

# The lower and upper approximations in a group

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**Abstract**—In this paper, we generalize some propositions in [C.Z. Wang, D.G. Chen, A short note on some properties of rough groups, Comput. Math. Appl. 59(2010)431-436.] and we give some equivalent conditions for rough subgroups. The notion of minimal upper rough subgroups is introduced and a equivalent characterization is given, which implies the rough version of Lagrange's Theorem.

**Keywords**—Lower approximations; Upper approximations; Rough sets; Rough groups; Lagrange's Theorem.

## I. INTRODUCTION

THE theory of rough sets was proposed by Pawlak [1] in 1982. The theory of rough sets is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. It is an expanding research area which stimulates explorations on both real-world applications and on the theory itself. According to Yao [2], there are two main generalized methods for the Pawlak rough set model: the constructive and the algebraic methods. The algebraic approach to rough sets was studied by some authors. Biswas and Nanda [3] applied the notion of rough sets to algebra and introduced the notion of rough subgroups. Kuroki [4] introduced the notion of a rough ideal in a semigroup. Kuroki and Wang [5] gave some properties of the lower and the upper approximations with respect to the normal subgroups. In addition, some properties of the lower and the upper approximations in a ring were studied in [6][7]. Rough modules have been investigated in [8]. In this paper, we generalize some propositions in [9] and we give some new propositions of the lower and the upper approximations in a group. Some equivalent conditions for rough subgroups and rough normal subgroups are given. The notion of minimal upper rough subgroups is introduced and a equivalent characterization is given, which implies the rough version of Lagrange's Theorem.

## II. PRELIMINARIES

We give some basic definitions and results which will be used later on. Throughout this paper, we shall denote a group by  $G$ , the identity of  $G$  by  $e$ .

**Definition 2.1**([9]) Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . Let

$$N_-(A) = \{x \in G : xN \subseteq A\}$$

$$\text{and } N^-(A) = \{x \in G : xN \cap A \neq \emptyset\}.$$

Then  $N_-(A)$  and  $N^-(A)$  are called lower and upper approximations of  $A$  with respect to the normal subgroup  $N$ , respectively.

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**Definition 2.2**([9]) Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . Then  $A$  is called a upper rough subgroup (respectively, normal subgroup) of  $G$  if  $N^-(A)$  is a subgroup (respectively, normal subgroup) of  $G$ . Similarly,  $A$  is called a lower rough subgroup (respectively, normal subgroup) of  $G$  if  $N_-(A)$  is a subgroup (respectively, normal subgroup) of  $G$ .

**Theorem 2.1**([9]) Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . Then  $N^-(A) = AN$ .

**Theorem 2.2**([10]) Let  $N$  and  $H$  be normal subgroups of a group  $G$ . Let  $A$  be a nonempty subset of  $G$ . Then the following assertions hold:

$$(1) N \subseteq H \text{ implies } N^-(A) \subseteq H^-(A).$$

$$(2) N \subseteq H \text{ implies } N_-(A) \supseteq H_-(A).$$

**Theorem 2.3**([11]) Let  $H, K, L$  be subgroups of a group  $G$  and assume that  $K \subseteq L$ . Then  $(HK) \cap L = (H \cap L)K$ .

**Theorem 2.4**([11]) Let  $H$  be a subset of a group  $G$ . Then  $H$  is a subgroup of  $G$  if and only if  $H$  is not empty and  $xy^{-1} \in H$  whenever  $x \in H$  and  $y \in H$ .

**Theorem 2.5**([11]) If  $H$  and  $K$  are subgroups of a group  $G$ , then  $HK$  is a subgroup if and only if  $HK = KH$ .

## III. SOME PROPERTIES OF THE LOWER AND THE UPPER APPROXIMATIONS IN A GROUP

In this section, we study and generalize the properties of the lower and the upper approximations in a group. These properties are important and useful for rough set model.

**Proposition 3.1.** Let  $N$  be a normal subgroup of  $G$  and  $A$  a nonempty subset of  $G$ . If  $e \in A$ , then  $N \subseteq N^-(A)$ .

**Proof:** By Theorem 2.1., we have  $N^-(A) = AN$ . Since  $e \in A$ , it follows that  $N \subseteq AN = N^-(A)$ . ■

Now, we give the generalized version of the proposition 3.2. in [1] as follows.

