be solved by any direct or iterative methods under the diagonal dominant condition, that is sum of off diagonal entry must less than the main diagonal of the coefficient matrix.

Now, we are analyzing the criteria of $\triangle t$, $\triangle x$, $\triangle y$, $\triangle \sigma$ and $\triangle \theta$ for existant and uniqueness solution of this linear system. Let us consider the diagonal dominant condition

$$\begin{aligned}
a_{i,j,l,m} &> \left|\tfrac{4\triangle t}{2\triangle x}c_x^+\right|_{i-1,j,l,m} + \left|\tfrac{\triangle t}{2\triangle x}c_x^+\right|_{i-2,j,l,m}\\
&+ \left|\tfrac{\triangle t}{2\triangle x}c_x^-\right|_{i+2,j,l,m} + \left|\tfrac{4\triangle t}{2\triangle x}c_x^-\right|_{i+1,j,l,m}\\
&+ \left|\tfrac{4\triangle t}{2\triangle y}c_y^+\right|_{i,j-1,l,m} + \left|\tfrac{\triangle t}{2\triangle y}c_y^+)\right|_{i,j-2,l,m}\\
&+ \left|\tfrac{\triangle t}{2\triangle y}c_y^-\right|_{i,j+2,l,m} + \left|\tfrac{4\triangle t}{2\triangle y}c_y^-)\right|_{i,j+1,l,m}\\
&+ \left|\tfrac{4\triangle t}{2\triangle \sigma}c_\sigma^+\right|_{i,j,l-1,m} + \left|\tfrac{\triangle t}{2\triangle \sigma}c_\sigma^+\right|_{i,j,l-2,m}\\
&+ \left|\tfrac{\triangle t}{2\triangle \sigma}c_\sigma^-\right|_{i,j,l+2,m} + \left|\tfrac{4\triangle t}{2\triangle \sigma}c_\sigma^-\right|_{i,j,l+1,m}\\
&+ \left|\tfrac{4\triangle t}{2\triangle \theta}c_\theta^+\right|_{i,j,l,m-1} + \left|\tfrac{\triangle t}{2\triangle \theta}c_\theta^+\right|_{i,j,l,m-2}\\
&+ \left|\tfrac{\triangle t}{2\triangle \theta}c_\theta^-\right|_{i,j,l,m+2} + \left|\tfrac{4\triangle t}{2\triangle \theta}c_\theta^-\right|_{i,j,l,m+1}\\
&\equiv cond
\end{aligned} \qquad (9)$$

where $i = 1,\dots,N_x;\ j = 1,\dots,N_y;\ l = 1,\dots,N_\sigma;$ $m = 1,\dots,N_\theta$.

In the same way as the previous section, we can simplify this stability criteria as follows. Since

$$\begin{aligned}
cond &\le \frac{5\triangle tM_x}{\triangle x} + \frac{5\triangle tM_y}{\triangle y} + \frac{5\triangle tM_\sigma}{\triangle \sigma} + \frac{5\triangle tM_\theta}{\triangle \theta}\\
&\le 20\triangle t\max\left\{\frac{M_x}{\triangle x}, \frac{M_y}{\triangle y}, \frac{M_\sigma}{\triangle \sigma}, \frac{M_\theta}{\triangle \theta}\right\}.
\end{aligned}$$

Since $|a_{i,j,l,m}| \ge 1$ and $20\triangle t\max\left\{\frac{M_x}{\triangle x}, \frac{M_y}{\triangle y}, \frac{M_\sigma}{\triangle \sigma}, \frac{M_\theta}{\triangle \theta}\right\} \ge cond$, we can choose

$$1 > 20\triangle t\max\left\{\frac{M_x}{\triangle x}, \frac{M_y}{\triangle y}, \frac{M_\sigma}{\triangle \sigma}, \frac{M_\theta}{\triangle \theta}\right\}.$$

Thus the condition of $\triangle t$ that satisfy the diagonal dominant of the linear system is following

$$\triangle t < \frac{1}{20\max\left\{\frac{M_x}{\triangle x}, \frac{M_y}{\triangle y}, \frac{M_\sigma}{\triangle \sigma}, \frac{M_\theta}{\triangle \theta}\right\}}.$$

III. The Splitting Upwind Schemes

In the previous section, the numerical solution of the spectral action balance equation was described with a very huge coefficient matrix that needs to be solved by any direct and iterative methods that take a lots of computer's memory and operation count.

