

# The Giant Component in a Random Subgraph of a Weak Expander

Yilun Shang

**Abstract**—In this paper, we investigate the appearance of the giant component in random subgraphs  $G_n(p)$  of a given large finite graph family  $G_n = (V_n, E_n)$  in which each edge is present independently with probability  $p$ . We show that if the graph  $G_n$  satisfies a weak isoperimetric inequality and has bounded degree, then the probability  $p$  under which  $G_n(p)$  has a giant component of linear order with some constant probability is bounded away from zero and one. In addition, we prove the probability of abnormally large order of the giant component decays exponentially. When a contact graph is modeled as  $G_n$ , our result is of special interest in the study of the spread of infectious diseases or the identification of community in various social networks.

**Keywords**—subgraph, expander, random graph, giant component, percolation.

## I. INTRODUCTION

**I**NFORMATION networks are often observed as subgraphs of some host graphs with orders prohibitively large or with incomplete information [8]. The property of a random subgraph of a given graph is thus very interesting. Let  $G_n = (V_n, E_n)$  be a finite graph with  $|V_n| = n$  vertices (or order  $n$ ) and  $G_n(p)$  be the spanning subgraph of  $G_n$  obtained by retaining each edge of  $G_n$  independently with probability  $p$ . If  $G_n$  is a complete graph, this model is known as the classical Erdős-Rényi random graph  $G(n, p)$  [6], [12], which can be regarded as percolation on (finite) complete graphs. Other examples of percolation on finite graphs are concerned with graphs of some symmetries such as regular graphs [10], [15], [16] and  $d$ -dimensional torus or box, which is closely related to percolation on corresponding infinite lattice graph  $\mathbb{Z}^d$  [4], [13]. Recently, random subgraph problem on general classes of finite graphs has also been investigated, see e.g. [1], [3], [5], [7], where isoperimetric inequalities instead of symmetry assumptions play a key role. In this paper, following the path of Alon et al. [1] and Benjamini et al. [3], we study the probability of emergence of the giant component in finite large graphs which satisfy a weak isoperimetric inequality (referred to as “weak expanders”).

For any two sets of vertices  $A$  and  $B$  in  $G_n$ , the set  $E_n(A, B)$  consists of all edges with one endpoint in  $A$  and the other in  $B$ . The edge-isoperimetric number,  $c(G_n)$ , (also called the Cheeger constant) is given by

$$\min_{0 < |A| \leq \frac{n}{2}} \frac{\partial_E A}{|A|},$$

where  $\partial_E A = E_n(A, V_n \setminus A)$  is the exterior edge-boundary of  $A$ . Let  $b$  and  $d$  be positive constants. A  $(b, d)$ -expander

graph is a graph  $G_n = (V_n, E_n)$  such that the maximal degree in  $G_n$  is not greater than  $d$ , and  $c(G_n) > b$ . In this paper, all asymptotics are as  $n \rightarrow \infty$ . We say that an event holds asymptotically almost surely (a.a.s.) if the probability that it holds tends to 1, following the notations in [12].

In [1], Alon, Benjamini and Stacey derived the precise critical probability for the emergence of a linear order giant component in expander graphs under the assumptions of regularity and high-girth:

**Theorem 1.**([1], Theorem 3.2) *Let  $d \geq 2$  and let  $G_n$  be a sequence of  $d$ -regular  $(b, d)$ -expander graphs with girth  $g_n \rightarrow \infty$ . If  $p > 1/(d-1)$ , then there exists a  $c > 0$  such that, asymptotically almost surely,*

*$G_n(p)$  contains a component of order at least  $cn$ .*

*If  $p < 1/(d-1)$ , then for any  $c > 0$ , asymptotically almost surely,*

*$G_n(p)$  does not contain a component of order at least  $cn$ .*

Recently, Benjamini, Boucheron, Lugosi and Rossignol [3] are able to show that in any  $(b, d)$ -expander graph, every giant component of given proportion emerges in an interval of length  $o(1)$ , removing the regularity and high-girth assumptions in Theorem 1. The result regarding the critical probability is the following.