**Proposition 3.2.** Let  $H, N$  be normal subgroups of a group  $G$  and  $A$  a nonempty subset of  $G$ . Then  $H^-(A)N^-(A) = (HN)^-(A)(HN)^-(A)$ .

**Proof:** By Theorem 2.1., we have that  $H^-(A) = AH$ ,  $N^-(A) = AN$  and  $(HN)^-(A) = AHN$ . Hence,  $H^-(A)N^-(A) = AHAN$ . Since  $H$  and  $N$  are normal subgroups of  $G$ , it follows that  $H = HH$ ,  $N = NN$ ,  $AH = HA$ ,  $AN = NA$ ,  $HN = NH$ . Thus,  $H^-(A)N^-(A) = AHAN = AHHANN = (AHN)(AHN) = (HN)^-(A)(HN)^-(A)$ . Therefore,  $H^-(A)N^-(A) = (HN)^-(A)(HN)^-(A)$ . ■

The product of two upper approximations is investigated by the following conclusion.

**Proposition 3.3.** Let  $H, N$  be normal subgroups of a group  $G$  and  $A$  a nonempty subset of  $G$ . If  $e \in A$ , then  $(HN)^-(A) \subseteq H^-(A)N^-(A)$ .

**Proof:** By Theorem 2.1., we have that  $H^-(A) = AH$ ,  $N^-(A) = AN$  and  $(HN)^-(A) = AHN$ . Hence,

$H^-(A)N^-(A) = AHAN$ . Since  $H$  is a normal subgroup of  $G$ , it follows that  $AH = HA$ . Thus,  $H^-(A)N^-(A) = AHAN = AAHN$ . Applying the condition  $e \in A$ , we conclude that  $(HN)^-(A) = AHN \subseteq A(AHN) = H^-(A)N^-(A)$ . Hence,  $(HN)^-(A) \subseteq H^-(A)N^-(A)$ . ■

Now, we give another generalized version of the proposition 3.2. in [1] as follows.

**Proposition 3.4.** *Let  $H$  and  $N$  be normal subgroups of a group  $G$ . If  $A$  is a upper rough subgroup of  $G$  with respect to  $HN$ , then  $(HN)^-(A) = H^-(A)N^-(A)$ .*

*Proof:* Since  $A$  is a upper rough subgroup of  $G$  with respect to  $HN$ , it follows from Definition 2.2. that  $(HN)^-(A)$  is a subgroup of  $G$  and so  $(HN)^-(A)(HN)^-(A) = (HN)^-(A)$ . Hence, by Proposition 3.2., we have  $H^-(A)N^-(A) = (HN)^-(A)(HN)^-(A) = (HN)^-(A)$ . This completes the proof of proposition. ■

Now, we give the generalized version of the proposition 3.4. in [1] as follows.

**Proposition 3.5.** *Let  $H, N$  be normal subgroups of  $G$  and  $A$  a nonempty subset of  $G$ . Then  $(HN)^-(A) = H^-(A)N^-(A)H$ .*

*Proof:* By Theorem 2.1., we have that  $H^-(A) = AH$ ,  $N^-(A) = AN$  and  $(HN)^-(A) = AHN$ . Thus,  $H^-(A)N^-(A) = AHN = (HN)^-(A)$ , that is,  $H^-(A)N^-(A) = (HN)^-(A)$ . Since  $H$  is a normal subgroup of  $G$ , it follows that  $HN = NH$ . Hence,  $N^-(A)H = ANH = AHN = (HN)^-(A)$ , that is,  $N^-(A)H = (HN)^-(A)$ . Consequently,  $(HN)^-(A) = H^-(A)N^-(A)H$ . ■

The following result explores the intersection of the upper approximation and a normal subgroup.