In this section, we will design a new numerical method that reduce the size of the original problem by splitting the original problem into four smaller problems. For each smaller problem can be solved easier than the original problem and take less computer's resource such as memory and operation counts. This method is called "The splitting method".

Let us consider the spectral action balance equation on a domain $\Omega \times \Gamma$ with boundary $\partial\Omega \times \partial\Gamma$:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}(c_xN) + \frac{\partial}{\partial y}(c_yN) + \frac{\partial}{\partial \sigma}(c_\sigma N) + \frac{\partial}{\partial \theta}(c_\theta N) = \frac{S}{\sigma}. \quad (10)$$

Let Λ_x, Λ_y, Λ_σ, and Λ_θ be approximation operator of $\frac{\partial}{\partial x}c_x(\cdot)$, $\frac{\partial}{\partial y}c_y(\cdot)$, $\frac{\partial}{\partial \sigma}c_\sigma(\cdot)$ and $\frac{\partial}{\partial \theta}c_\theta(\cdot)$, respectively. For each point $(x_i, y_j, \sigma_l, \theta_m, t_k)$, the approximate operators are represented as following

$$\begin{aligned}
\tfrac{\partial}{\partial x}(c_xN)\Big|^k_{ijlm} &\approx \Lambda_xN^k_{ijlm},\\
\tfrac{\partial}{\partial y}(c_yN)\Big|^k_{ijlm} &\approx \Lambda_yN^k_{ijlm},\\
\tfrac{\partial}{\partial \sigma}(c_\sigma N)\Big|^k_{ijlm} &\approx \Lambda_\sigma N^k_{ijlm},\\
\tfrac{\partial}{\partial \theta}(c_\theta N)\Big|^k_{ijlm} &\approx \Lambda_\theta N^k_{ijlm}.
\end{aligned}$$

Therefore, the equation (10) can be approximated at each point $(x_i, y_j, \sigma_l, \theta_m, t)$ by the following

$$\frac{\partial N}{\partial t}\bigg|_{ijlm} + \Lambda_xN_{ijlm} + \Lambda_yN_{ijlm} + \Lambda_\sigma N_{ijlm} + \Lambda_\theta N_{ijlm} = \frac{S_{ijlm}}{\sigma_l}.$$

For convenient writing, the indices i,j,l,m are neglected, yields

$$\frac{\partial N}{\partial t} + \Lambda_xN + \Lambda_yN + \Lambda_\sigma N + \Lambda_\theta N = \frac{S}{\sigma}.$$

Let us consider the backward time of spectral action balance equation at point $(x_i, y_j, \sigma_l, \theta_m)$

$$\frac{N^{k+1}-N^k}{\tau}+\Lambda_x N^{k+1}+\Lambda_y N^{k+1}+\Lambda_\sigma N^{k+1}+\Lambda_\theta N^{k+1}=\frac{S^{k+1}}{\sigma}. \quad (11)$$

and we introduce the splitting scheme

$$\frac{N^{k+\frac{1}{4}}-N^k}{\tau}+\Lambda_x N^{k+\frac{1}{4}} = 0 \quad (12)$$
$$\frac{N^{k+\frac{2}{4}}-N^{k+\frac{1}{4}}}{\tau}+\Lambda_y N^{k+\frac{2}{4}} = 0 \quad (13)$$
$$\frac{N^{k+\frac{3}{4}}-N^{k+\frac{2}{4}}}{\tau}+\Lambda_\sigma N^{k+\frac{3}{4}} = 0 \quad (14)$$
$$\frac{N^{k+1}-N^{k+\frac{3}{4}}}{\tau}+\Lambda_\theta N^{k+1} = \frac{S^{k+1}}{\sigma}. \quad (15)$$

Now, we will prove that equations (12)-(15) are consistent with the equation (11) by rearranging equations (12)-(15), yields

$$-N^k+(I+\tau\Lambda_x)N^{k+\frac{1}{4}} = 0 \quad (16)$$
$$-N^{k+\frac{1}{4}}+(I+\tau\Lambda_y)N^{k+\frac{2}{4}} = 0 \quad (17)$$
$$-N^{k+\frac{2}{4}}+(I+\tau\Lambda_\sigma)N^{k+\frac{3}{4}} = 0 \quad (18)$$
$$-N^{k+\frac{3}{4}}+(I+\tau\Lambda_\theta)N^{k+1} = \frac{\tau S^{k+1}}{\sigma} \quad (19)$$

