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# The Research and Application of M/M/1/N Queuing Model with Variable Input Rates, Variable Service Rates and Impatient Customers

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*Abstract*—How to maintain the service speeds for the business to make the biggest profit is a problem worthy of study, which is discussed in this paper with the use of queuing theory. An M/M/1/N queuing model with variable input rates, variable service rates and impatient customers is established, and the following conclusions are drawn: the stationary distribution of the model, the relationship between the stationary distribution and the probability that there are $n$ customers left in the system when a customer leaves (not including the customer who leaves himself), the busy period of the system, the average operating cycle, the loss probability for the customers not entering the system while they arriving at the system, the mean of the customers who leaves the system being for impatient, the loss probability for the customers not joining the queue due to the limited capacity of the system and many other indicators. This paper also indicates that the following conclusion is not correct: the more customers the business serve, the more profit they will get. At last, this paper points out the appropriate service speeds the business should keep to make the biggest profit.

*Keywords*—variable input rates, impatient customer, variable service rates, profit maximization.

## I. INTRODUCTION

CUSTOMERS' coming to the enterprise to seek for service may constitute a queuing system. The arrival process of the customers is the input process; the enterprise is the service agency, and we rule on the principle that first-come, first-served. Customers always hope that the queue length for service is as short as possible when they arrive; otherwise, they may refuse to enter the system and leave immediately, even customers who are already in the system may also leave the system due to impatience, which requires the system adjust service speeds flexibly[1]−[5]: raise the service speed to reduce the loss of customers when the queue is long and reduce the service speeds to reduce the cost of services. The service speeds which should the system keep is discussed in this paper.

## II. MODEL HYPOTHESIS

(1) There is only one service window in the system, and its capacity is $N(N > m > 0\ )$, first come first served.

(2) Exponential distribution with the parameter$\lambda_n = \frac{1}{1+n}\lambda$ is applied to time intervals that the customers arrive at system, among which $n$ is the queue length, $n = 0, 1, 2\cdots$N-1. If we suppose $\alpha_n = \frac{1}{1+n}$ obviously we can get $\alpha_0 = 1, \alpha_n > \alpha_{n+1}, \lim\limits_{n\to+\infty} \alpha_n = 0$

(3) Exponential distribution with the parameter $\mu_n$ is applied to the service time $T$ for each customer, among which $\mu_n = \begin{cases} n\mu, 1 \le n \le m \\ m\mu, m < n \le N \end{cases}$

(4)The customers who are already in the system may leave the system due to impatience, its intensity is $\delta_n$ , and $\delta_n = n(n+1)\delta$ , among which $n$ is the queue length,$n = 0, 1, 2\cdots N$, $\delta > 0$. Obviously we can get: $\delta_0 = 0, \delta_n < \delta_{n+1}, \lim\limits_{n\to+\infty} \delta_n = +\infty$

(5) The arrival process of customers and system service process are independent respectively.

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## III. MATHEMATICAL MODEL

Stationary distribution of the system $p_n = P(N(t) = n)$ exists due to limited state of the system. From the model hypothesis we can get the Kolmogorov equations:

For state 0: $\lambda p_0 = (\mu + 2\delta)p_1 \Rightarrow p_1 = \frac{\lambda}{\mu+2\delta}p_0$

For state 1: $\lambda p_0 + (2\mu + 6\delta)p_2 = [(\mu + 2\delta) + \frac{\lambda}{2})]p_1 \Rightarrow p_2 = (\frac{1}{2!})^2 \frac{\lambda^2}{(\mu+2\delta)(\mu+3\delta)}p_0$

For state 2: $\frac{\lambda}{2}p_1 + (3\mu + 12\delta)p_3 = [(2\mu + 6\delta) + \frac{\lambda}{3}]p_2 \Rightarrow p_3 = (\frac{1}{3!})^2 \frac{\lambda^3}{(\mu+2\delta)(\mu+3\delta)(\mu+4\delta)}p_0$

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For state m-1: $\frac{\lambda}{m-1}p_{m-2} + [m\mu + m(m+1)\delta]p_m = [(m-1)\mu + (m-1)m\delta + \frac{\lambda}{m}]p_{m-1} \Rightarrow p_m = (\frac{1}{m!})^2 \frac{\lambda^m}{(\mu+2\delta)(\mu+3\delta)\cdots[\mu+(m+1)\delta]}p_0 = \frac{\lambda^m}{(m!)^2}\prod\limits_{i=2}^{m+1}\frac{1}{\mu+i\delta}p_0$