**Proposition 3.6.** *Let  $H, L$  be normal subgroups of  $G$  and  $A$  a subgroup of  $G$ . If  $A \subseteq L$ , then  $H^-(A) \cap L = (H \cap L)^-(A)$ .*

*Proof:* Since  $H$  and  $L$  are normal subgroups of  $G$ , it follows that  $H \cap L$  is also a normal subgroup of  $G$ . Hence by Theorem 2.1., we have that  $(H \cap L)^-(A) = A(H \cap L)$ ,  $H^-(A) = AH$ . Since  $H \trianglelefteq G$  and  $H \cap L \trianglelefteq G$ , it follows that  $A(H \cap L) = (H \cap L)A$ ,  $AH = HA$  and so  $(H \cap L)^-(A) = (H \cap L)A$ ,  $H^-(A) = HA$ . By the condition  $A \subseteq L$  and Theorem 2.3., we have  $(HA) \cap L = (H \cap L)A$  and so  $H^-(A) \cap L = (HA) \cap L = (H \cap L)A = (H \cap L)^-(A)$ . This completes the proof of proposition. ■

The lower approximations can be represented as follows:

**Proposition 3.7.** *Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a lower rough subgroup of  $G$ . Then  $N_-(A) = \cup\{H \leq G \mid N \subseteq H \subseteq A\}$ .*

*Proof:* Since  $A$  is a lower rough subgroup of  $G$ , it follows from Definition 2.2 that  $N_-(A)$  is a subgroup of  $G$ . Hence,  $N_-(A) \neq \emptyset$  and by Definition 2.2, we have  $N \subseteq N_-(A) \subseteq A$ . Thus,  $N_-(A) \in \{H \leq G \mid N \subseteq H \subseteq A\}$  and so  $N_-(A) \subseteq \cup\{H \leq G \mid N \subseteq H \subseteq A\}$ . On the other hand, for all  $H \in \{H \leq G \mid N \subseteq H \subseteq A\}$ , we shall prove that  $H \subseteq N_-(A)$ . For all  $h \in H$ , since  $H \leq G$  and  $N \subseteq H \subseteq A$ , it follows that  $hN \subseteq H \subseteq A$ , that is,  $hN \subseteq A$ . Therefore, by Definition 2.1., we have  $h \in N_-(A)$ . Hence,  $H \subseteq N_-(A)$ . We have proved that  $\forall H \in \{H \leq G \mid N \subseteq H \subseteq A\}$ ,  $H \subseteq N_-(A)$ . Thus,  $\cup\{H \leq G \mid N \subseteq H \subseteq A\} \subseteq N_-(A)$ . Consequently,  $N_-(A) = \cup\{H \leq G \mid N \subseteq H \subseteq A\}$ . ■

Now, we give the generalized version of the proposition 3.6. in [1] as follows.

**Proposition 3.8.** *Let  $H, N$  be normal subgroups of  $G$ . If  $A$  is a lower rough subgroup of  $G$  with respect to  $H$  and  $A$  is also a lower rough subgroup of  $G$  with respect to  $N$ . Then  $H_-(A)N_-(A) = (HN)_-(A)$  if and only if  $(HN)_-(A) = H_-(A) = N_-(A)$ .*

*Proof:* We first prove the necessity. Since  $A$  is a lower rough subgroup of  $G$  with respect to  $H$  and  $A$  is also a lower rough subgroup of  $G$  with respect to  $N$ , it follows by Definition 2.2. that  $H_-(A)$  and  $N_-(A)$  are subgroups of  $G$ . By the condition  $H_-(A)N_-(A) = (HN)_-(A)$ , we conclude that  $H_-(A) \subseteq H_-(A)N_-(A) = (HN)_-(A)$ . That is,  $H_-(A) \subseteq (HN)_-(A)$ . On the other hand, clearly,  $H \leq HN$ , hence by Theorem 2.2, we have  $H_-(A) \supseteq (HN)_-(A)$ . Therefore,  $H_-(A) = (HN)_-(A)$ . Similarly,  $N_-(A) = (HN)_-(A)$ . Hence,  $(HN)_-(A) = H_-(A) = N_-(A)$ .

Conversely, since  $A$  is a lower rough subgroup of  $G$  with respect to  $H$  and  $A$  is also a lower rough subgroup of  $G$  with respect to  $N$ , it follows by Definition 2.2. that  $H_-(A)$  and  $N_-(A)$  are subgroups of  $G$ . Therefore, by the condition  $H_-(A) = N_-(A)$ , we conclude that  $H_-(A)N_-(A) = H_-(A)$ . Applying the condition  $(HN)_-(A) = H_-(A)$ , we have that  $H_-(A)N_-(A) = (HN)_-(A)$ . ■

#### IV. ROUGH GROUPS AND MINIMAL UPPER ROUGH SUBGROUPS

By means of the properties given by Section III, we present some equivalent characterizations of rough subgroups in this section. Furthermore, the notion of minimal upper rough subgroups is introduced, and its properties are explored. The studies have done much to deepen our understanding of rough set models.