where I is an identity approximation operator. To eliminate $N^{k+\frac{1}{4}}$ in equations (16) and (17), we multiply the equation (17) by $(I+\tau\Lambda_x)$ and adding the result to the equation (16), then we obtain

$$-N^k+(I+\tau\Lambda_x)(I+\tau\Lambda_y)N^{k+\frac{2}{4}}=0. \quad (20)$$

To eliminate $N^{k+\frac{2}{4}}$ in equations (18) and (20), we multiply the equation (18) by $(I+\tau\Lambda_x)(I+\tau\Lambda_y)$ and adding the result to the equation (20) then

$$-N^k+(I+\tau\Lambda_x)(I+\tau\Lambda_y)(I+\tau\Lambda_\sigma)N^{k+\frac{3}{4}}=0. \quad (21)$$

Similarly to eliminate $N^{k+\frac{3}{4}}$ in equations (19) and (21), we multiply the equations (19) by $(I+\tau\Lambda_x)(I+\tau\Lambda_y)(I+\tau\Lambda_\sigma)$ and adding the result to the equation (21) then

$$\begin{aligned}-N^k+(I+\tau\Lambda_x)(I+\tau\Lambda_y)(I+\tau\Lambda_\sigma)(I+\tau\Lambda_\theta)N^{k+1}\\ =(I+\tau\Lambda_x)(I+\tau\Lambda_y)(I+\tau\Lambda_\sigma)(\frac{\tau S^{k+1}}{\sigma}).\end{aligned} \quad (22)$$

Since

$$(I+\tau\Lambda_x)(I+\tau\Lambda_y)(I+\tau\Lambda_\sigma)=I+\tau(\Lambda_\sigma+\Lambda_x+\Lambda_y)+O(\tau^2) \quad (23)$$

and

$$\begin{aligned}(I+\tau\Lambda_x)(I+\tau\Lambda_y)(I+\tau\Lambda_\sigma)(I+\tau\Lambda_\theta)\\ =I+\tau(\Lambda_x+\Lambda_y+\Lambda_\sigma+\Lambda_\theta)+O(\tau^2),\end{aligned} \quad (24)$$

substituting equations (23) and (24) into the equation (22),

$$\begin{aligned}-N^k+[I+\tau(\Lambda_x+\Lambda_y+\Lambda_\sigma+\Lambda_\theta)+O(\tau^2)]N^{k+1}\\ =[I+\tau(\Lambda_x+\Lambda_y+\Lambda_\sigma)+O(\tau^2)]\left[\tfrac{\tau S^{k+1}}{\sigma}\right].\end{aligned} \quad (25)$$

Then

$$\begin{aligned}\tfrac{N^{k+1}-N^k}{\tau}+\Lambda_x N^{k+1}+\Lambda_y N^{k+1}\\ +\Lambda_\sigma N^{k+1}+\Lambda_\theta N^{k+1}=\tfrac{S^{k+1}}{\sigma}.\end{aligned} \quad (26)$$

Therefore, the scheme (26) and the equivalent schemes (12)-(15) approximate the spectral action balance equation with the same accuracy $O(\tau)$ as the scheme (11).

A. The First-Order Splitting Upwind Scheme

For the stability criteria for each system, we must choose the type of approximate operator $\Lambda_x, \Lambda_y, \Lambda_\sigma$ and Λ_θ. In this section, we choose these approximate operators as the first-order upwind approximation. We apply the first-order upwind approximation with equations (12)-(15), yields

$$\left.\frac{N^{n+\frac{1}{4}}-N^n}{\tau_x}\right|_{i,j,l,m} + \left.\frac{c_x^+}{\triangle x}(N_i-N_{i-1})\right|_{j,l,m}^{n+\frac{1}{4}} + \left.\frac{c_x^-}{\triangle x}(N_{i+1}-N_i)\right|_{j,l,m}^{n+\frac{1}{4}} = 0 \quad (27)$$

$$\left.\frac{N^{n+\frac{2}{4}}-N^{n+\frac{1}{4}}}{\tau_y}\right|_{i,j,l,m} + \left.\frac{c_y^+}{\triangle y}(N_j-N_{j-1})\right|_{i,l,m}^{n+\frac{2}{4}} + \left.\frac{c_y^-}{\triangle y}(N_{j+1}-N_j)\right|_{i,l,m}^{n+\frac{2}{4}} = 0 \quad (28)$$