**Theorem 2.**([3], Proposition 3.1) *Let  $G_n$  be a  $(b, d)$ -expander graph and let  $c \in (0, 1)$ . There exist constants  $q_1 = q_1(d) > 0$  and  $q_2 = q_2(c) \in (q_1, 1)$ , such that for any  $\beta \in (0, 1)$ , for all  $n$  large enough,  $p_{n,\beta}(c) \in (q_1, q_2(c))$ . Here,  $p_{n,\beta}(c)$  is the probability under which the probability of the giant component of  $G_n(p_{n,\beta}(c))$  has order at least  $cn$  is equal to  $\beta$ .*

*Furthermore, for any  $c \in (0, 1)$ , there are constants  $C_1 > 0$  and  $C_2 > 0$ , depending only on  $b$  and  $d$ , such that for any  $p \geq q_2(c)$ , the probability that the giant component has order at least  $cn$  is large than  $1 - C_1 e^{-C_2 n}$ .*

In this paper, we move a further step beyond Theorem 2 by allowing a weaker assumption on the expansion property of  $G_n$ . We introduce a weak isoperimetric inequality, which can be viewed as a weaker version of the open question proposed in [3]. We prove the probability that the giant component has linear order is bounded away from zero and one, as in Theorem 2 above, and the probability of abnormally large order of the giant component has exponential decay. Note that if  $G_n$  is modeled as a contact graph, consisting of edges formed by pairs of people with possible contact, our result is of special

Y. Shang is with the Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China e-mail: shylmath@hotmail.com

interest in the study of the spread of infectious diseases or the identification of community in various social networks.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries and then present our main result. Section 3 is devoted to the proof. It is often that several key lemmas in Section 3 are to be found as pieces of a long proof of a big statement in [1], [3], [4] and so the validity of these technical lemmas under weaker assumption needs to be carefully checked. We include the proofs of them, more or less as they were presented in [1], [3], [4], not only for the convenience of the reader but also to convince the reader that they do hold in our setting.

## II. SOME NOTATIONS AND MAIN RESULT

We introduce some notations that will be used throughout the paper. For any  $\alpha_n \in (0, 1]$ , we define the a weak isoperimetric inequality as

$$\min_{0 < \alpha_n \leq 1} \frac{\partial_E A}{|A|^{\alpha_n}} > b,$$

where  $\partial_E A = E_n(A, V_n \setminus A)$  is the exterior edge-boundary of  $A$ . Clearly, if  $\alpha_n \equiv 1$ , we recover the ordinary isoperimetric inequality. We remark that there is a more typical definition of weak isoperimetric inequality [2] which is in a different manner from ours.

In what follows, denote by  $G_n = (V_n, E_n)$  a  $(b, d, \alpha_n)$ -expander graph such that the maximum degree is not larger than  $d$  and the above weak expansion property holds. Each point configuration  $x \in \{0, 1\}^E$  is identified with the subgraph of  $G_n$  with vertex set  $V_n$  and edge set obtained by removing from  $E_n$  all edges  $e$  such that  $x(e) = 0$ . For  $p \in [0, 1]$ , we equip the space  $\{0, 1\}^E$  with the product probability measure  $\mu_{n,p}$  under which each  $x(e)$  is independently 1 with probability  $p$  and 0 with probability  $1 - p$ . We denote by  $E_{n,p}(f) = \int f(x) d\mu_{n,p}(x)$  the mean of random variable  $f : \{0, 1\}^E \rightarrow \mathbb{R}$ . For  $x \in \{0, 1\}^E$ , let  $C_n^{(1)} = C_n^{(1)}(x)$  be the largest connected component in the configuration  $x$ , and let  $L_n^{(1)} = L_n^{(1)}(x) = |C_n^{(1)}(x)|$ . Denote by  $C(v)$  the connected component containing a vertex  $v \in V_n$ .

