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The Pell Equation $x^2 - Py^2 = Q$

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***Abstract*—**Let p be a prime number such that $p \equiv 1(mod\ 4)$, say $p = 1 + 4k$ for a positive integer k. Let $P = 2k + 1$ and $Q = k^2$. In this paper, we consider the integer solutions of the Pell equation $x^2 - Py^2 = Q$ over $\mathbf{Z}$ and also over finite fields $\mathbf{F}_p$. Also we deduce some relations on the integer solutions (x_n, y_n) of it.

***Keywords*—**Pell equation, solutions of Pell equation.

I. Preliminary Facts on Diophantine and Pell Equations.

A Diophantine equation is an indeterminate polynomial equation that allows the variables to be integers only. Diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations. In more technical language, they define an algebraic curve, algebraic surface or more general object, and ask about the lattice points on it. The word Diophantine refers to the Hellenistic mathematician of the 3rd century, Diophantus of Alexandria, who made a study of such equations and was one of the first mathematicians to introduce symbolism into algebra. The mathematical study of Diophantine problems Diophantus initiated is now called Diophantine analysis. A linear Diophantine equation is an equation between two sums of monomials of degree zero or one. While individual equations present a kind of puzzle and have been considered throughout history, the formulation of general theories of Diophantine equations was an achievement of the twentieth century. For example, the equation $ax + by = 1$ is known the linear Diophantine equation. In general, the Diophantine equation is the equation given by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{1}$$

Also for $n = 2$, there are infinitely many solutions (x, y, z) of the Diophantine equation $x^n + y^n = z^n$. For larger values of n, Fermat's last theorem (see [4]) states that no positive integer solutions x, y, z satisfying the equation exist. In [18], [19], [21], we considered some specific Diophantine equations and their integer solutions.

Let $D \neq 1$ be a positive non-square integer and N be any fixed positive integer. Then the Diophantine equation

$$x^2 - Dy^2 = \pm N \tag{2}$$

is known as Pell equation and is named after John Pell (1611-1685), a mathematician who searched for integer solutions to equations of this type in the seventeenth century. Ironically, Pell was not the first to work on this problem, nor did he contribute to our knowledge for solving it. Euler (1707-1783), who brought us the ψ-function, accidentally named the equation after Pell, and the name stuck. For $N = 1$, the Pell equation

$$x^2 - Dy^2 = \pm 1 \tag{3}$$

is known as the classical Pell equation and was first studied by Brahmagupta (598-670) and Bhaskara (1114-1185), (see [1]). Its complete theory was worked out by Lagrange (1736-1813), not Pell. It is often said that Euler (1707-1783) mistakenly attributed Brouncker's (1620-1684) work on this equation to Pell. However the equation appears in a book by Rahn (1622-1676) which was certainly written with Pell's help: some say entirely written by Pell. Perhaps Euler knew what he was doing in naming the equation. Baltus [2], Kaplan and Williams [5], Lenstra [6], Matthews [7], Mollin (also Poorten and Williams) [8], Stevenhagen [10] considered some specific Pell (and Diophantine) equations and their integer solutions (Further details on Pell equations can be found in [2], [3], [9]).

The Pell equation in (3) has infinitely many integer solutions (x_n, y_n) for $n \geq 1$. The first non-trivial positive integer solution (x_1, y_1) (in this case x_1 or $x_1 + y_1\sqrt{D}$ is minimum) of this equation is called the fundamental solution, because all other solutions can be (easily) derived from it. In fact, if (x_1, y_1) is the fundamental solution of $x^2 - Dy^2 = 1$, then the n-th positive solution of it, say (x_n, y_n), is defined by the equality

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n \tag{4}$$

for integer $n \geq 2$. (Furthermore, all nontrivial solutions can be obtained considering the four cases $(\pm x_n, \pm y_n)$ for $n \geq 1$). There are several methods for finding the fundamental solution of Pell's equation $x^2 - Dy^2 = 1$ for a positive non-square integer D, e.g., the cyclic method known in India (12-th century), or the slightly less efficient but more regular English method (17-th century) which produce all solutions of $x^2 - Dy^2 = 1$. But the most efficient method for finding the fundamental solution is based on the simple finite continued fraction expansion of $\sqrt{D}$. We can describe it as follows: Let

$$[a_0; \overline{a_1, a_2, \cdots, a_r, 2a_0}]$$

be the simple continued fraction of $\sqrt{D}$, where $a_0 = \lfloor \sqrt{D} \rfloor$. Let $p_0 = a_0$, $p_1 = 1 + a_0a_1$, $q_0 = 1$ and $q_1 = a_1$. In general

$$\begin{aligned} p_n &= a_np_{n-1} + p_{n-2} \\ q_n &= a_nq_{n-1} + q_{n-2} \end{aligned} \tag{5}$$

