# The ratios between the spectral norm, the numerical radius and the spectral radius 

Kui Du


#### Abstract

Recently, Uhlig [Numer. Algorithms, 52(3):335-353, 2009] proposed open questions about the ratios between the spectral norm, the numerical radius and the spectral radius of a square matrix. In this note, we provide some observations to answer these questions.


Keywords—Spectral norm, Numerical radius, Spectral radius, Ratios

## I. Introduction

THE numerical radius $w(A)$ of an $n \times n$ matrix $A$ is the real number

$$
w(A)=\max _{z \in F(A)}|z|
$$

where $F(A)$ denotes the field of values (or numerical range) of $A$, defined by

$$
F(A)=\left\{x^{*} A x: x \in \mathbb{C}^{n},\|x\|_{2}=1\right\} .
$$

The spectral radius $\rho(A)$ of $A$ is the real number

$$
\rho(A)=\max _{z \in \sigma(A)}|z|,
$$

where $\sigma(A)$ denotes the spectrum of $A$. The spectral norm of $A$ is defined by

$$
\|A\|_{2}=\max _{\|x\|_{2}=1}\|A x\|_{2} .
$$

In this note, we consider the ratios

$$
s(A)=\|A\|_{2} / w(A)
$$

and

$$
\tau(A)=w(A) / \rho(A)
$$

It is well known that

$$
0 \leq \rho(A) \leq w(A) \leq\|A\|_{2} \leq 2 w(A)
$$

Thus,

$$
1 \leq s(A) \leq 2
$$

and

$$
1 \leq \tau(A) \leq \infty
$$

Here we employ the convention that $\tau(A)=\infty$ for $\rho(A)=0$. Obviously, $s(z A)=s(A)$ and $\tau(z A)=\tau(A)$ for all $z \neq 0$. It follows from $\rho\left(A^{m}\right)=[\rho(A)]^{m}$ and $w\left(A^{m}\right) \leq[w(A)]^{m}$ that $\tau\left(A^{m}\right) \leq[\tau(A)]^{m}$.

Recently, Uhlig [13] proposed the questions: What do the ratios $s(A)$ and $\tau(A)$ indicate about the matrix $A$ ? What can one conclude about $A$ when these ratios are large or very different, or when they are nearly equal? In this note, we provide some observations to answer these questions.

Kui Du, Institute of Mathematics, Aalto University, P.O.Box 11100, FI00076 Aalto, Finland. Email: kuidumath@yahoo.com

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## II. The ratios between the spectral norm, the nUMERICAL RADIUS AND THE SPECTRAL RADIUS

A. The extreme cases $\tau(A)=1, s(A)=1$ and $s(A)=2$

In this subsection, we review the existing results for the extreme cases $\tau(A)=1, s(A)=1$ and $s(A)=2$, respectively. We focus on the relation between $s(A)$ and $\tau(A)$.

A matrix $A$ is said to be spectral if $w(A)=\rho(A)$, i.e., $\tau(A)=1$. The spectral matrices have been investigated by several researchers. We have the following results (see [5] and [9, p.60]).
Proposition 1. Let $A \in \mathbb{C}^{n \times n}$ such that $\tau(A)=1$.

- If $n \leq 2$, then $s(A)=1$, and $A$ is a normal matrix.
- If $n>2$, then $A$ is unitarily similar to a triangle matrix of the form

$$
\left[\begin{array}{cc}
\Lambda_{k} & 0  \tag{1}\\
0 & B
\end{array}\right],
$$

where $1 \leq k \leq n$,

$$
\begin{gathered}
\Lambda_{k}=\left[\begin{array}{lll}
\lambda_{1} & & \\
& \ddots & \\
& & \lambda_{k}
\end{array}\right], \\
B=\left[\begin{array}{ccc}
\lambda_{k+1} & * & * \\
& \ddots & * \\
& & \lambda_{n}
\end{array}\right],
\end{gathered}
$$

and

$$
\begin{gathered}
\rho(B)<\rho(A)=\left|\lambda_{1}\right|=\cdots=\left|\lambda_{k}\right|, \\
w(B) \leq \rho(A) .
\end{gathered}
$$

Furthermore, if $\rho(A)<\|B\|_{2}$,

$$
1<s(A) \leq 2
$$

otherwise, $s(A)=1$.
A matrix $A$ is said to be radial if $w(A)=\|A\|_{2}$, i.e., $s(A)=1$. We have the following results (see [9, p.45]).

Proposition 2. Let $A \in \mathbb{C}^{n \times n}$ such that $s(A)=1$. Then $\tau(A)=1$.