Note that, for any  $c \in (0, 1)$ ,  $\mu_{n,p}\{L_n^{(1)} \geq cn\}$  is a strictly increasing polynomial of  $p$ . Therefore, for any  $\beta \in [0, 1]$ , we define  $p_{n,\beta}(c)$  as the unique real number  $p \in [0, 1]$  such that

$$\mu_{n,p}\{L_n^{(1)} \geq cn\} = \beta.$$

We sometimes suppress the subscript  $n$  if no ambiguity will be caused. Our main result reads as follows.

**Theorem 3.** *Let  $c \in (0, 1)$  and  $b \in (0, 1)$ . Suppose that  $1 - \alpha_n = O(1/\ln n)$ . There exist two constants  $q_1 = q_1(d) > 0$  and  $q_2 = q_2(c) \in (q_1, 1)$ , such that for any  $\beta \in (0, 1)$ , for all  $n$  large enough,  $p_{n,\beta}(c) \in (q_1, q_2(c))$ .*

*Moreover, for any  $c \in (0, 1)$ , there are positive constants  $C_1$  and  $C_2$ , depending only on  $b$  and  $d$ , such that for any  $p \geq q_2(c)$ ,*

$$\mu_{n,p}\{L_n^{(1)} \geq cn\} \geq 1 - C_1 e^{-C_2 n}. \quad (1)$$

## III. PROOFS

To prove Theorem 3 we need the following two lemmas, the proofs of which are extended from [1] (Lemma 2.2, Proposition 3.1) and [4] (Theorem 2).

**Lemma 1.** *There exist constants  $0 < p_0(b) < 1$ ,  $a(b) > 0$  and  $C(b, d) > 0$ , such that for any  $p \geq p_0$  and large enough  $n$ ,*

$$\mu_{n,p}\{L_n^{(1)} \geq an\} \geq 1 - e^{-Cn}.$$

**Proof.** We first will show that if  $p > 1 - b$ , then there is some  $\delta > 0$ , depending only on  $p - 1 + b$ , such that for any  $v \in V_n$

$$\mu_{n,p}\left\{|C(v)| \geq \frac{n}{2}\right\} \geq \delta. \quad (2)$$

To see this, we explore the giant component  $C(v)$  and its edge-boundary  $W(v)$  as follows. We order the edges in  $E_n$  and let  $C_1 = \{v\}$ ,  $W_1 = \emptyset$ . At each step,  $C_k$  is a subset of  $C(v)$  and  $W_k$  is a subset of  $W(v)$ . At step  $k$ , we explore the first edge  $e_k = (y, z)$  from  $E_n \setminus W_k$  which is adjacent to a vertex  $y \in C_k$  and to a vertex  $z \in V_n \setminus C_k$ , if there exists such an edge. Otherwise,  $C_{k+1} = C_k$  and  $W_{k+1} = W_k$ . If the edge  $(y, z)$  is open (i.e.,  $x(e_k) = 1$ ), let  $C_{k+1} = C_k \cup \{z\}$  and  $W_{k+1} = W_k$ . If the edge  $(y, z)$  is closed (i.e.,  $x(e_k) = 0$ ), let  $C_{k+1} = C_k$  and  $W_{k+1} = W_k \cup \{e_k\}$ . Hence, we have  $C(v) = \cup_{k=1}^{\infty} C_k$ . If  $|C(v)| < n$ , there exists a smallest  $N$  such that  $W_N = \partial_E C_N$ . In addition, we have  $N = |C_N| + |W_N|$  and  $C_N = C(v)$ . Therefore, by the weak expansion property, if  $|C_N| \leq n/2$ ,

$$\begin{aligned} N - |W_N| &= |C_N| \leq \frac{|W_N|}{b} |C_N|^{1-\alpha} \\ &\leq \frac{N^{1-\alpha}}{b} |W_N|. \end{aligned}$$