We first give the equivalent characterizations of rough subgroups.

**Theorem 4.1.** *Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . Then the following assertions are equivalent:*

- (1)  $A$  is a upper rough subgroup of  $G$ .
- (2)  $AN$  is a subgroup of  $G$ .
- (3) For all  $x, y \in A$ ,  $xy^{-1} \in AN$ .

*Proof:* (1)  $\Leftrightarrow$  (2) The proof follows from Theorem 2.1. and Definition 2.2..

(2)  $\Rightarrow$  (3) The proof is obvious.

(3)  $\Rightarrow$  (2) For all  $a, b \in AN$ . Then there exist  $x_1, x_2 \in A$ ,  $n_1, n_2 \in N$  such that  $a = x_1n_1$ ,  $b = x_2n_2$ . Hence,  $ab^{-1} = x_1n_1(x_2n_2)^{-1} = x_1x_2^{-1}(x_2n_1n_2^{-1}x_2^{-1})$ . By assertion (3), we have  $x_1x_2^{-1} \in AN$ . Hence there exist  $x \in A$ ,  $n \in N$  such that  $x_1x_2^{-1} = xn$ . Thus,  $ab^{-1} = x_1x_2^{-1}(x_2n_1n_2^{-1}x_2^{-1}) = xn(x_2n_1n_2^{-1}x_2^{-1})$ . Since  $N$  is a normal subgroup of  $G$ , it follows that  $x_2n_1n_2^{-1}x_2^{-1} \in N$  and so  $n(x_2n_1n_2^{-1}x_2^{-1}) \in N$ . Thus,  $ab^{-1} = xn(x_2n_1n_2^{-1}x_2^{-1}) \in AN$ . Therefore, by Theorem 2.4., we conclude that  $AN$  is a subgroup of  $G$ . ■

In order to describe the rough subgroups, we present the following definition.

**Definition 4.1.** *Let  $N$  be a normal subgroup of a group  $G$  and  $A, B$  nonempty subsets of  $G$ . Then  $A, B$  are upper*

rough commutative with respect to  $N$  if  $N^-(A)N^-(B) = N^-(B)N^-(A)$ . Similarly,  $A, B$  are lower rough commutative with respect to  $N$  if  $N_-(A)N_-(B) = N_-(B)N_-(A)$ .

We can give the following result by means of the above definition.

**Theorem 4.2.** Let  $N$  be a normal subgroup of a group  $G$  and  $A, B$  nonempty subsets of  $G$ . Then the following assertions hold:

(1) If  $A$  and  $B$  are upper rough subgroups, then  $AB$  is a upper rough subgroup of  $G$  if and only if  $A$  and  $B$  are upper rough commutative.

(2) If  $A$  and  $B$  are upper rough subgroups, then  $AB$  is a lower rough subgroup of  $G$  if and only if  $A, B$  are lower rough commutative.

*Proof:* The proof follows from Definition 2.2. and Theorem 2.5.. ■

By above result, it is easy to give the following corollaries.

**Corollary 4.1** Let  $N$  be a normal subgroup of a group  $G$ ,  $A$  a upper rough subgroup of  $G$  and  $B$  a upper normal subgroup of  $G$ . Then  $AB$  is a upper rough subgroup of  $G$ .

*Proof:* Since  $A$  is a upper rough subgroup of  $G$  and  $B$  is a upper normal subgroup of  $G$ , it follows from Definition 2.2. that  $N^-(A)$  is a subgroup of  $G$  and  $N^-(B)$  is a normal subgroup of  $G$ . Hence  $N^-(A)N^-(B) = N^-(B)N^-(A)$  and so  $A$  and  $B$  are upper rough commutative. Therefore by Theorem 4.2, we have that  $AB$  is a upper rough subgroup of  $G$ . ■

**Corollary 4.2.** Let  $N$  be a normal subgroup of  $G$ ,  $A$  a lower rough subgroup of  $G$  and  $B$  a lower normal subgroup of  $G$ . Then  $AB$  is a lower rough subgroup of  $G$ .