- If $n \leq 2$, then $A$ is a normal matrix.
- If $n>2$, then $A$ is unitarily similar to a block diagonal matrix of the form (1) such that $\rho(B)<\rho(A)$ and $\|B\|_{2} \leq \rho(A)$.
Remark 3. Note that $s(A) \approx 1$ does not imply that $\tau(A) \approx 1$. See Example 4.


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Example 4 (A scaled Jordan block). Let

$$
J_{n}^{\alpha}(\lambda)=\left[\begin{array}{cccc}
\lambda & \alpha & &  \tag{2}\\
& \lambda & \ddots & \\
& & \ddots & \alpha \\
& & & \lambda
\end{array}\right]_{n \times n}=\lambda I+N
$$

be a matrix of order $n>1$. Then $F\left(J_{n}^{\alpha}(\lambda)\right)$ is a disk centered at $\lambda$ with radius $|\alpha| \cos \frac{\pi}{n+1}$ (see [10, Theorem 2.1]). We have $\rho\left(J_{n}^{\alpha}(\lambda)\right)=|\lambda|, w(N)=|\alpha| \cos \frac{\pi}{n+1}, w\left(J_{n}^{\alpha}(\lambda)\right)=|\lambda|+$ $|\alpha| \cos \frac{\pi}{n+1}$ and $\left\|J_{n}^{\alpha}(\lambda)\right\|_{2} \leq|\lambda|+|\alpha|$. Then

$$
\begin{gathered}
\tau\left(J_{n}^{\alpha}(\lambda)\right)=1+\frac{|\alpha|}{|\lambda|} \cos \frac{\pi}{n+1}, \\
s\left(J_{n}^{\alpha}(\lambda)\right) \leq \frac{1+|\alpha / \lambda|}{1+|\alpha / \lambda| \cos \frac{\pi}{n+1}} \leq \frac{1}{\cos \frac{\pi}{n+1}} .
\end{gathered}
$$

Thus, when $n \rightarrow \infty$ and $|\alpha| /|\lambda| \rightarrow \infty, s\left(J_{n}^{\alpha}(\lambda)\right) \rightarrow 1$. However, $\tau\left(J_{n}^{\alpha}(\lambda)\right) \rightarrow \infty$.

Propositions 1 and 2 give the answers of the questions $((1)(2))$ of [13, p.352]. When $s(A)=2$, we have the following result (see [8, p.18-7]).
Proposition 5. Let $A \in \mathbb{C}^{n \times n}$ such that $s(A)=2$. Then $A$ is unitarily similar to a block diagonal matrix of the form

$$
\left[\begin{array}{ll}
\|A\|_{2} J_{2}(0) & \\
& B
\end{array}\right],
$$

where $J_{2}(0)=\left[\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right]$ and $w(B) \leq \frac{\|A\|_{2}}{2}$.
By Proposition 5, it is easy to show that in the case $s(A)=$ $2,1 \leq \tau(A) \leq \infty$.

## B. Upper bounds for $s(A)$ and $\tau(A)$

Let

$$
\begin{equation*}
A=U \Sigma V^{*}, \quad \Sigma=\operatorname{diag}\left(\sigma_{1}, \sigma_{2}, \ldots, \sigma_{n}\right) \tag{3}
\end{equation*}
$$

be a singular value decomposition of $A$, where $U$ and $V$ are unitary and $\sigma_{1} \geq \sigma_{2} \geq \cdots \geq \sigma_{n} \geq 0$. Denote the 2-norm condition number of a nonsingular matrix $A$ by

$$
\kappa(A):=\|A\|_{2}\left\|A^{-1}\right\|_{2}=\sigma_{1} / \sigma_{n}
$$

Proposition 6. Let $A \in \mathbb{C}^{n \times n}$ be as in (3) such that $\sigma_{n}>$ 0 . Then $s(A) \leq \kappa(A)$ and $\tau(A) \leq \kappa(A)$. In particular, the following statements are equivalent:
(i) $s(A)=\kappa(A)$.
(ii) $\tau(A)=\kappa(A)$.
(iii) $\kappa(A)=1$.
(iv) $A$ is a nonzero multiple of a unitary matrix.

Proposition 7. Let $A \in \mathbb{C}^{n \times n}$ be as in (3) such that $\sigma_{n}>0$ and $s(A)=\tau(A)$. Then $s(A)=\tau(A) \leq \sqrt{\kappa(A)}$ and the equality holds if and only if $\kappa(A)=1$.