Thus,

$$|W_N| \geq \frac{Nb}{b + N^{1-\alpha}}.$$

Under the measure  $\mu_{n,p}$ ,  $(x(e_1), \dots, x(e_N))$  can be completed so as to form an infinite *i.i.d.* sequence of Bernoulli random variables with parameter  $p$ . To construct  $C(v)$ , we flipped  $N-1$  independent coins (with probabilities  $p$  and  $1-p$ ) and at least  $Nb/(b + N^{1-\alpha})$  among them turned out zero. Let  $A_n$  be the event that in the first  $n^\alpha$  terms there are at least  $(n+1)b/(b + (n+1)^{1-\alpha+1})$  terms equal zero. When  $n$  is large enough, we can write

$$A_n = \left\{ \frac{\sum_{i=1}^n x(e_i)}{n^\alpha} < 1 - b \right\}.$$

Involving the Kolmogorov strong law of large numbers, we obtain

$$\frac{\sum_{i=1}^n x(e_i)}{n^\alpha} \rightarrow p$$

a.a.s. Thus, it is easy to see if  $p > 1 - b$ , then with positive probability  $\delta$ , depending only on  $p - 1 + b$ , a random infinite sequence of *i.i.d.* Bernoulli variables of parameter  $p$  does not have an  $n$  such that at least  $(n+1)b/(b + (n+1)^{1-\alpha+1})$

among the  $n^\alpha$  first coordinates equal zero. Consequently, for any  $v \in V_n$ , since

$$\mu_{n,p} \left\{ |C(v)| \leq \frac{n}{2} \right\} \leq \mu_{n,p} \left\{ \exists N, \text{ s.t. } W_N \geq \frac{Nb}{b + N^{1-\alpha}} \right\},$$

we deduce

$$\begin{aligned} & \mu_{n,p} \left\{ |C(v)| > \frac{n}{2} \right\} \\ & \geq 1 - \mu_{n,p} \left\{ \exists N, \text{ s.t. } W_N \geq \frac{Nb}{b + N^{1-\alpha}} \right\} \\ & = 1 - \left( 1 - \mu_{n,p} \left\{ \exists ! n, \text{ s.t. } W_n \geq \frac{nb}{b + n^{1-\alpha}} \right\} \right) \\ & = \mu_{n,p} \{ \exists ! n, \text{ s.t. } A_n \text{ occurs} \} \\ & = \delta, \end{aligned}$$

which concludes the proof of (2).

Next, we fix  $q \in (1 - b, 1)$ , and let  $R$  be a positive real number to be determined later. Denote by  $S_n$  the number of vertices which belong to a component of order at least  $R/2$ . That is,

$$S_n = \sum_{v \in V} X_v,$$

where  $X_v = 1_{\{|C(v)| > R/2\}}$ . Note that  $X_v$  and  $X_{v'}$  are independent as soon as  $d(v, v') > R$  where  $d(v, v')$  is the distance of vertices  $v$  and  $v'$  according to the shortest path metric in  $G_n$ . Since the maximum degree of  $G_n$  is less than  $d$ , the maximum degree in the dependency graph of  $(X_v)_{v \in V}$  is less than  $d^R$ . Recall that the dependency graph [12] of the random variables  $(X_v)_{v \in V}$  is given by the vertex set  $V_n$  and the edge set satisfying that if for two disjoint sets of vertices  $A$  and  $B$  there is no edge between  $A$  and  $B$  then the families  $(X_v)_{v \in A}$  and  $(X_v)_{v \in B}$  are independent. By Theorem 2.1 [11], for every  $t > 0$ ,

$$\mu_{n,p} \{ S_n < E_{n,p}(S_n) - t \} \leq e^{-\frac{t^2}{2}}.$$

Using (2) we know that if  $p \geq q$ ,  $E_{n,p}(S_n) = \sum_{v \in V} \mu_{n,p} \{ |C(v)| > R/2 \} \geq \delta n$ . Choosing  $t = E_{n,p}(S_n)/2$  in the above concentration inequality yields, for any  $p \geq q$ ,

$$\mu_{n,p} \left\{ S_n < \frac{\delta n}{2} \right\} \leq e^{-\frac{2}{2}}.$$

Thus, with probability at least  $1 - e^{-\frac{2}{2}}$ , there are at least  $\delta n/2$  vertices which belong to components of size at least  $R/2$ .