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for $n \geq 2$. Then the fundamental solution of $x^2 - Dy^2 = 1$ is

$$(x_1, y_1) = \begin{cases} (p_r, q_r) & if\ r\ is\ odd \\ (p_{2r+1}, q_{2r+1}) & if\ r\ is\ even. \end{cases}$$

On the other hand, in connection with (2) and (3), it is well known that if (X_1, Y_1) and (x_{n-1}, y_{n-1}) are integer solutions of $x^2 - Dy^2 = \pm N$ and $x^2 - dy^2 = 1$, respectively, then (X_n, Y_n) is also a positive solution of $x^2 - Dy^2 = \pm N$, where

$$X_n + DY_n = (x_{n-1} + Dy_{n-1})(X_1 + DY_1) \tag{6}$$

for $n \geq 2$.

II. The Pell Equation $x^2 - Py^2 = Q$.

In [11], [12], [13], [14], [15], [16] and [17], we considered some specific Pell equations and their integer solutions. Also we deduced some recurrence relations on the integer solutions (x_n, y_n) of these Pell equations.

In the present paper, we consider the very specific Pell equation and its integer solutions. P and Q be two non-zero integers and let $D = P^2 - 4Q$ be the discriminant such that $D \neq 0$. In [22], we defined a new sequence $A = A_n(P, Q)$ with parameters P and Q defined by $A_0 = 0$, $A_1 = 1$ and $A_n = PA_{n-1} - QA_{n-2}$ for $n \geq 2$ and derived some algebraic identities on it. Also we showed that every prime number $p \equiv 1(mod\ 4)$ can be written of the form $P^2 - 4Q$. Indeed, let $p = 1 + 4k$. Then the quadratic equation $P^2 - 4D = p$ has a solution for

$$P = 2k + 1 \quad and \quad Q = k^2. \tag{7}$$

In this work, we will consider the Pell equation

$$x^2 - Py^2 = Q \tag{8}$$

over $\mathbf{Z}$ and also over finite fields $\mathbf{F}_p$, where P and Q is defined in (7). Note that for $p = 5$, we have $k = 1$ and hence $P = 3$ and $Q = 1$. So (8) becomes $x^2 - 3y^2 = 1$ which is the classical Pell equation. For the other values of $p > 5$, the Pell equation $x^2 - Py^2 = Q$ is not a classical Pell equation. So we have to consider these two conditions.

Theorem 2.1: Let $p = 5$, then for the classical Pell equation $x^2 - 3y^2 = 1$, we have

1) The continued fraction expansion of $\sqrt{3}$ is $[1; \overline{1, 2}]$.
2) The fundamental solution is $(x_1, y_1) = (2, 1)$.
3) Define the sequence $\{(x_n, y_n)\}$, where

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

for $n \geq 1$. Then (x_n, x_n) is a solution of $x^2 - 3y^2 = 1$.
4) The solutions (x_n, y_n) satisfy $x_n = 2x_{n-1} + 3y_{n-1}$ and $y_n = x_{n-1} + 2y_{n-1}$ for $n \geq 2$.
5) The solutions (x_n, y_n) satisfy the recurrence relations

$$\begin{aligned} x_n &= 3(x_{n-1} + x_{n-2}) - x_{n-3} \\ y_n &= 3(y_{n-1} + y_{n-2}) - y_{n-3} \end{aligned}$$

for $n \geq 4$.

6) The $n-$th solution (x_n, y_n) can be given by

$$\frac{x_n}{y_n} = \left[1; (\overline{1, 2})_{n-2}, 1, 3\right] \tag{10}$$

for $n \geq 2$, where $(\overline{1, 2})_{n-2}$ means that there are $n - 2$ successive terms "$1, 2$".