Note that $\sigma_{n} \leq \rho(A) \leq w(A) \leq\|A\|_{2}=\sigma_{1}$ and if $\rho(A)=$ $\sigma_{n}>0$ then $\sigma_{1}=\cdots=\sigma_{n}$. The proofs of Propositions 6 and 7 are trivial.

By Proposition 6, a large $\tau(A)$ implies that $A$ is ill conditioned and if $A$ is well conditioned, i.e., $\kappa(A) \approx 1$, then $s(A) \approx 1$ and $\tau(A) \approx 1$. For a diagonalizable (singular or nonsingular) nonzero matrix $A=X \Lambda X^{-1}$, we have $s(A), \tau(A) \leq\|A\|_{2} /\|\Lambda\|_{2} \leq \kappa(X)$. Thus, a large $\tau(A)$ implies that any eigenvector basis of the diagonalizable matrix $A$ is ill conditioned.
Remark 8. Let $A \in \mathbb{C}^{n \times n}$ be singular. The generalized 2norm condition number is defined by $\kappa^{\dagger}(A)=\sigma_{1} / \sigma_{r}$, where $r=\operatorname{rank}(A)$ is the rank of $A$. In general, we do not have $s(A) \leq \kappa^{\dagger}(A)$ and $\tau(A) \leq \kappa^{\dagger}(A)$. For example, let

$$
A=\left[\begin{array}{ccc}
\epsilon & 1 & 0 \\
0 & \epsilon & 1 \\
0 & 0 & 0
\end{array}\right]
$$

We have $s(A) \rightarrow \sqrt{2}, \tau(A) \rightarrow \infty$ and $\kappa^{\dagger}(A) \rightarrow 1$ as $\epsilon \rightarrow 0$.
Let

$$
\begin{equation*}
A=U(\Lambda+N) U^{*} \tag{4}
\end{equation*}
$$

be a Schur decomposition of $A$, where $U$ is a unitary matrix, $\Lambda$ is a diagonal matrix whose diagonal elements are the eigenvalues of $A$ and $N$ is a strictly upper triangular matrix.

Proposition 9. Let $A \in \mathbb{C}^{n \times n}$ be as in (4). Then

$$
w(A) \leq \rho(A)+w(N),
$$

and if $\rho(A) \neq 0$,

$$
\tau(A) \leq 1+w(N) / \rho(A) .
$$

Proof: Since $U$ is unitary $F(A)=F(\Lambda+N)$ [9, p.11]. Then

$$
\begin{aligned}
w(A) & =\max _{\|x\|_{2}=1}\left|x^{*}(\Lambda+N) x\right| \\
& \leq \max _{\|x\|_{2}=1}\left|x^{*} \Lambda x\right|+\max _{\|x\|_{2}=1}\left|x^{*} N x\right| \\
& =\rho(A)+w(N) .
\end{aligned}
$$

The proof of the second inequality is trivial.
The bound in Proposition 9 is attainable. For example, let $J_{n}^{\alpha}(\lambda)$ be as in (2). Assume $\lambda \neq 0$. We have

$$
\tau\left(J_{n}^{\alpha}(\lambda)\right)=1+\frac{w(N)}{\rho\left(J_{n}^{\alpha}(\lambda)\right)} .
$$

Another obvious bound for $\tau(A)$ is $\tau(A) \leq\|A\|_{2} / \rho(A)$. When $\tau(A)=\|A\|_{2} / \rho(A)$, i.e., $s(A)=1$, we have $\tau(A)=1$ (see Proposition 2).

## III. A SUFFICIENT CONDITION FOR $0 \in F(A)$

For the matrix $J_{n}^{\alpha}(\lambda)$ in Example 4 with $\lambda \neq 0$, if $|\alpha|$ is sufficiently small, then $0 \notin F\left(J_{n}^{\alpha}(\lambda)\right)$. And when $\tau\left(J_{n}^{\alpha}(\lambda)\right) \geq$ $2,0 \in F\left(J_{n}^{\alpha}(\lambda)\right)$. So a natural question is: Does there exist a constant $c>1$ s.t., if $\tau(A) \geq c$, then $0 \in F(A)$ ? The answer to this question is positive. It was proved in [2, Lemma 2.6] that for any $A \in \mathbb{C}^{n \times n}$ if 0 is not an interior point of $F(A)$, then $\tau(A) \leq n$. Here we give a slightly different version. For completeness we include its proof, which is similar to that of Lemma 2.6 of [2].