For state m: $\frac{\lambda}{m}p_{m-1} + [m\mu + (m+1)(m+2)\delta]p_{m+1} = [m\mu + m(m+1)\delta + \frac{\lambda}{m+1}]p_m \Rightarrow p_{m+1} = \frac{\lambda^{m+1}}{m!(m+1)!}(\prod\limits_{i=2}^{m+1}\frac{1}{\mu+i\delta})\frac{1}{m\mu+(m+1)(m+2)\delta}p_0$

For state m+1: $\frac{\lambda}{m+1}p_m + [m\mu + (m+2)(m+3)\delta]p_{m+2} = [m\mu + (m+1)(m+2)\delta + + \frac{\lambda}{m+1}]p_m \Rightarrow p_{m+1} = \frac{\lambda^{m+1}}{m!(m+1)!}(\prod\limits_{i=2}^{m+1}\frac{1}{\mu+i\delta})\frac{1}{m\mu+(m+1)(m+2)\delta}p_0 \Rightarrow p_{m+2} = \frac{\lambda^{m+2}}{m!(m+2)!}(\prod\limits_{i=2}^{m+1}\frac{1}{\mu+i\delta})\frac{1}{m\mu+(m+1)(m+2)\delta}\frac{1}{m\mu+(m+2)(m+3)\delta}p_0$

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For state N-1: $\frac{\lambda}{N-1}p_{N-2} + [m\mu + N(N+1)\delta]p_N = [m\mu + (N-1)N\delta + \frac{\lambda}{N}]p_{N-1} \Rightarrow p_N = (\frac{\lambda^N}{m!N!})(\prod\limits_{i=2}^{m+1}\frac{1}{\mu+i\delta})(\prod\limits_{i=1}^{N-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta})p_0$

In accordance with the regularity we can get $\sum_{k=0}^{N} p_k = 1 \Rightarrow$

$$p_0 = \{1+\sum_{n=1}^{m}[\frac{\lambda^n}{(n!)^2}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta}]+\sum_{n=m+1}^{N}[\frac{\lambda^n}{m!n!}(\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta})(\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta})]\}^{-1}$$

We obtain the following theorem based on the above derivation:

**Claim 1.** Stationary distribution of the system exists, and $p_n =$

$$\begin{cases} (\frac{1}{n!})^2\lambda^n(\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta})p_0, 1 \le n \le m \\ \frac{\lambda^n}{m!n!}(\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta})(\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta})p_0, m < n \le N \end{cases}$$

among which

$$p_0 = \{1+\sum_{n=1}^{m}[\frac{\lambda^n}{(n!)^2}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta}]+\sum_{n=m+1}^{N}[\frac{\lambda^n}{m!n!}(\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta})(\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta})]\}^{-1}$$

To get theorem 2, we give the following lemma:

**Lemma .** Let $X(t) = \sum_{n=1}^{N(t)} X_n$ is the total amount of remuneration before the moment $t$ , if $E[X] < \infty$ $E[T] < \infty$ ,then(1) $\lim_{t\to\infty} P(\frac{X(t)}{t}) = \frac{E[X]}{E[T]} = 1$ (2) $\lim_{t\to\infty} E(\frac{X(t)}{t}) = \frac{E[X]}{E[T]}$

**Claim 2.** If $b_n = P$ (there are $n$ customers left in the system when a customer leaves, not including the customer who leaves himself),$v_n = \begin{cases} n\mu + n(n+1)\delta, 1 \le n \le m \\ m\mu + n(n+1)\delta, m < n \le N \end{cases}$ , then $p_n = [\frac{\lambda_n}{\lambda_0}b_0 + \lambda_n \sum_{n=0}^{N} v_n p_n]^{-1} b_n$, $n = 0, 1, 2\cdots N$.