Fix  $p_0 \in (q, 1)$  and a set of at most  $r = \delta n/R$  components of order at least  $R/2$  which contain together at least  $\delta n/2$  vertices. If  $\varepsilon = 1 - (1 - p_0)/(1 - q)$ ,  $G(p_0)$  has the same law as  $G(q) \cup G(\varepsilon)$ , where  $G(q)$  and  $G(\varepsilon)$  are independent. Hence, we claim that there is some  $C$  depending only on  $b, d$  and  $\varepsilon$  such that with probability at least  $1 - e^{-Cn}$ , in the random graph  $G(\varepsilon)$ , there is no way of splitting these components into two parts  $A$  and  $B$ , each containing at least  $\delta n/6$  vertices, with no path of  $G(\varepsilon)$  connecting the two parts. By the method of reduction to absurdity, we see that the above comment implies that, with the required probability,  $G(q) \cup G(\varepsilon)$  contains a connected component consisting of at least  $\delta n/6$  vertices. In

what follows, we will show the claim. Let us fix two parts  $A$  and  $B$  of the components above, each containing at least  $\delta n/6$  vertices. By virtue of the Menger theorem, there are at least  $b(\delta n)^\alpha / 6^\alpha$  edge-disjoint paths between  $A$  and  $B$ . Since the total number of edges is less than  $dn/2$ , at least half of these paths are of length not larger than  $6^\alpha dn^{1-\alpha} / (b\delta^\alpha)$ . Thus, the probability that there is no path between  $A$  and  $B$  in  $G(\varepsilon)$  is at most

$$\left( 1 - \varepsilon^{\frac{1}{2 \cdot 6}} \right)^{\frac{1}{2 \cdot 6}} \leq e^{-\frac{1}{2 \cdot 6}} \cdot \varepsilon^{\frac{1}{2 \cdot 6}}.$$

There are at most  $2^r = 2^{\delta n/R}$  ways to choose  $A$  and  $B$ . Thus, the probability that there is a way to split the components into two parts  $A$  and  $B$ , each containing at least  $\delta n/6$  vertices, with no path of  $G(\varepsilon)$  connecting the two parts is at most

$$2^r e^{-\frac{1}{2 \cdot 6}} \cdot \varepsilon^{\frac{1}{2 \cdot 6}} \leq e^{-\frac{1}{4 \cdot 6}} \cdot \varepsilon^{\frac{1}{2 \cdot 6}},$$

as long as

$$R \geq \frac{4\delta n^{1-\alpha} 6^\alpha \ln 2}{b\delta^\alpha \varepsilon^{\frac{1}{2 \cdot 6}}},$$

which is a finite number since  $1 - \alpha_n = O(1/\ln n)$ . This finishes the proof of the claim.

Wrapping up the arguments, we have proved that if  $R$  is chosen large enough, there is some positive constant  $C$  depending only on  $b, d, \delta, p_0$  and  $q$  such that with probability at least  $1 - e^{-\frac{2}{2}} - e^{-Cn}$ , for  $n$  large enough, there is a component of size at least  $\delta n/6$  in  $G_n(p_0)$ . Lemma 1 then follows readily.  $\square$

**Lemma 2.** For any  $a_1 \in (0, 1/2)$  and  $a_2 \in (1/2, 1)$ , there is a constant  $0 < q_3(a_1, a_2) < 1$ , depending only on  $a_1, a_2, b$  and  $d$ , such that, for any  $p > q_3(a_1, a_2)$ ,

$$\begin{aligned} \mu_{n,p} \{ G_n(p) \text{ contains a component of order in } [a_1 n, a_2 n] \} \\ \leq 4 \left( 1 + \frac{1}{a_1} \right) e^{-n}. \end{aligned}$$

**Proof.** From [9] (pp. 68) we know that an infinite  $d$ -regular rooted tree contains  $\frac{1}{(d-1)r+1} \binom{dr}{r} \leq (de)^r$  rooted subtrees of order  $r$ . Given a vertex  $v \in G_n$ , one may associate a subtree of the infinite  $d$ -regular tree rooted at  $v$  by considering the self-avoiding paths issued from  $v$  in  $G_n$ . Therefore, any connected subgraph of order  $r$  in  $G_n$  containing  $v$  can correspond to a different subtree of order  $r$ . Thus, the total number of connected subsets of order  $r$  in  $V_n$  is less than  $n(de)^r/r$ .