Proof: 1) It is easily seen that $\sqrt{3} = [1; \overline{1, 2}]$.

2) The fundamental (minimal) solution of $x^2 - 3y^2 = 1$ is $(x_1, y_1) = (2, 1)$ since $2^2 - 3.1^1 = 1$.

3) We prove the theorem by induction on n. Let $n = 1$. Then

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

which is the fundamental solution. Let us assume that this equation is satisfied for $n - 1$, that is, $x_{n-1}^2 - 3y_{n-1}^2 = 1$. We will show that it is also satisfied for n. Clearly we deduce that

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 2x_{n-1} + 3y_{n-1} \\ x_{n-1} + 2y_{n-1} \end{pmatrix}. \end{aligned} \tag{11}$$

Hence

$$\begin{aligned} x_n^2 - 3y_n^2 &= (2x_{n-1} + 3y_{n-1})^2 - 3(x_{n-1} + 2y_{n-1})^2 \\ &= 4x_{n-1}^2 + 12x_{n-1}y_{n-1} + 9y_{n-1}^2 \\ &\quad -3x_{n-1}^2 - 12x_{n-1}y_{n-1} - 12y_{n-1}^2 \\ &= x_{n-1}^2 - 3y_{n-1}^2 \\ &= 1. \end{aligned}$$

So (x_n, y_n) is also a solution of $x_n^2 - 3y_n^2 = 1$.

4) It is clear from (11) that $x_n = 2x_{n-1} + 3y_{n-1}$ and $y_n = x_{n-1} + 2y_{n-1}$ for $n \geq 2$.

5) We only prove by induction that $x_n = 3(x_{n-1} + x_{n-2}) - x_{n-3}$. Let $n = 4$. Then from (9) we get $x_1 = 2, x_2 = 7, x_3 = 26$ and $x_4 = 97$. So

$$x_4 = 3(x_3 + x_2) - x_1 = 3(26 + 7) - 2 = 97,$$

that is, $x_n = 3(x_{n-1} + x_{n-2}) - x_{n-3}$ is true for $n = 4$. Let us assume that this relation is satisfied for $n - 1$, that is,

$$x_{n-1} = 3(x_{n-2} + x_{n-3}) - x_{n-4}. \tag{12}$$

Then from (11) and (12), we obtain $x_n = 3(x_{n-1} + x_{n-2}) - x_{n-3}$ for $n \geq 4$.

6) Similarly it can be shown by induction on n that the $n-$th solution (x_n, y_n) can be given by

$$\frac{x_n}{y_n} = \left[1; (\overline{1, 2})_{n-2}, 1, 3\right] \tag{13}$$

for $n \geq 2$. ■

Example 2.1: Some integer solutions of $x^2 - 3y^2 = 1$ are

$$\begin{aligned}
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \\
\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 26 \\ 15 \end{pmatrix} \\
\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^4\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 97 \\ 56 \end{pmatrix} \\
\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^5\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 362 \\ 209 \end{pmatrix} \\
\begin{pmatrix} x_6 \\ y_6 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^6\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1351 \\ 780 \end{pmatrix}.
\end{aligned}$$

Also

$$\begin{aligned}
\frac{7}{4} &= [1;1,3] \\
\frac{26}{15} &= [1;1,2,1,3] \\
\frac{97}{56} &= [1;1,2,1,2,1,3] \\
\frac{362}{209} &= [1;1,2,1,2,1,2,1,3] \\
\frac{1351}{780} &= [1;1,2,1,2,1,2,1,2,1,3].
\end{aligned}$$

Now we consider the the Pell equation $x^2 - Py^2 = Q$ for $p > 5$.

Theorem 2.2: Let $p > 5$, then for the Pell equation $x^2 - Py^2 = Q$, we get

1) The fundamental solution is $(x_1, y_1) = (k+1, 1)$.
2) The continued fraction expansion of $\sqrt{P}$ is

$$\sqrt{P} = \begin{cases} [t;\overline{2t}] & if\ p = 2t^2+1 \\ [t;\overline{t,2t}] & if\ p = 2t^2+3 \\ [t-1;\overline{1,t-2,1,2t-2}] & if\ p = 2t^2-5. \end{cases}$$

3) Define the sequence $\{(x_n, y_n)\}$, where

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} k+1 & 2k+1 \\ 1 & k+1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (14)$$

for $n \geq 1$. Then $x_n^2 - Py_n^2 = Q^n$ for $n \geq 1$.
4) The solutions (x_n, y_n) satisfy

$$\begin{aligned} x_n &= (k+1)x_{n-1} + (2k+1)y_{n-1} \\ y_n &= x_{n-1} + (k+1)y_{n-1} \end{aligned}$$

for $n \geq 2$.