*Proof:* The proof is similar to that of Corollary 4.1.. ■

The following definition will be used.

**Definition 4.2**([12]). Let  $H$  be a subgroup of a group  $G$  and  $T \subseteq G$ . Then  $T$  is a transversal of  $H$  in  $G$  if  $T$  contains exactly one element of every right coset  $Hx$ ,  $x \in G$ .

**Notation 4.1.** Let  $A$  be a nonempty subset of a group  $G$ . Then let  $\langle A \rangle$  denote the subgroup generated by  $A$ .

Now we present the definition of minimal upper rough subgroups.

**Definition 4.3.** Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a upper rough subgroup of  $G$ . Then  $A$  is called a minimal upper rough subgroup of  $G$  if for all  $Y \subsetneq A$ , we have  $N^-(A) \neq N^-(Y)$ .

In the following, we study the equivalent characterizations of the minimal upper rough subgroups. For this purpose, we first give a lemma.

**Lemma 4.1.** Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . Then  $AN$  is a subgroup of  $G$  if and only if there exists a transversal  $T$  of  $N$  in  $\langle A, N \rangle$  such that  $T \subseteq A$ .

*Proof:* We first prove the sufficiency. Let  $T$  be a transversal of  $N$  in  $\langle A, N \rangle$  such that  $T \subseteq A$ . Then  $\langle A, N \rangle = \cup\{tN \mid t \in T\} = TN \subseteq AN$ , that is,  $\langle A, N \rangle \subseteq AN$ . On the other hand, Clearly,  $AN \subseteq \langle A, N \rangle$ . Hence  $AN = \langle A, N \rangle$ , that is,  $AN$  is a subgroup of  $G$ .

Conversely, since  $AN$  is a subgroup of  $G$ , it follows that  $AN = \langle A, N \rangle$ . Let  $T$  be a transversal of  $N$  in  $\langle A, N \rangle$ . For all  $t \in T$ , since  $T \subseteq \langle A, N \rangle = AN$ , it follows that  $t \in AN$ .

Hence there exist  $x \in A$ ,  $n \in N$  such that  $t = xn$  and so  $tN = xnN = xN$ . We have proved that for all  $t \in T$ , there exists  $x \in A$  such that  $tN = xN$ , which implies that there exists  $T' \subseteq A$  such that  $\{tN \mid t \in T\} = \{xN \mid x \in T'\}$ . That is,  $T'$  is a transversal of  $N$  in  $\langle A, N \rangle$  and  $T' \subseteq A$ . This completes the proof of the necessity. ■

**Theorem 4.3.** Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . Then  $A$  is a upper rough subgroup if and only if there exists a transversal  $T$  of  $N$  in  $\langle A, N \rangle$  such that  $T \subseteq A$ .

*Proof:* The proof follows from Lemma 4.1. and Theorem 4.1.. ■

In fact, the following result gives a representation of minimal upper rough subgroups.

**Theorem 4.4.** Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . Then  $A$  is a minimal upper rough subgroup of  $G$  if and only if  $A$  is a transversal of  $N$  in  $\langle A, N \rangle$ .

*Proof:* We first prove the necessity. Since  $A$  is a upper rough subgroup of  $G$ , it follows from Theorem 4.3. that there exists a transversal  $T$  of  $N$  in  $\langle A, N \rangle$  such that  $T \subseteq A$ . Hence  $\langle A, N \rangle = \cup\{tN \mid t \in T\} = TN$ . Applying Theorem 4.1., we conclude that  $AN$  is a subgroup of  $G$ . Thus,  $AN = \langle A, N \rangle$  and so  $AN = TN$ , which implies  $N^-(A) = N^-(T)$  by Theorem 2.1.. Since  $A$  is a minimal upper rough subgroup of  $G$ , it follows by Definition 4.3. that  $T = A$ . Therefore  $A$  is a transversal of  $N$  in  $\langle A, N \rangle$ .