Thanks to the weak expansion property, for any subset  $U \subset V_n$  of order  $r$ , the probability that all edges in  $\partial_E U$  are absent is at most  $(1-p)^{br}$  if  $r \leq n/2$ ; and at most  $(1-p)^{b(n-r)}$  if  $r > n/2$ . Hence, for any  $n \in \mathbb{N}$ , the probability of having

a connected component of order in  $[a_1n, a_2n]$  is at most

$$\begin{aligned}
 & \sum_{r=\lceil a_1n \rceil}^{\lfloor n/2 \rfloor} \frac{n(de)^r}{r} (1-p)^{br} \\
 & + \sum_{r=\lfloor n/2 \rfloor + 1}^{\lfloor a_2n \rfloor} \frac{n(de)^r}{r} (1-p)^{b(n-r)} \\
 = & \sum_{r=\lceil a_1n \rceil}^{\lfloor n/2 \rfloor} \frac{n}{r} (de(1-p)^b)^r (1-p)^{b(n-r)} \\
 & + (1-p)^{bn} \sum_{r=\lfloor n/2 \rfloor + 1}^{\lfloor a_2n \rfloor} \frac{n}{r} \left( \frac{de}{(1-p)^b} \right)^r \\
 & \cdot (1-p)^{b((n-r) - (n-r))} \\
 \leq & \frac{1}{a_1} \cdot \frac{(de(1-p)^b)^{a_1n}}{1 - (de(1-p)^b)} \\
 & + 2(1-p)^{nb} \frac{(de(1-p)^{-b})^{a_2n+1}}{de(1-p)^{-b} - 1} \\
 \leq & \frac{4}{a_1} e^{-n} + 4e^{-n},
 \end{aligned}$$

provided that

$$de(1-p)^{-b} \geq 2, \quad (de)^{a_2}(1-p)^{b(1-a_2)} \leq e^{-1} \quad (3)$$

and

$$(de(1-p)^b)^{a_1} \leq e^{-1}. \quad (4)$$

The conditions (3) and (4) are satisfied if  $p$  is larger than some  $q_3(a_1, a_2)$ , which is bounded away from 1.  $\square$

Now we will show Theorem 3 by exploiting Lemma 1 and Lemma 2.

**Proof of Theorem 3.** First, we show the lower bound of  $p_{n,\beta}(c)$ . Fix  $0 < q_1 < 1/(d-1)$  and  $p \leq q_1$ . Consider the subcritical Galton-Watson process with the first offspring distribution  $\text{Bin}(d, p)$  and other offspring distributions  $\text{Bin}(d-1, p)$ . Since the maximum degree of  $G_n$  is at most  $d$ , the connected component  $C(v)$  containing a vertex  $v \in V_n$  has order no more than  $S$ , where  $S$  is the total number of descendants of the above branching process with root  $v$ . It is well-known (e.g. [14] pp. 172) that there are some  $\lambda > 0$ ,  $M < \infty$ , depending only on  $d$  and  $q_1$ , such that, for any  $n$  and  $p \leq q_1$ ,

$$E_{n,p}(e^{\lambda S}) \leq M.$$

Hence, by Markovian inequality, we have for any  $t > 0$  and  $p \leq q_1$ ,

$$\begin{aligned}
 \mu_{n,p}\{L_n^{(1)} \geq t\} & \leq n\mu_{n,p}\{S \geq t\} \\
 & \leq \frac{nE_{n,p}(e^{\lambda S})}{e^{\lambda t}} \\
 & \leq nMe^{-\lambda t}.
 \end{aligned}$$

We obtain

$$\begin{aligned}
 \mu_{n,p}\{L_n^{(1)} \geq cn\} & \leq \mu_{n,p}\left\{L_n^{(1)} \geq \frac{2}{\lambda} \ln(nM^{1/2})\right\} \\
 & \leq \frac{1}{n}.
 \end{aligned}$$

Taking into account the fact that  $\mu_{n,p}\{L_n^{(1)} > cn\}$  is increasing with respect to  $p$ , we have  $p_{n,\beta}(c) > q_1$  for any  $\alpha \in (0, 1)$  and large enough  $n$ .