$$\left.\frac{N^{n+\frac{3}{4}}-N^{n+\frac{2}{4}}}{\tau_\sigma}\right|_{i,j,l,m} + \left.\frac{c_\sigma^+}{\triangle \sigma}(N_l-N_{l-1})\right|_{i,j,m}^{n+\frac{3}{4}} + \left.\frac{c_\sigma^-}{\triangle \sigma}(N_{l+1}-N_l)\right|_{i,j,m}^{n+\frac{3}{4}} = 0 \quad (29)$$

$$\left.\frac{N^{n+1}-N^{n+\frac{3}{4}}}{\tau_\theta}\right|_{i,j,l,m} + \left.\frac{c_\theta^+}{\triangle \theta}(N_m-N_{m-1})\right|_{i,j,l}^{n+1} + \left.\frac{c_\theta^-}{\triangle \theta}(N_{m+1}-N_m)\right|_{i,j,l}^{n+1} = \left.\frac{S^{n-1}}{\sigma_l}\right|_{i,j,l,m} \quad (30)$$

From the splitting scheme, we get four tridiagonal systems and for each system has a unique solution when it satisfies the diagonal dominant condition. The conditions of $\tau_x, \tau_y, \tau_\sigma$ and τ_θ that satisfy the diagonal dominant condition of the splitting scheme are following

$$\tau_x<\frac{\triangle x}{2M_x}, \quad \tau_y<\frac{\triangle y}{2M_y}, \quad \tau_\sigma<\frac{\triangle \sigma}{2M_\sigma}, \quad \tau_\theta<\frac{\triangle \theta}{2M_\theta}.$$

Thus the condition of τ that satisfy the diagonal dominant of all linear systems in the splitting scheme is following

$$\tau<\frac{1}{2}\min\left\{\frac{\triangle x}{M_x},\frac{\triangle y}{M_y},\frac{\triangle \sigma}{M_\sigma},\frac{\triangle \theta}{M_\theta}\right\}.$$

B. The Second-Order Splitting Upwind Scheme

For the stability criteria for each system, we must choose the type of approximate operators $\Lambda_x, \Lambda_y, \Lambda_\sigma$ and Λ_θ. In this section, we choose these approximate operators as the second-order upwind approximation. We apply the second-order upwind approximation with equations (12)-(15), yields

$$\begin{aligned}&\left.\frac{N^{n+\frac{1}{4}}-N^{n}}{\tau_x}\right|_{i,j,l,m}\\ &+\frac{c_x^{+}}{2\triangle x}(3N_i-4N_{i-1}+N_{i-2})\Big|_{j,l,m}^{n+\frac{1}{4}}\\ &+\frac{c_x^{-}}{2\triangle x}(-N_{i+2}+4N_{i+1}-3N_i)\Big|_{j,l,m}^{n+\frac{1}{4}}=0\end{aligned} \quad (31)$$

$$\begin{aligned}&\left.\frac{N^{n+\frac{2}{4}}-N^{n+\frac{1}{4}}}{\tau_y}\right|_{i,j,l,m}\\ &+\frac{c_y^{+}}{2\triangle y}(3N_j-4N_{j-1}+N_{j-2})\Big|_{i,l,m}^{n+\frac{2}{4}}\\ &+\frac{c_y^{-}}{2\triangle y}(-N_{j+2}+4N_{j+1}-3N_j)\Big|_{i,l,m}^{n+\frac{2}{4}}=0\end{aligned} \quad (32)$$

$$\begin{aligned}&\left.\frac{N^{n+\frac{3}{4}}-N^{n+\frac{2}{4}}}{\tau_\sigma}\right|_{i,j,l,m}\\ &+\frac{c_\sigma^{+}}{2\triangle \sigma}(3N_l-4N_{l-1}+N_{l-2})\Big|_{i,j,m}^{n+\frac{3}{4}}\\ &+\frac{c_\sigma^{-}}{2\triangle \sigma}(-N_{l+2}+4N_{l+1}-3N_l)\Big|_{i,j,m}^{n+\frac{3}{4}}=0\end{aligned} \quad (33)$$

$$\begin{aligned}&\left.\frac{N^{n+1}-N^{n+\frac{3}{4}}}{\tau_\theta}\right|_{i,j,l,m}\\ &+\frac{c_\theta^{+}}{2\triangle \theta}(3N_m-4N_{m-1}+N_{m-2})\Big|_{i,j,l}^{n+1}\\ &+\frac{c_\theta^{-}}{2\triangle \theta}(-N_{m+2}+4N_{m+1}-3N_m)\Big|_{i,j,l}^{n+1}=\left.\frac{S^{n-1}}{\sigma_l}\right|_{i,j,l,m}.\end{aligned} \quad (34)$$