Proof: Recall that $P = 2k+1$ and $Q = k^2$. So we have $x^2 - (2k+1)y^2 = k^2$.

1) The fundamental solution is $(x_1, y_1) = (k+1, 1)$ since $(k+1)^2 - (2k+1).1^2 = k^2 + 2k + 1 - 2k - 1 = k^2$.

2) Let $p = 2t^2 + 1$ for some positive integer t. Then we get

$$4k + 1 = 2t^2 + 1 \Leftrightarrow k = \frac{t^2}{2}$$

and hence $P = 2k + 1 = t^2 + 1$. It is easily seen that

$$\begin{aligned}
\sqrt{t^2+1} &= t + (\sqrt{t^2+1} - t) = t + \frac{1}{\frac{1}{(\sqrt{t^2+1}-t)}} \\
&= t + \frac{1}{\frac{\sqrt{t^2+1}+t}{1}} = t + \frac{1}{2t + (\sqrt{t^2+1} - t)}.
\end{aligned}$$

So $\sqrt{P} = [t;\overline{2t}]$. Similarly it can be shown that if $p = 2t^2+3$, then $\sqrt{P} = [t;\overline{t,2t}]$ and if $p = 2t^2 - 5$, then $\sqrt{P} = [t-1;\overline{1,t-2,1,2t-2}]$.

3) We prove it by induction on n. Let $n = 1$. Then

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} k+1 & 2k+1 \\ 1 & k+1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k+1 \\ 1 \end{pmatrix}$$

which is the fundamental solution. Let us assume that the equation $x^2 - Py^2 = Q^n$ is satisfied for $n-1$, that is, $x_{n-1}^2 - Py_{n-1}^2 = Q^{n-1}$. For n, we obtain

$$\begin{aligned}
\begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} k+1 & 2k+1 \\ 1 & k+1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} k+1 & 2k+1 \\ 1 & k+1 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} (k+1)x_{n-1} + (2k+1)y_{n-1} \\ x_{n-1} + (k+1)y_{n-1} \end{pmatrix}. \qquad (15)
\end{aligned}$$

So

$$\begin{aligned}
x_n^2 - Py_n^2 &= [(k+1)x_{n-1} + (2k+1)y_{n-1}]^2 \\
&\quad -(2k+1)[x_{n-1} + (k+1)y_{n-1}]^2 \\
&= (k+1)^2x_{n-1}^2 + 2(k+1)(2k+1)x_{n-1}y_{n-1} \\
&\quad +(2k+1)^2y_{n-1}^2 \\
&\quad -(2k+1)x_{n-1}^2 - 2(2k+1)(k+1)x_{n-1}y_{n-1} \\
&\quad -(2k+1)(k+1)^2y_{n-1}^2 \\
&= x_{n-1}^2[(k+1)^2 - (2k+1)] \\
&\quad +y_{n-1}^2[(2k+1)^2 - (2k+1)(k+1)^2] \\
&= k^2[x_{n-1}^2 - (2k+1)y_{n-1}^2] \\
&= k^2(Q^{n-1}) \\
&= k^2(k^2)^{n-1} \\
&= k^{2n} \\
&= Q^n.
\end{aligned}$$

Therefore $x_n^2 - Py_n^2 = Q^n$ as we claimed. ■

Example 2.2: 1) Let $t = 6$. Then $p = 2t^2 + 1 = 73$ is a prime and $k = 18$. So we have the Pell equation $x^2 - 37y^2 = 324$. Note that $\sqrt{37} = [6, \overline{12}]$. The fundamental solution is $(x_1, y_1) = (19, 1)$ and for

$$\begin{aligned}
\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 19 & 37 \\ 1 & 19 \end{pmatrix}^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 398 \\ 38 \end{pmatrix} \\
\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 19 & 37 \\ 1 & 19 \end{pmatrix}^3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8968 \\ 1120 \end{pmatrix} \\
\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} &= \begin{pmatrix} 19 & 37 \\ 1 & 19 \end{pmatrix}^4\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 211832 \\ 30243 \end{pmatrix} \\
\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} &= \begin{pmatrix} 19 & 37 \\ 1 & 19 \end{pmatrix}^5\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5143984 \\ 786544 \end{pmatrix}
\end{aligned}$$

we have $x_n^2 - 37y_n^2 = 324^n$, and etc.