Next, the upper bound of  $p_{n,\beta}(c)$  can be shown by choosing (recall Lemma 1 and Lemma 2)

$$q_2(c) = \max \left\{ q_3 \left( \min \left\{ \frac{1}{4}, a \right\}, \max \left\{ \frac{3}{4}, c \right\} \right), p_0(b) \right\}.$$

In fact, we can show this by the reduction to absurdity. Suppose that  $p_{n,\beta}(c) \geq q_2(c)$ , i.e.,  $p_{n,\beta}(c) \geq q_4(\min\{1/4, a\}, \max\{3/4, c\})$  and  $p_{n,\beta}(c) \geq p_0(b)$ . If  $c_n < 3/4$ ,

$$(0, 1) \ni \beta = \mu_{n,p}\{L_n^{(1)} \geq cn\} \geq \mu_{n,p}\left\{L_n^{(1)} \geq \frac{3}{4}n\right\}, \quad (5)$$

and if  $c \geq 3/4$ ,

$$(0, 1) \ni \beta = \mu_{n,p}\{L_n^{(1)} \geq cn\}. \quad (6)$$

Involving Lemma 1 and Lemma 2, the right-hand sides of (5) and (6) tend to 1 as  $n \rightarrow \infty$ , which is a contradiction. Hence, we have  $p_{n,\beta}(c) < q_2(c)$ .

Finally, we show the statement (1). This can be proved by comparing  $c$  with  $a$  in Lemma 1. Suppose that  $p \geq q_2(c)$ .

**Case (i):**  $c \leq a$ . By Lemma 1, we have

$$\mu_{n,p}\{L_n^{(1)} \geq cn\} \geq \mu_{n,p}\{L_n^{(1)} \geq an\} \geq 1 - e^{-Cn}.$$

**Case (ii):**  $c > a \geq 1/2$ . Choosing  $a_1 = 1/4$  and  $a_2 = c$ , we have by Lemma 1 and Lemma 2,

$$\begin{aligned}
 \mu_{n,p}\{L_n^{(1)} \geq cn\} & = \mu_{n,p}\{L_n^{(1)} \geq an\} \\
 & \quad - \mu_{n,p}\{an \leq L_n^{(1)} < cn\} \\
 & \geq 1 - e^{-Cn} - 4 \left( 1 + \frac{1}{a_1} \right) e^{-n} \\
 & = 1 - e^{-Cn} - 20e^{-n}.
 \end{aligned}$$

**Case (iii):**  $c > 1/2 > a$ . Choosing  $a_1 = a$  and  $a_2 = c$ , we have by Lemma 1 and Lemma 2,

$$\begin{aligned}
 \mu_{n,p}\{L_n^{(1)} \geq cn\} & = \mu_{n,p}\{L_n^{(1)} \geq an\} \\
 & \quad - \mu_{n,p}\{an \leq L_n^{(1)} < cn\} \\
 & \geq 1 - e^{-Cn} - 4 \left( 1 + \frac{1}{a} \right) e^{-n}.
 \end{aligned}$$

**Case (iv):**  $1/2 \geq c > a$ . Choosing  $a_1 = a$  and  $a_2 = 3/4$ , we have by Lemma 1 and Lemma 2,

$$\begin{aligned}
 \mu_{n,p}\{L_n^{(1)} \geq cn\} & = \mu_{n,p}\{L_n^{(1)} \geq an\} \\
 & \quad - \mu_{n,p}\{an \leq L_n^{(1)} < cn\} \\
 & \geq 1 - e^{-Cn} - 4 \left( 1 + \frac{1}{a} \right) e^{-n}.
 \end{aligned}$$

The proof of Theorem 3 is thus complete.  $\square$

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