From the splitting scheme, we get four penta-diagonal systems and for each system has a unique solution when it satisfies the diagonal dominant condition. The conditions of $\tau_x, \tau_y, \tau_\sigma$ and τ_θ that satisfy the diagonal dominant condition of the splitting scheme are following

$$\tau_x < \frac{\triangle x}{5M_x}, \quad \tau_y < \frac{\triangle y}{5M_y}, \quad \tau_\sigma < \frac{\triangle \sigma}{5M_\sigma}, \quad \tau_\theta < \frac{\triangle \theta}{5M_\theta}.$$

Thus the condition of τ that satisfy the diagonal dominant of all linear systems in the splitting scheme is following

$$\tau < \frac{1}{5}\min\left\{\frac{\triangle x}{M_x}, \frac{\triangle y}{M_y}, \frac{\triangle \sigma}{M_\sigma}, \frac{\triangle \theta}{M_\theta}\right\}.$$

IV. Numerical Experiments

In this section, we collect some results calculated using the first and second order splitting upwind schemes and compare to the central difference scheme in [2]. We wish to emphasize the diversity of the possible applications.

We begin with spectral action balance equation:

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial x}(c_x N)+\frac{\partial}{\partial y}(c_y N)+\frac{\partial}{\partial \sigma}(c_\sigma N)+\frac{\partial}{\partial \theta}(c_\theta N)=\frac{S}{\sigma}, \quad (35)$$

$$\forall (x,y,\sigma,\theta)\in \Omega\times\Gamma$$

where $t \in [0,T]$, and the initial and boundary conditions are defined as follows:

$$\begin{aligned} N|_{t=0} &= N_0(x,y,\sigma,\theta), \quad \forall (x,y,\sigma,\theta)\in\Omega\times\Gamma; \quad (36)\\ \frac{\partial N}{\partial n} &= 0, \quad \forall (x,y,\sigma,\theta)\in\partial\Omega\times\partial\Gamma,\ t\in[0,T]. \quad (37)\end{aligned}$$

The specific parameters used in our calculations are as follows:

$$\begin{array}{llll} x_l \le x \le x_r, & y_l \le y \le y_r, & \sigma_l \le \sigma \le \sigma_r, & \theta_l \le \theta \le \theta_r,\\ x_l=-1, & x_r=1, & y_l=-1, & y_r=1,\\ \sigma_l=0.04, & \sigma_r=1, & \theta_l=0, & \theta_r=2\pi,\\ N_x=20, & N_y=20, & N_\sigma=20, & N_\theta=20,\\ \triangle x=\frac{x_r-x_l}{N_x-1}, & \triangle y=\frac{y_r-yx_l}{N_y-1}, & \triangle \sigma=\frac{\sigma_r-\sigma_l}{N_\sigma-1}, & \triangle \theta=\frac{\theta_r-\theta_l}{N_\theta-1}, \end{array} \quad (38)$$

source terms:

$$\begin{aligned} S(0,9:11,N_y-2,N_\sigma/2,N_\theta/2) &= 100,\\ S(0,14:17,N_y-2,N_\sigma/2,N_\theta/2) &= 100,\\ S(0,3:10,3,N_\sigma/2,N_\theta/2) &= 100,\\ S(t,x,y,\sigma,\theta) &= 0,\\ \forall x,y,\sigma,\theta\in\Omega\times\Gamma, &\quad t>0. \end{aligned}$$

and propagation velocity terms:

$$\begin{aligned} c_x(i,j,N_\sigma/2,N_\theta/2) &= \cos(\pi(i+j)/N_x)\\ c_x(i,j,N_\sigma/2,N_\theta/2) &= \sin(\pi(i+j)/N_y)\\ c_\sigma(:,:,:,:) &= 0.01\\ c_\theta(:,:,:,:) &= 0.01 \end{aligned}$$