Proof: Let $A_n =$ the interval that from $N(t) = n+1$ to $N(t) = n$ twice, $B_n =$ the number of the customers served during the time $A_n$ , $C_0 =$ the number of times that there is no customer left in the system when a customer leaves during the time $A_n$ , then that the $N(t)$ from $n+1$ to $n$ each time can be regarded as an update event,$A_n$ can be regarded as a update interval. The time that $N(t)$ among $A_n$ stay in the state $n$ is a random variable, and it obeys the exponential distribution with the parameter $\lambda_n$. By the lemma, if we regard the time that $N(t)$ stay in the state $n$ as remuneration, then $p_n = \frac{1/\lambda_n}{E[A_n]}$. Only one customer leaves the system among $B_n$ , and we can get that $N(t)$ stay in the state $n$; if we regard $B_n$ as update interval length, then by the lemma we know that $b_n = \frac{1}{E[B_n]}$, $b_0 = \frac{E[C_0]}{E[B_n]}$. When$N(t) = 0$ , the next customer will arrive after an interval with an average length of $\lambda_0^{-1}$ , and $A_n$ can be viewed that it is formed by $C_0$ such length and $B_n$ intervals (the length of the interval is equal to the service time for each customer), so $E[A_n] = \frac{1}{\lambda_0}E[C_0] + E[T]E[B_n] \Rightarrow E[A_n] = \frac{1}{\lambda_0}\frac{b_0}{b_n} + E[T]\frac{1}{b_n}$. Because $E[T] = \sum_{n=0}^{N} v_n p_n$, we can get $E[A_n] = \frac{1}{\lambda_0}\frac{b_0}{b_n} + \frac{1}{b_n}\sum_{n=0}^{N} v_n p_n$, further,we can get:$p_n = \frac{1/\lambda_n}{E[A_n]} = [\frac{\lambda_n}{\lambda_0}b_0 + \lambda_n \sum_{n=0}^{N} v_n p_n]^{-1} b_n$, $n = 0, 1, 2\cdots N$.

**Inference 1.** For M/M/1queuing system with parameters $\begin{cases} \lambda_i = \lambda \\ \mu_i = \mu \end{cases}$ , then $p_n = b_n$ , $n = 0, 1, 2\cdots N$.

Proof: If $v_n \equiv \mu$ , then $E[T] = \frac{1}{\mu}$ . If $\lambda_n \equiv \lambda$,$n = 0$, then $p_0 = [b_0 + \frac{\lambda}{\mu}]^{-1} b_0$ . We have known that for M/M/1 queuing system with the parameters $\begin{cases} \lambda_i = \lambda \\ \mu_i = \mu \end{cases}$ , then $p_0 = 1 - \frac{\lambda}{\mu}$ .So we can get $b_0 = 1 - \frac{\lambda}{\mu}$ , put above results into $p_n = [\frac{\lambda_n}{\lambda_0}b_0 + \lambda_n \sum_{n=1}^{N} v_n p_n]^{-1} b_n$, we can get $p_n = b_n$ , $n = 0, 1, 2\cdots N$

**Claim 3.** Let $W_n =$ the time during which the state of the system transfers to $n$ till the state transfers to 0 for the first time, $\omega_n = E(W_n)$,$n = 1, 2\cdots N$, then $\omega_n = \sum_{n=1}^{N}\rho_n + \sum_{m=1}^{n-1}(\prod_{k=1}^{m}\frac{v_k}{\lambda_k})\sum_{i=m+1}^{N}\rho_i$, among which $\lambda_n = \frac{1}{1+n}\lambda$, $\rho_1 = \frac{1}{\mu+2\delta}$, $v_n = \begin{cases} n\mu + n(n+1)\delta, 1 \le n \le m \\ m\mu + n(n+1)\delta, m < n \le N \end{cases}$ , $\rho_n = \begin{cases} \frac{\lambda^n}{n!\prod_{k=1}^{n}[k\mu+k(k+1)\delta]}, 1 \le n \le m \\ \frac{\lambda^n}{n!\prod_{k=1}^{m}[k\mu+k(k+1)\delta]\prod_{k=m+1}^{n}[m\mu+k(k+1)\delta]}, m < n \le N \end{cases}$ .