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Theorem 10. Let $A \in \mathbb{C}^{n \times n}$ such that $\tau(A) \geq n$. Then $0 \in$ $F(A)$.

Proof: It is sufficient to prove if $0 \notin F(A)$ then $\tau(A)<n$. It is well known that if $0 \notin F(A)$ then there exists a real number $\theta$ such that the Hermitian matrix $H\left(e^{\mathrm{i} \theta} A\right)=\left(e^{\mathrm{i} \theta} A+e^{-\mathrm{i} \theta} A^{*}\right) / 2$ is positive definite; see, e.g., [9, p.21]. By rotating $A$, we assume that the Hermitian part $H(A)=\left(A+A^{*}\right) / 2$ of $A$ is positive definite. Since $F(A)$ is unitary similarity invariant [9, p.11], we also assume $A$ is in upper triangular form

$$
\left[\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1 n} \\
& \ddots & \ddots & \vdots \\
& & \ddots & a_{n-1, n} \\
& & & a_{n n}
\end{array}\right]
$$

The positive definiteness of $H(A)$ implies that for all $i, j=$ $1, \cdots, n$,

$$
\frac{1}{2}\left[\begin{array}{cc}
2 \operatorname{Re}\left(a_{i i}\right) & a_{i j} \\
\overline{a_{i j}} & 2 \operatorname{Re}\left(a_{j j}\right)
\end{array}\right]
$$

are positive definite. Then

$$
\left|a_{i j}\right|<2 \sqrt{\operatorname{Re}\left(a_{i i}\right) \operatorname{Re}\left(a_{j j}\right)} \leq 2 \rho(A) .
$$

Let $|A|=\left(\left|a_{i j}\right|\right)$. By Gershgorin circle theorem, $\rho(|A|+$ $\left.|A|^{T}\right)<2 n \rho(A)$. Then $\tau(A)<n$ follows from $w(A) \leq$ $w(|A|)=\rho\left(|A|+|A|^{T}\right) / 2$ (see [9, p.44]).
Remark 11 (A geometric interpretation of the $2 \times 2$ case). If $A \in \mathbb{C}^{2 \times 2}$ has eigenvalues $\lambda_{1}$ and $\lambda_{2}$, then $F(A)$ is an (possibly degenerate) elliptical disk with foci $\lambda_{1}$ and $\lambda_{2}$. Since any elliptical disk can be covered by two circular disks centered at $\lambda_{1}$ and $\lambda_{2}$ with radius $a$, where $a$ is the length of the semi-major axis of the elliptical disk (see Figure 1), we have $w(A) \leq \rho(A)+a$. Thus, $\tau(A) \geq 2$ means $\rho(A) \leq a$. We have $\left|\lambda_{1}\right|+\left|\lambda_{2}\right| \leq 2 \rho(A) \leq 2 a$. Therefore, $0 \in F(A)$.


Fig. 1. Any elliptical disk can be covered by two circular disks centered at foci with radius $a$, where $a$ is the length of the semimajor axis of the elliptical disk.

By Remark 11, if the numerical range of an $n \times n$ matrix $A$ is an elliptical disk with foci $c_{1}, c_{2} \in \sigma(A)$, the condition in Theorem 10 can be reduced to $\tau(A) \geq 2$.
Corollary 12. Let $A \in \mathbb{C}^{n \times n}$ such that $\tau(A) \geq n$. Then the Hermitian matrix $H\left(e^{\mathrm{i} \theta} A\right)$ is neither positive nor negative definite. Furthermore, if $\tau(A)>n$ then $H\left(e^{\mathrm{i} \theta} A\right)$ is indefinite.

Note that $w(A)=w\left(e^{\mathrm{i} \theta} A\right)$ and $\rho(A)=\rho\left(e^{\mathrm{i} \theta} A\right)$. The proof of Corollary 12 is easy. The following example shows that $H(A)$ may be semi-definite when $\tau(A)=n$.
Example 13 (An upper triangular Toeplitz matrix $Z_{n}$ satisfying $\left.\tau\left(Z_{n}\right)=n\right)$. Let

$$
Z_{n}=\left[\begin{array}{cccc}
1 & 2 & \cdots & 2 \\
& 1 & \ddots & \vdots \\
& & \ddots & 2 \\
& & & 1
\end{array}\right]_{n \times n}
$$

The matrix $Z_{n}$ is given in [2] to show that the bound in Lemma 2.6 of [2] is sharp. We have $w\left(Z_{n}\right)=\rho\left(H\left(Z_{n}\right)\right)=n$, $\rho\left(Z_{n}\right)=1$ and $\tau\left(Z_{n}\right)=n$. The origin is on the boundary of $F\left(Z_{n}\right)$. Obviously, $H\left(Z_{n}\right)$ is the matrix with all the entries being 1 and is positive semidefinite.