In this experiment, we simulate a spectral action balance equation in a square domain. The physical configuration consists of a square container filled with wave energy. The splitting central difference scheme and the first and second order splitting upwind schemes are presented. Firstly, we set the initial values of N as zero for very nodes in the domain $\Omega \times \Gamma$. At the initial time, we filled the wave energy into the domain. The numerical results by these three methods are shown in Figures 1 - 3. From these three figures, we can see that the result from a central difference scheme is unphysical oscillations. The reason for this lies in the fact that, for grid spaces are too large, certain properties of the continuous equations are no longer correctly captured by discrete equation. But the result from the first and second order splitting upwind schemes is very stable with the same grid spaces as a central difference scheme and moves along the direction field of the propagation velocities. At the first time step for the first and second order upwind schemes, the energy peaked at those grid point and after that its moves along the direction field of the propagation velocities. The solution of the first order upwind scheme is more stable than other numerical schemes but less precision than the second order upwind scheme. However, the second order upwind scheme is the most precision but the numerical solution near the boundary is very complicated to compute because it has to use the three points forward or backward difference schemes and need more boundary conditions.

V. Conclusion and Discussion

The spectral action balance equation is an equation that used to simulate short-crested wind-generated waves in shallow water areas such as coastal regions and inland waters. This equation consist of two spatial dimensions, wave direction, and wave frequency which can be solved by finite difference method. When this equation with dominating convection term are discretized using central differences, stability problems occur when the grid spacing is chosen too coarse. We have analyzed the first and second order of the splitting upwind schemes for numerical solution of the spectral action balance equation with time splitting. Expression for the leading term of the first and second order accuracy of the splitting schemes, respectively. These numerical schemes were adopted to split the wave spectral action balance equation into four one-dimensional problems, which for each small problem obtains the independently tridiagonal linear systems. Therefore, we can solve these systems by direct or iterative methods at the same time which is very fast when performed by a multi-processors computer.

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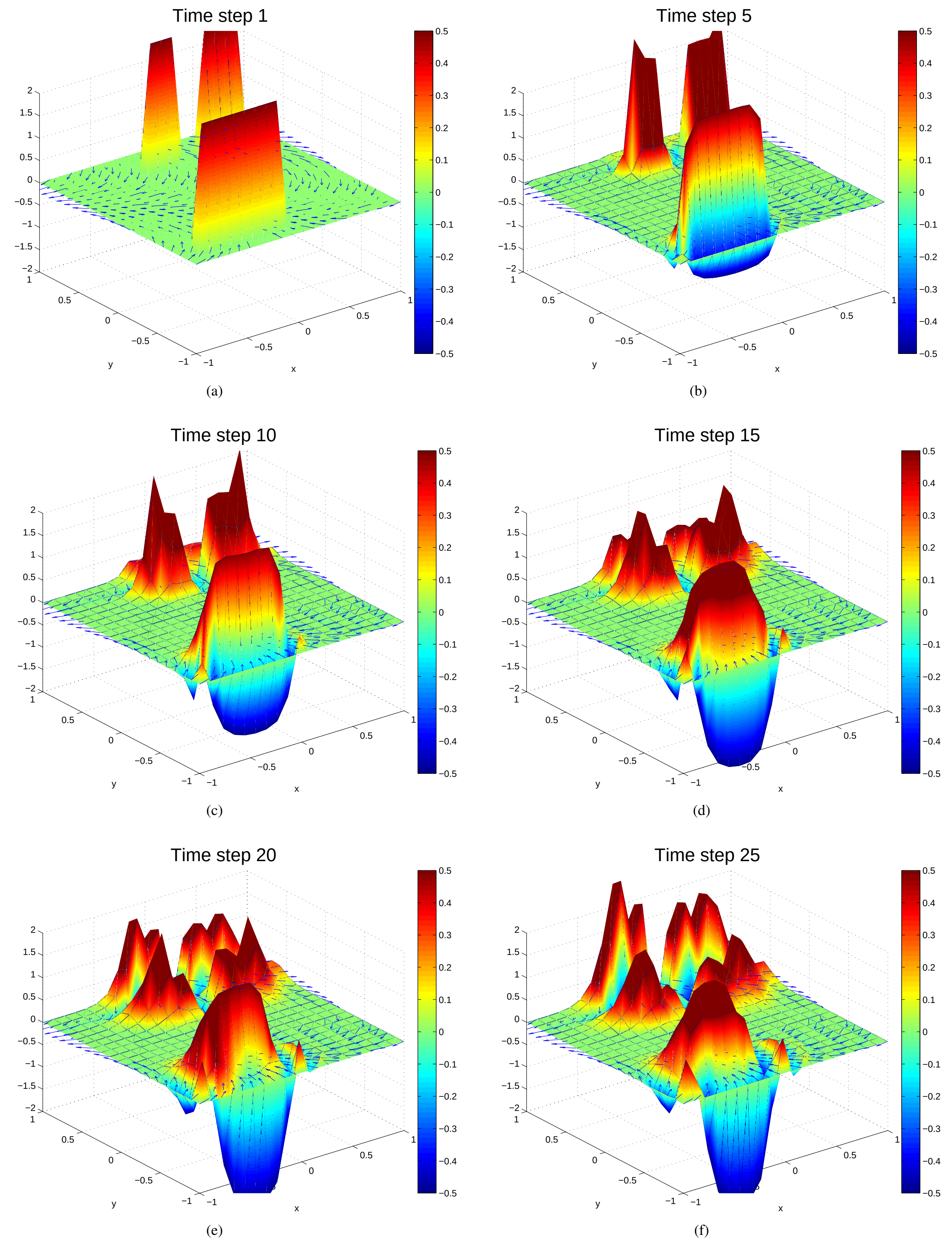