Proof: Let $\beta_n$ = the time during which the state of the system transfers to $n$ till the next customer arrives. $\gamma_n$ = the time during the state of the system transfers to $n$ till the next customer leaves. From the moment that the state transfers to $n$ , the states will change at the probability of 1, either $n \to n+1$ or $n \to n-1$ , also we depend on the mathematical expectation formula and the nature of exponential distribution, we can get: $\omega_n = E(W_n) = E[\min(\beta_n, \gamma_n)] + E[W_n | \beta_n < \gamma_n]P\{\beta_n < \gamma_n\} + E[W_n | \beta_n > \gamma_n]P\{\beta_n > \gamma_n\} = \frac{1}{\lambda_n+v_n} + \frac{\lambda_n}{\lambda_n+v_n}\omega_{n+1} + \frac{v_n}{\lambda_n+v_n}\omega_{n-1}$, namely $\omega_{n+1} - \omega_n = -\frac{1}{\lambda_n} + \frac{v_n}{\lambda_n}(\omega_n - \omega_{n-1})$(*). Because $\omega_0 = 0$, from the above equation recursion we can get $\omega_{n+1} - \omega_n = -\frac{1}{\lambda_n\rho_n}\sum_{i=1}^{n}\rho_n + \frac{1}{\lambda_n\rho_n}\omega_1$. Let $Z_n = \omega_{n+1} - \omega_n$, $u_n = \frac{z_n\lambda_1\lambda_2\cdots\lambda_n}{v_1v_2\cdots v_n}$, $u_0 = Z_0$, then depending on(*)we can get: $Z_n = -\frac{1}{\lambda_n} + \frac{v_n}{\lambda_n}Z_{n-1}$, $Z_0 = \omega_1 \Rightarrow u_n\frac{v_1v_2\cdots v_n}{\lambda_1\lambda_2\cdots\lambda_n} = -\frac{1}{\lambda_n} + \frac{v_1v_2\cdots v_n}{\lambda_1\lambda_2\cdots\lambda_n}u_{n-1} \Rightarrow u_n - u_{n-1} = -\frac{1}{\lambda_n}\frac{\lambda_1\lambda_2\cdots\lambda_n}{v_1v_2\cdots v_n} = -\rho_n < 0 \Rightarrow u_n < u_{n-1}$, $\Rightarrow u_n$ is a monotone decreasing function, also $Z_n > 0, \lambda_n > 0, v_n > 0$,thus $u_n = \lambda_n Z_n \rho_n > 0$, $\Rightarrow u_n$ has the lower bound, $\Rightarrow \lim_{n\to\infty} u_n$ exists. Also $u_0 - u_n = \sum_{i=1}^{n}\rho_i \Rightarrow \omega_1 = u_0 = \lim_{n\to\infty} u_n + \sum_{i=1}^{\infty}\rho_i$. It is easy for us to get: $\lim_{n\to\infty}\lambda_n = 0$, $\lim_{n\to\infty}\rho_n = 0$. The state of system is limited, so the system is positive recurrent.$\forall$ $n$, $0 < \omega_n < \infty$ $\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty}\lambda_n Z_n \rho_n = 0$. $\omega_1 = 0 + \sum_{n=1}^{\infty}\rho_n = \sum_{n=1}^{N}\rho_n + \sum_{n=N+1}^{\infty}\rho_n = \sum_{n=1}^{N}\rho_n + 0 = \sum_{n=1}^{N}\rho_n(\lambda_n = \frac{1}{1+n}\lambda$, among which $n$ is the queue length, $n = 0, 1, 2\cdots N-1$, namely $\mu_n = \begin{cases} \frac{1}{1+n}\lambda, 0 \le n \le N-1 \\ 0, n > N+1 \end{cases} \Rightarrow \rho_n = 0$, n> N-1$\Rightarrow \sum_{n=N+1}^{\infty}\rho_n = 0$). Put $\omega_1 = \sum_{n=1}^{N}\rho_n$ into $\omega_{n+1} - \omega_n = -\frac{1}{\lambda_n\rho_n}\sum_{i=1}^{n}\rho_n + \frac{1}{\lambda_n\rho_n}\omega_1$, we can get