Let $\mathbb{P}_{k}$ denote the set of (complex) polynomials of degree at most $k$. The polynomial numerical hull of $A$ of degree $k$,

$$
\mathcal{V}^{k}(A):=\left\{z \in \mathbb{C}:|p(z)| \leq\|p(A)\|_{2}, \forall p \in \mathbb{P}_{k}\right\}
$$

is introduced by Nevanlinna [11, p.41]. We have (see [11], [7])

$$
\begin{equation*}
\mathcal{V}^{1}(A)=F(A) \tag{5}
\end{equation*}
$$

By (5), we have

$$
F(A)=\bigcap_{z \in \mathbb{C}}\left\{\lambda \in \mathbb{C}:|\lambda-z| \leq\|A-z I\|_{2}\right\}
$$

Then $0 \in F(A)$ implies $|z| \leq\|A-z I\|_{2}$ for all $z \in \mathbb{C}$. We have the following corollary.

Corollary 14. Let $A \in \mathbb{C}^{n \times n}$ such that $\tau(A) \geq n$. Then $\min _{z \in \mathbb{C}}\|I-z A\|_{2}=1$.

Proof: Since $\tau(A) \geq n$, we have $|z| \leq\|A-z I\|_{2}$ for all $z \in \mathbb{C}$. Then $\|I-A / z\|_{2} \geq 1$ for all $z \neq 0$. Note that $\|I-z A\|_{2}=1$ when $z=0$. The proof is completed.

## IV. CONCLUDING REMARKS

In this note, we discuss the existing results for the ratios $s(A)$ and $\tau(A)$. We also provide several upper bounds for them. If no further known conditions for $A$ are given, there is no obvious relation between $s(A)$ and $\tau(A)$ except that $s(A)=1$ implies $\tau(A)=1$. If $\tau(A) \gg 1$, the matrix $A$ is extremely ill conditioned and highly non-normal.

We complete this note by discussing the convergence of GMRES [12] for the linear system

$$
\begin{equation*}
A x=b, \quad A \in \mathbb{C}^{n \times n}, \quad b \in \mathbb{C}^{n} \tag{6}
\end{equation*}
$$

Given an initial guess $x_{0}$ for the solution of (6), at the iteration step $m(\geq 1)$, GMRES yields the approximate solution $x_{m}$ in the affine subspace $x_{0}+\mathcal{K}_{m}\left(A, r_{0}\right)$ such that

$$
\left\|r_{m}\right\|_{2}:=\left\|b-A x_{m}\right\|_{2}=\min _{y \in x_{0}+\mathcal{K}_{m}\left(A, r_{0}\right)}\|b-A y\|_{2}
$$

where

$$
\mathcal{K}_{m}\left(A, r_{0}\right):=\operatorname{span}\left\{r_{0}, A r_{0}, \cdots, A^{m-1} r_{0}\right\}
$$

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is the $m$ th Krylov subspace generated by the matrix $A$ and the initial residual vector $r_{0}:=b-A x_{0}$. For a diagonalizable matrix $A=X \Lambda X^{-1}$, one has the estimate (see, e.g., [6, p.54])

$$
\begin{equation*}
\frac{\left\|r_{m}\right\|_{2}}{\left\|r_{0}\right\|_{2}} \leq \kappa(X) \min _{\substack{\phi_{m \in \mathbb{P}}, \phi_{m}(0)=1}} \max _{1 \leq i \leq n}\left|\phi_{m}\left(\lambda_{i}\right)\right| . \tag{7}
\end{equation*}
$$

Obviously, this bound (7) does not always yield satisfactory results when $\tau(A) \gg 1$ due to $\kappa(X) \geq \tau(A)$. For a general matrix, one has the estimate (see [3, Corollary 6.2])

$$
\begin{equation*}
\frac{\left\|r_{m}\right\|_{2}}{\left\|r_{0}\right\|_{2}} \leq\left[1-\nu(A) \nu\left(A^{-1}\right)\right]^{m / 2}, \tag{8}
\end{equation*}
$$

where $\nu(A)=\min \{|z|: z \in F(A)\}$ is the distance from the origin to $F(A)$. Thus, this bound (8) is useless when $\tau(A) \geq$ $n$. The same conclusion also applies to the bound (3.3) in [4] and the bound (2.1) in [1].

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