Fig. 1: Numerical results for every 5 time steps by using a central difference scheme.

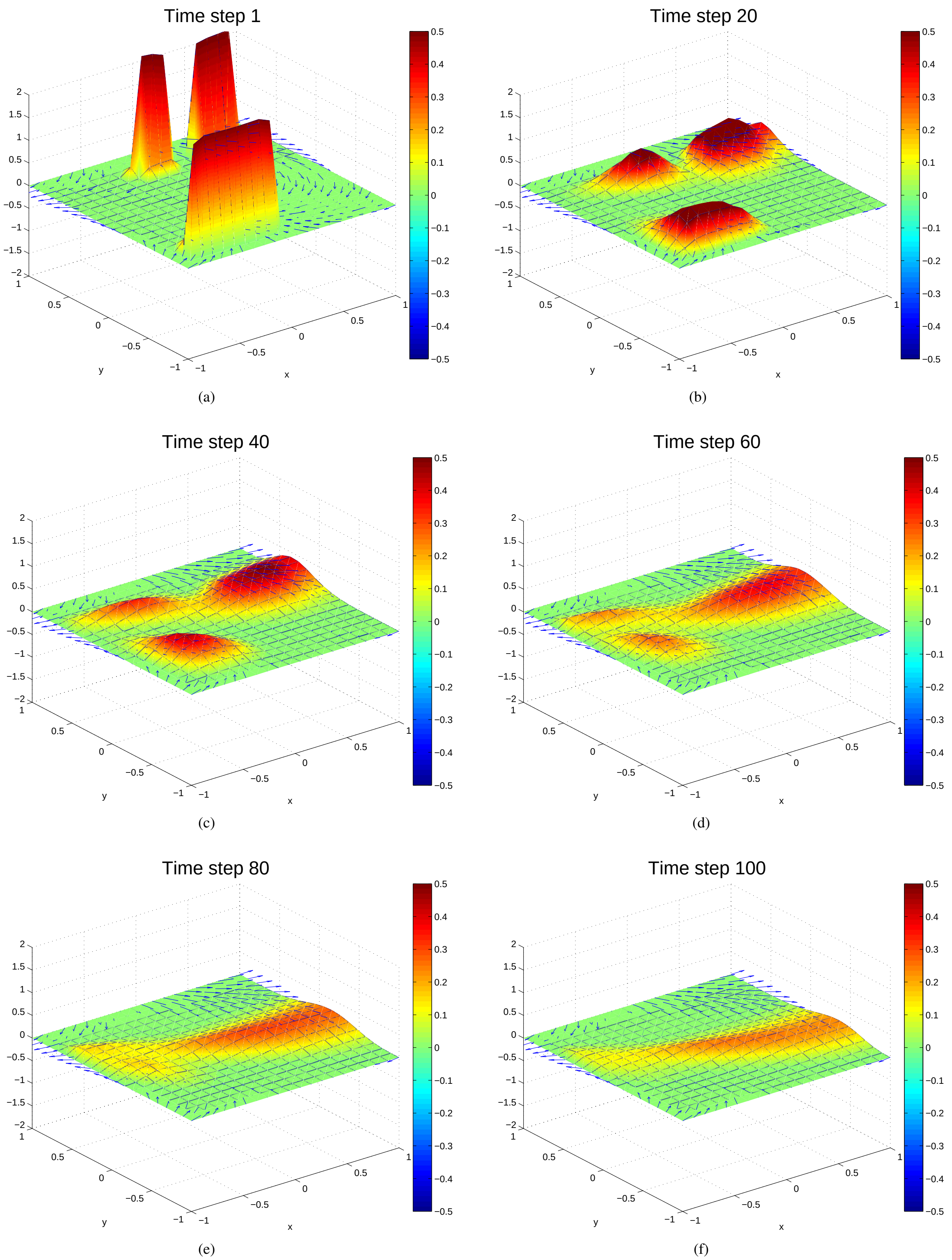

Fig. 2: Numerical results for every 20 time steps by using the first order splitting upwind scheme.