$\omega_{n+1}-\omega_n = \frac{1}{\lambda_n\rho_n}\sum_{i=n+1}^{N}\rho_i = \prod_{k=1}^{n}\frac{v_k}{\lambda_k}\sum_{i=n+1}^{N}\rho_i \Rightarrow \omega_n = \prod_{k=1}^{n-1}\frac{v_k}{\lambda_k}\sum_{i=n}^{N}\rho_i+\omega_{n-1}$, from this equation recursion we can get $\omega_n=\sum_{n=1}^{N}\rho_n+\sum_{m=1}^{n-1}(\prod_{k=1}^{m}\frac{v_k}{\lambda_k})\sum_{i=m+1}^{N}\rho_i,\ 1\le n\le N.$

**Inference 2.** Busy period $\xi$ is the time from the moment the first customer arrives at the system to the moment the last customer leaves the system. The average busy period of the system $E(\xi)=\frac{1}{\lambda}\{\sum_{n=1}^{m}[\frac{\lambda^n}{(n!)^2}\prod\frac{1}{\mu+i\delta}]+\sum_{n=m+1}^{N}[\frac{\lambda^n}{m!n!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}]\}$

Proof: $E(\xi)=\omega_1$, also depending on the proof of the theorem 3: $\omega_1=\sum_{n=1}^{N}\rho_n\Rightarrow E(\xi)=\sum_{i=1}^{N}\rho_i=\sum_{i=1}^{N}\frac{\lambda_1\lambda_2\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}=\frac{1}{\lambda_0}\sum_{i=1}^{N}\frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}=\frac{1}{\lambda_0}\left[\frac{1}{p_0}-1\right]=\frac{1}{\lambda}\{\sum_{n=1}^{m}[\frac{\lambda^n}{(n!)^2}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta}]+\sum_{n=m+1}^{N}[\frac{\lambda^n}{m!n!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}]\}$

**Inference 3.** Idle period $I$ is the time when there is no customer in the system. The operating cycle is the time when the state of the system from 0 until the next time they transferred to the state 0. The average operating cycle of the system $E(I+\xi)=\frac{1}{\lambda}\{1+\sum_{n=1}^{m}[\frac{\lambda^n}{(n!)^2}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta}]+\sum_{n=m+1}^{N}[\frac{\lambda^n}{m!n!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}]\}$

Proof: $\beta_n$ = the time during the state of the system transfers to $n$ till the next customer arrives. When the state of system transfers to the state 0, it will transfer to state 1 in $\beta_0$ time, and the state 1 will return 0 in $W_1$ time. Thus $I=\beta_0$, $E(I+\xi)=E(\beta_0+W_1)=\frac{1}{\lambda_0}+\frac{1}{\lambda_0}\left(\frac{1}{p_0}-1\right)=\frac{1}{\lambda_0 p_0}=\frac{1}{\lambda}\{1+\sum_{n=1}^{m}[\frac{\lambda^n}{(n!)^2}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta}]+\sum_{n=m+1}^{N}[\frac{\lambda^n}{m!n!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}]\}$

## IV. The other relevant indicators of system

(1) When a customer reaches the system and finds that there are $n$ customers in the system, if he enters the system at the probability of $\alpha_n=\frac{1}{1+n}$ , and he leaves the system at the probability of $1-\alpha_n$ , then the loss probability of the system because of the customers not entering the system $P_{loss}=\sum_{n=0}^{N}p(X=n)\times(1-\alpha_n)=\sum_{n=0}^{N}p_n-\sum_{n=0}^{N}\alpha_n p_n=1-\{p_0+p_0\sum_{n=1}^{m}[\frac{\lambda^{n+1}}{(n+1)!n!}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta}]+p_0\sum_{n=m+1}^{N}[\frac{\lambda^{n+1}}{m!(n+1)!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}]\}$

2) Some customers may leave system before being served due to impatience. The mean of those customers who leaves the system because of impatience $L_I=\sum_{n=0}^{N}\delta_n p_{n+1}=\sum_{n=0}^{N}n(n+1)\delta p_{n+1}=p_0\sum_{n=0}^{m-1}[\frac{\delta\lambda^{n+1}}{(n+1)!(n-1)!}\prod_{i=2}^{n+2}\frac{1}{\mu+i\delta}]\ p_0\sum_{n=m}^{N}[\frac{\delta\lambda^{n+1}}{m!(n-1)!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\prod_{i=1}^{n-m+1}\frac{1}{m\mu+(m+i)(m+i+1)\delta}]$