Conversely, since  $A$  is a transversal of  $N$  in  $\langle A, N \rangle$ , it follows that  $\langle A, N \rangle = \{xN \mid x \in A\} = AN$ . Hence  $AN$  is a subgroup of  $G$ . By Theorem 4.1., we have that  $A$  is a upper rough subgroup of  $G$ . We now show that  $A$  is a minimal upper rough subgroup of  $G$ . Otherwise, suppose that there exists  $Y \subsetneq A$  such that  $N^-(Y) = N^-(A)$ , which implies  $YN = AN$  by Theorem 2.1.. Choosing  $x \in A \setminus Y$ , thus  $x \in A \subseteq AN = YN$ , that is,  $x \in YN$ , which means that there exist  $y \in Y$ ,  $n \in N$  such that  $x = yn$ . Hence,  $xN = ynN = yN$ , that is,  $xN = yN$ . Since  $x, y \in A$  and  $A$  is a transversal of  $N$  in  $\langle A, N \rangle$ , it follows that  $x = y$ . Hence,  $x \in Y$ , a contradiction. Therefore  $A$  is a minimal upper rough subgroup of  $G$ . ■

Now, we give the rough version of Lagrange's Theorem as follows.

**Corollary 4.3.** Let  $G$  be a finite group and  $N$  a normal subgroup of  $G$ . If  $A$  is a minimal upper rough subgroup of  $G$ , then  $|A||G|$ .

*Proof:* Since  $A$  is a minimal upper rough subgroup of  $G$ , it follows from Theorem 4.4. that  $A$  is a transversal of  $N$  in  $\langle A, N \rangle$ . Hence  $|A| = |\langle A, N \rangle : N|$  and so  $|A||G|$ . ■

**Corollary 4.4.** Let  $G$  be a finite group and  $N$  a normal subgroup of  $G$ . If  $A$  is a upper rough subgroup of  $G$ , then there exists  $T \subseteq A$  such that  $N^-(T) = N^-(A)$  and  $T$  is a minimal upper rough subgroup of  $G$ .

*Proof:* Since  $A$  is a upper rough subgroup of  $G$ , it follows from Theorem 4.3. that there exists a transversal  $T$  of  $N$  in  $\langle A, N \rangle$  such that  $T \subseteq A$ . Hence,  $TN = \cup\{tN \mid t \in T\} = \langle A, N \rangle$ . By Theorem 4.1., we conclude that  $AN$  is a subgroup of  $G$  and so  $AN = \langle A, N \rangle$ . Thus,  $TN = AN$ . Applying Theorem 2.1., we have  $N^-(T) = N^-(A)$ . In addition, since

$T$  is a transversal of  $N$  in  $\langle A, N \rangle$ , it follows by Theorem 4.4. that  $T$  is a minimal upper rough subgroup of  $G$ . ■

## V. ROUGH NORMAL GROUPS

In this section, we study the equivalent characterizations of rough normal subgroups.

**Theorem 5.1.** *Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . Then the following assertions are equivalent:*

- (1)  $A$  is a upper rough normal subgroup of  $G$ .
- (2)  $AN$  is a subgroup of  $G$  and for all  $g \in G$  and all  $x \in A$ ,  $g x g^{-1} \in AN$ .
- (3)  $AN$  is a subgroup of  $G$  and for all  $g \in G$ ,  $A^g \subseteq AN$ .

*Proof:* (1)  $\Rightarrow$  (2) By Theorem 2.1. and Definition 2.2., we have that  $AN$  is a normal subgroup of  $G$ . Hence (2) holds.

(2)  $\Rightarrow$  (1) Let  $g \in G$  and  $y \in AN$ . Then there exist  $x \in A$ ,  $n \in N$  such that  $y = xn$ . Thus,  $g y g^{-1} = g x n g^{-1} = (g x g^{-1})(g n g^{-1})$ . Since  $N$  is a normal subgroup of  $G$ , it follows that  $g n g^{-1} \in N$ . By assertion (2), we conclude that  $AN$  is a subgroup of  $G$  and  $g x g^{-1} \in AN$ . Hence  $g y g^{-1} = (g x g^{-1})(g n g^{-1}) \in AN$ . Therefore  $AN$  is a normal subgroup of  $G$ . Hence by Theorem 2.1. and Definition 2.2., we have that  $A$  is a upper rough normal subgroup of  $G$ .