2) Let $t = 23$. Then $p = 2t^2 + 3 = 1061$ is a prime and $k = 265$. So we have the Pell equation $x^2 - 531y^2 = 70225$. Note that $\sqrt{531} = [23; \overline{23, 46}]$. The fundamental solution is $(x_1, y_1) = (265, 1)$ and for

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 71287 \\ 532 \end{pmatrix}$$
$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 19244834 \\ 212799 \end{pmatrix}$$
$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 5232122113 \\ 75849368 \end{pmatrix}$$
$$\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = \begin{pmatrix} 1432020496466 \\ 25408054001 \end{pmatrix}$$

we have $x_n^2 - 531y_n^2 = 70225^n$, and etc.

3) Let $t = 9$. Then $p = 2t^2 - 5 = 157$ is a prime and $k = 39$. So we have the Pell equation $x^2 - 79y^2 = 1521$. Note that $\sqrt{79} = [8, \overline{1, 7, 1, 16}]$. The fundamental solution is $(x_1, y_1) = (40, 1)$ and for

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 40 & 79 \\ 1 & 40 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1679 \\ 80 \end{pmatrix}$$
$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 40 & 79 \\ 1 & 40 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 73480 \\ 4879 \end{pmatrix}$$
$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 40 & 79 \\ 1 & 40 \end{pmatrix}^4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3324641 \\ 268640 \end{pmatrix}$$
$$\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = \begin{pmatrix} 40 & 79 \\ 1 & 40 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 154208200 \\ 14070241 \end{pmatrix}$$

we have $x_n^2 - 79y_n^2 = 1521^n$, and etc.

III. The Pell Equation $x^2 - Py^2 = Q$ over $\mathbf{F}_p$

In [20], we considered the Pell Equations $x^2 - ky^2 = N$ and $x^2 + xy - ky^2 = N$ over finite fields $\mathbf{F}_p$. In this section, we will consider the integer solutions of $x^2 - Py^2 = Q$ over finite fields $\mathbf{F}_p$. Let

$$D_p = \{(x, y) \in \mathbf{F}_p \times \mathbf{F}_p : x^2 - Py^2 \equiv Q(mod\ p)\}.$$

Then we can give the following theorem.

Theorem 3.1: For the Pell equation $x^2 - Py^2 = Q$, we have

$$\#D_p = \begin{cases} p+1 & if\ p \equiv 5(mod\ 8) \\ p-1 & if\ p \equiv 1(mod\ 8). \end{cases}$$

Proof: Let $p \equiv 5(mod\ 8)$. If $y = 0$, the the quadratic equation $x^2 \equiv k^2(mod\ p)$ has two solutions $x = k$ and $x = p - k$. If $x = 0$, then the quadratic equation $-(2k+1)y^2 \equiv k^2(mod\ p)$ has no solution y. Now let $S_p = \mathbf{F}_p - \{k, p-k\}$. Then there are $\frac{p-1}{2}$ elements x in S_p such that $\frac{x^2-Q}{P}$ is a square. Let $\frac{x^2-Q}{P} = u^2$ for some $u \neq 0$. Then we get $y^2 \equiv u^2(mod\ p)$ and hence $y = u$ and $y = -u$, that is, there are two integer solutions (x, u) and $(x, p-u)$. So there are $2(\frac{p-1}{2}) = p - 1$ solutions. We see as above that this equation has also two solutions $(k, 0)$ and $(p-k, 0)$. So $x^2 - Py^2 = Q$ has $p - 1 + 2 = p + 1$ integer solutions.

Similarly it can be shown that if $p \equiv 1(mod\ 8)$, then $x^2 - Py^2 = Q$ has $p - 1$ integer solutions. ■

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