(3) When a customer arrives at the system and finds that there are N customers in the system, he can not enter the system. The loss probability of the system because of limited capacity $P_N=\frac{\lambda^N}{m!N!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\prod_{i=1}^{N-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}p_0$

(4) The average input rate of system $\overline{\lambda}=(\sum_{n=0}^{N}\lambda_n p_n)(1-P_N)=p_0\{\lambda+\sum_{n=1}^{m}[\frac{\lambda^{n+1}}{n!(n+1)!}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta}]+\sum_{n=m+1}^{N}[\frac{\lambda^{n+1}}{m!(n+1)!}(\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta})(\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta})]\}\{1-p_0\frac{\lambda^N}{m!N!}[\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}][\prod_{i=1}^{N-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}]\}$

(5) The average service intensity of the system is $\bar{\rho}=\frac{\widetilde{\lambda}}{\overline{\mu}}$ , $\widetilde{\lambda}$ is the average input rate of those customers who enter the system and receive the service, $\overline{\mu}$ is the average service rate. If $\overline{\gamma}=\frac{1}{\overline{\mu}}$ , then $\overline{\gamma}$ is the average service time of the system. The average number of the customers who arrive and receive the service during the time $t$ is $\widetilde{\lambda}t$ . The free time of the system is $p_0 t$ . The busy time of the system is $t-p_0 t$ . The number of customers who receive service during the time $t$ is $\frac{t-p_0 t}{\overline{\gamma}}$ . The number of customers entering the system and receive the service, and the number of customers who are serviced in unit time are the same under the balance condition , so $\widetilde{\lambda}t=\frac{t-p_0 t}{\overline{\gamma}}$ . If $t\to\infty$ , then $\widetilde{\lambda}\overline{\gamma}=1-p_0$ , so $\bar{\rho}=1-p_0$.

(6)The average queue length of the system $L_s=\sum_{n=0}^{N}np_n=p_0\sum_{n=1}^{m}[\frac{\lambda^n}{(n-1)!n!}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta}]+p_0\sum_{n=m+1}^{N}[\frac{\lambda^n}{m!(n-1)!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}]$

## V. An example

For an M/M/1/3 queue system, exponential distribution with the parameter $\lambda$ is applied to time intervals that the customers enter the system, and $\lambda=3.6$ ; the average service cost for each customer is E, and E=1; the average income for serving a customer is G, and G =2; $\mu$ is the service rate; the probability that customers entering the system is $P_n=\frac{(1-\rho)\rho^n}{1-\rho^{N+1}}$ ; $n=0,1,2,3$ N=3, then the income of the system is $\lambda(1-p_N)G$ , net income $F$=$\lambda(1-p_N)G-E\mu$ . Compare the net income $F$ and loss probability $P_n$ when $\mu=6$ and $\mu=3$ .

Analysis: $\mu=6\Rightarrow\rho=0.6$,$P_3=\frac{(1-0.6)0.6^3}{1-0.6^4}=0.0993$,$F$=$2\times3.6[1-\frac{(1-0.6)0.6^3}{1-0.6^4}]-1\times6$=0.46; $\mu=3\Rightarrow\rho=1.21$,$P_3=\frac{(1-1.21)1.21^3}{1-1.21^4}]=0.32$, $F$=$2\times3.6[1-\frac{(1-1.21)1.21^3}{1-1.21^4}]-1\times3$= 1.86.

From above analysis we can see that when $\mu=3$ , 32% of the customers do not enter the system and the net income $F$=1.86, while $\mu=6$, less than10% of the customers do not enter the system, but $F$= 0.46, from which we can get: the most profitable strategy is not necessarily the strategy to serve the largest number of customers. Then what the service speeds should the business keep to make the biggest profit?