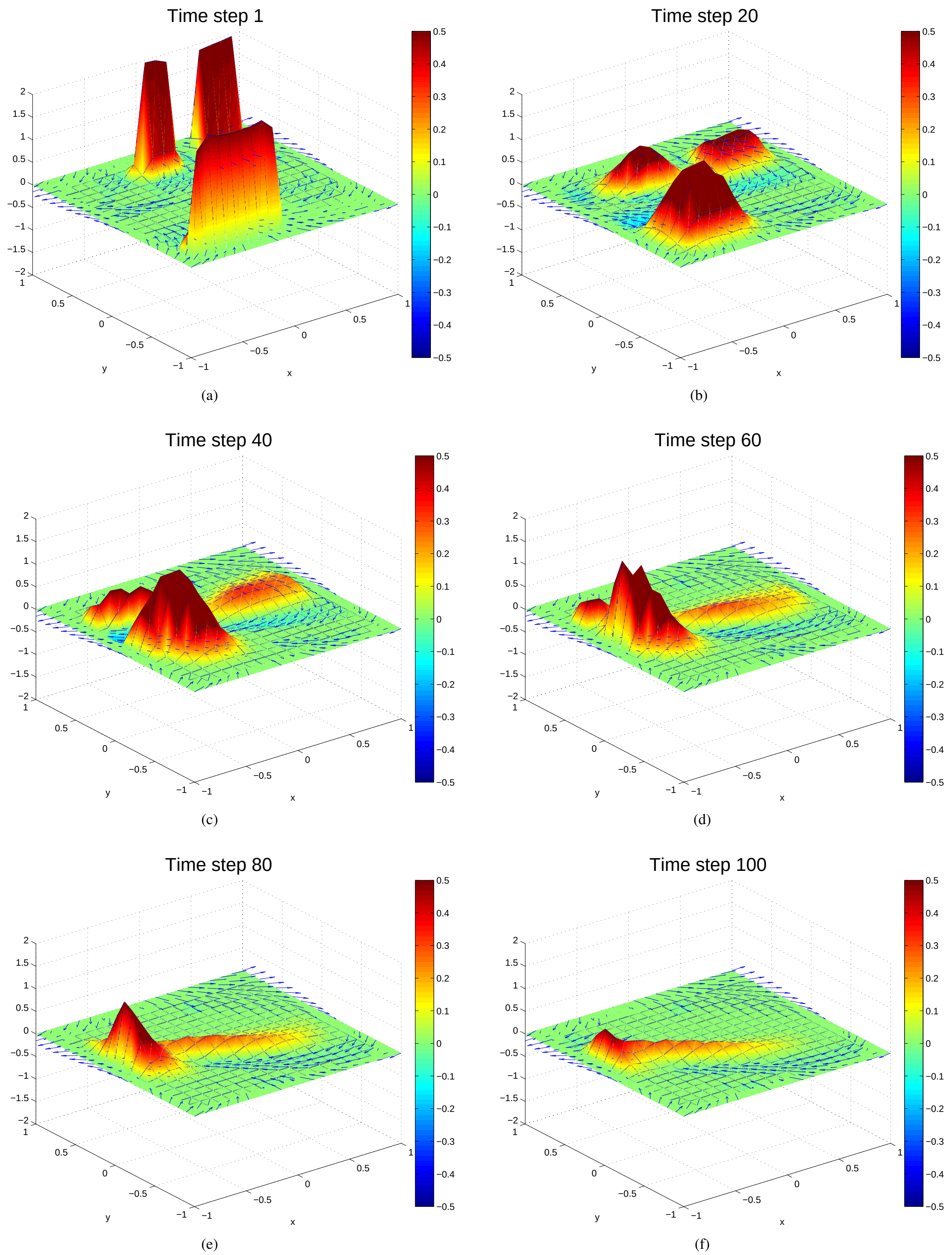

Fig. 3: Numerical results for every 20 time steps by using the second order splitting upwind scheme.