(2)  $\Leftrightarrow$  (3) The proof is obvious. ■

**Lemma 5.1.** *Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . If  $AN$  is a normal subgroup of  $G$ , then  $\forall g \in G$ ,  $A^g N = AN$ .*

*Proof:* Let  $g \in G$ . Since  $AN$  is a normal subgroup of  $G$ , it follows that  $(AN)^g = AN$  and  $A^g \subseteq AN$ . On the one hand,  $A^g \subseteq AN$  implies  $A^g N \subseteq AN$ . On the other hand, for all  $y \in AN$ , since  $AN = (AN)^g$ , it follows that there exist  $x \in A$ ,  $n \in N$  such that  $y = (xn)^g = g x n g^{-1} = g x g^{-1} g n g^{-1} = x^g g n g^{-1}$ . Since  $N$  is a normal subgroup of  $G$ , it follows that  $g n g^{-1} \in N$ . Hence,  $y = x^g g n g^{-1} \in A^g N$ . Thus  $AN \subseteq A^g N$ . Therefore,  $A^g N = AN$ . ■

**Theorem 5.2.** *Let  $N$  be a normal subgroup of a group  $G$ ,  $A$  a nonempty subset of  $G$  and  $g \in G$ . Then  $A$  is a upper rough normal subgroup of  $G$  if and only if  $A^g$  is a upper rough normal subgroup of  $G$ .*

*Proof:* We first proof the necessity. Since  $A$  is a upper rough normal subgroup of  $G$ , it follows by Definition 2.2. that  $N^-(A)$  is a normal subgroup of  $G$ . By Theorem 2.1., we have  $N^-(A) = AN$  and hence  $AN$  is a normal subgroup of  $G$ . By Lemma 5.1., we conclude that  $A^g N = AN$  and thus  $A^g N$  is a normal subgroup of  $G$ . By Theorem 2.1., we have  $N^-(A^g) = A^g N$ . Therefore  $N^-(A^g)$  is a normal subgroup of  $G$ . Hence, by Definition 2.2., we conclude that  $A^g$  is a upper rough normal subgroup of  $G$ .

Conversely, since  $A^g$  is a upper rough normal subgroup of  $G$ , it follows by Definition 2.2. that  $N^-(A^g)$  is a normal subgroup of  $G$ . By Theorem 2.1., we have  $N^-(A^g) = A^g N$ . Therefore  $A^g N$  is a normal subgroup of  $G$ . By Lemma 5.1., we have  $(A^g)^{g^{-1}} N = A^g N$ , that is,  $AN = A^g N$ . Hence  $AN$  is a normal subgroup of  $G$ . By Theorem 2.1., we have  $N^-(A) = AN$  and thus  $N^-(A)$  is a normal subgroup of  $G$ . Hence, by Definition 2.2., we conclude that  $A$  is a upper rough normal subgroup of  $G$ . ■

The above conclusions show that we can give many equivalent descriptions for upper rough normal subgroups. However, we only give a necessity of lower rough normal subgroups as follows:

**Theorem 5.3.** *Let  $N$  be a normal subgroup of a group  $G$  and  $A$  a nonempty subset of  $G$ . If  $A$  is a lower rough normal subgroup of  $G$ , then  $N_-(A) = \cap_{g \in G} N_-(A^g)$ .*

*Proof:* Let  $g \in G$ . By  $N_-(A) \subseteq A$ , we have  $(N_-(A))^g \subseteq A^g$ . Since  $A$  is a lower rough normal subgroup of  $G$ , it follows from Definition 2.2. that  $N_-(A)$  is a normal subgroup of  $G$ . Thus  $(N_-(A))^g = N_-(A)$  and so  $N_-(A) \subseteq A^g$ . Therefore,  $N_-(A) = N_-(N_-(A)) \subseteq N_-(A^g)$ , that is,  $N_-(A) \subseteq N_-(A^g)$ . Hence,  $N_-(A) \subseteq \cap_{g \in G} N_-(A^g)$ . On the other hand,  $\cap_{g \in G} N_-(A^g) \subseteq N_-(A^e) = N_-(A)$ . Consequently,  $N_-(A) = \cap_{g \in G} N_-(A^g)$ . ■

## VI. CONCLUSIONS

There are few researches on the lower and upper approximation in a group. In this paper, we studied the properties of the two approximation operations. By means of these properties, we gave some equivalent representations for rough (normal) subgroups and minimal upper rough subgroups. These studies have many other problems to discuss, and deserve the further consideration, such as, the axiomatization problems of the two approximation operations, the reduction problems etc.

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