## VI. Optimization of the model

1. Selection of $(\mu,m)$ among $\mu_n(\mu,m)$

The net income of the system $F = G\widetilde{\lambda} - \frac{G}{M_1}\overline{\mu} - \frac{G}{M_2}N$ , among which G is the average income for serving a customer, $\frac{G}{M_1}$ is the average service cost for each customer, N is the total number of seats for customers, and $\frac{G}{M_2}$ is loss cost of each seat, $\widetilde{\lambda}$ is the average input rate of those customers who enter the system and receive the service, and $\widetilde{\lambda} = \sum_{n=0}^{N}(\lambda_n p_n - \delta_n p_{n+1})(1 - P_N) = p_0\{\lambda + \sum_{n=1}^{m-1}[\frac{\lambda^{n+1}}{n!(n+1)!}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta} - \frac{\lambda^{n+1}\delta}{(n-1)!(n+1)!}\prod_{i=2}^{n+2}\frac{1}{\mu+i\delta}] + \frac{\lambda^{m+1}P_0}{m!(m+1)!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta} - \frac{\lambda^{m+1}\delta P_0}{(m-1)!m!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\frac{1}{m\mu+(m+1)(m+2)\delta}] + \sum_{n=m+1}^{N}[\frac{\lambda^{n+1}}{m!(n+1)!}(\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta})(\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}) - \frac{\lambda^{n+1}\delta P_0}{m!(n-1)!}(\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta})(\prod_{i=2}^{n-m+1}\frac{1}{m\mu+(m+i)(m+i+1)\delta})]\}(\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta})(\prod_{i=2}^{n-m+1}\frac{1}{m\mu+(m+i)(m+i+1)\delta})]\}\{1 - p_0\frac{\lambda^N}{m!N!}[\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}][\prod_{i=1}^{N-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}]\}$, $\overline{\mu}$ is the average service rates, and $\overline{\mu} = \sum_{n=1}^{N}\mu_n p_n = p_0\mu\sum_{n=1}^{m}\frac{\lambda^n}{(n-1)!n!}\prod_{i=2}^{n+1}\frac{1}{\mu+i\delta} + p_0\mu\sum_{n=m+1}^{N}\{\frac{\lambda^n}{n!(m-1)!}\prod_{i=2}^{m+1}\frac{1}{\mu+i\delta}\prod_{i=1}^{n-m}\frac{1}{m\mu+(m+i)(m+i+1)\delta}\}$

We want to get $\max_F$ . Considering both $m$ and $\mu$ are limited, we can put the possible values of $(\mu, m)$ into $F = G\widetilde{\lambda} - \frac{G}{M_1}\overline{\mu} - \frac{G}{M_2}N$ , and compare these values, if $(\mu_0, m_0)$ makes the biggest, then $(\mu_0, m_0)$ is what we want.

2. Optimization of *N* among M/M/1/N

To get $\max_F$ , N should satisfy $\begin{cases} F(N-1) \le F(N) \\ F(N+1) \le F(N) \end{cases}$

$$\Leftrightarrow \begin{cases} G\widetilde{\lambda}(N-1) - \frac{G}{M_1}\overline{\mu}(N-1) - (N-1)\frac{G}{M_2} \\ \le G\widetilde{\lambda}(N) - \frac{G}{M_1}\overline{\mu}(N) - N\frac{G}{M_2} \\ G\widetilde{\lambda}(N+1) - \frac{G}{M_1}\overline{\mu}(N+1) - (N+1)\frac{G}{M_2} \\ \le G\widetilde{\lambda}(N) - \frac{G}{M_1}\overline{\mu}(N) - N\frac{G}{M_2} \end{cases}$$

$$\Leftrightarrow \begin{cases} \widetilde{\lambda}(N-1) - \frac{1}{M_1}\overline{\mu}(N-1) - (N-1)\frac{1}{M_2} \\ \le \widetilde{\lambda}(N) - \frac{1}{M_1}\overline{\mu}(N) - N\frac{1}{M_2} \\ \widetilde{\lambda}(N+1) - \frac{1}{M_1}\overline{\mu}(N+1) - (N+1)\frac{1}{M_2} \\ \le \widetilde{\lambda}(N) - \frac{1}{M_1}\overline{\mu}(N) - N\frac{1}{M_2} \end{cases}$$

$\Leftrightarrow [M_1\widetilde{\lambda}(N+1) - \overline{\mu}(N+1)] - [M_1\widetilde{\lambda}(N) - \overline{\mu}(N)] \le \frac{M_1}{M_2}$
$\le [M_1\widetilde{\lambda}(N) - \overline{\mu}(N)] - [M_1\widetilde{\lambda}(N-1) - \overline{\mu}(N-1)]$ , with different value ranges, we can get N.

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