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# The Finite Difference Scheme for the Suspended String Equation with the Nonlinear Damping Term

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***Abstract*****—**A numerical solution of the initial boundary value problem of the suspended string vibrating equation with the particular nonlinear damping term based on the finite difference scheme is presented in this paper. The investigation of how the second and third power terms of the nonlinear term affect the vibration characteristic. We compare the vibration amplitude as a result of the third power nonlinear damping with the second power obtained from previous report provided that the same initial shape and initial velocities are assumed. The comparison results show that the vibration amplitude is inversely proportional to the coefficient of the damping term for the third power nonlinear damping case, while the vibration amplitude is proportional to the coefficient of the damping term in the second power nonlinear damping case.

***Keywords*****—**Finite-difference method, the nonlinear damped equation, the numerical simulation, the suspended string equation

## I. Introduction

In this work, the numerical simulation of a heavy and flexible vibrating suspended string with finite length *a* with the particular nonlinear damping term is studied.

The vibration equation can be shown as [1]

$$\begin{aligned} &u_{tt}-\left(xu_{xx}+u_x\right)+\alpha\left|u_t\right|^{c-1}u_t=0 && \alpha>0, c\geq 1\\ &u(a,t)=0, && t\in[0,T], \qquad (1)\\ &u(x,0)=\phi(x),\quad u_t(x,0)=\psi(x), && x\in[0,a], \end{aligned}$$

where $u(x,t)$ is the horizontal displacement of the string at $(x,t)$, $\alpha$ is the coefficient of the damping term and a positive number, $c\geq 1$.

The string being investigated is assumed to be heavy and flexible with the length of $a$. In addition, the string is assumed to have a uniform density and be suspended with the upper end fixed and the lower end free.

It is known that Eq. (1) is used to explain the vibration of the suspended string without taking into account the damping term. To consider the damping term, [2] showed the existence of time-periodic solution for the suspended string equation with the linear damping term by assuming the periodic initial function. The global solution of Eq. (1) for an energy decay with the nonlinear external force was also shown in [3]. The numerical solution without the damping term was studied by using the finite difference method and proposed in [4]. The solutions were found to agree with those obtained by the Crank-Nicolson method in [5].

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The numerical solution of (1) with the first power linear damping term (*c*=1) and the second power nonlinear term (*c*=2) is studied in [6]. The purpose of this paper to further investigate the effect of the second power nonlinear term to the suspended string vibration. In this work, we use a finite difference scheme to find the numerical solution of vibrating equation in Eq. (1) with the third power nonlinear damping term, i.e. $c=3$. The results are compared with the solutions of the vibrating equation with the second power nonlinear damping term.

## II. The Method of Solution

To apply the finite difference method to Eq. (1), the initial condition has been modified as follows

$$\begin{aligned} &u_{tt}-\left((m\Delta x)u_{xx}+u_x\right)+\alpha\left|u_t\right|^{c-1}u_t=0, && \alpha<0, c\geq 1\\ &u(0,t)=0,\qquad u_x(a,t)=0, && t\in[0,T], \qquad (2)\\ &u(x,0)=\phi(x),\quad u_t(x,0)=\psi(x), && x\in[0,a]. \end{aligned}$$

The solution domains (0 < *x* < 1, *t* > 0) are divided into subintervals $\Delta x$ and $\Delta t$ in the direction of the position $x$ and of the time *t*, respectively. The numerical solution at the grid point is by substituting $u_{tt}$, $u_{xx}$, $u_x$ and $u_t$ in Eq. (2) by the central finite difference as

$$\left(\frac{u_m^{n+1}-2u_m^n+u_m^{n-1}}{k^2}\right)-\left[mh\left(\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{h^2}\right)+\left(\frac{u_{m+1}^n-u_{m-1}^n}{2h}\right)\right]$$
$$+\alpha\left|\frac{u_m^{n+1}-u_m^{n-1}}{2k}\right|^{c-1}\left(\frac{u_m^{n+1}-u_m^{n-1}}{2k}\right)=0, \qquad (3)$$

where $u_m^{n+1}=u\left(x,t+k\right)$, $u_m^n=u\left(x,t\right)$, $u_m^{n-1}=u\left(x,t-k\right)$, $u_{m+1}^n=u\left(x+h,t\right)$, $u_{m-1}^n=u\left(x-h,t\right)$, $u_{m+1}^{n+1}=u\left(x+h,t+k\right)$, $u_m^{n+1}=u\left(x,t+1\right)$, $u_{m-1}^{n+1}=u\left(x-h,t+k\right)$, $m$ is the position step (*m* = 1,…, *M*) and *n* is the time step (*n* = 1,…, *N* ), while *h* and *k* are the mesh size in *x* and *t*, respectively.

For the third power nonlinear damping case (*c* = 3), the last term of left hand side of Eq. (3) can be shown as

$$\left|u_m^{n+1}-u_m^{n-1}\right|^3=\left(u_m^{n+1}\right)^3-3\left(u_m^{n+1}\right)^2u_m^{n-1}+3u_m^{n+1}\left(u_m^{n-1}\right)^2-\left(u_m^{n-1}\right)^3. \qquad (4)$$

Substituting Eq. (4) into Eq. (3), we have

$$\begin{aligned} &\left(u_m^{n+1}\right)^3-3u_m^{n-1}\left(u_m^{n+1}\right)^2+\left(\frac{4}{\alpha}-3\left(u_m^{n-1}\right)^2\right)u_m^{n+1}\\ &=\frac{2}{\alpha}\left(2mp+p\right)u_{m+1}^n+\frac{2}{\alpha}\left(4-4mp\right)u_m^n \qquad (5)\\ &+\frac{2}{\alpha}\left(2mp-p\right)u_{m-1}^n+\left(\left(u_m^{n-1}\right)^2-\frac{4}{\alpha}\right)u_m^{n-1}. \end{aligned}$$

where $p=k^2/h$

Eq. (5) can be classified into 4 different cases depending on the values of $n$ and $m$. We then obtain the finite difference schemes for the numerical solution as follows:

**Case 1:** $n = 0$ and $m = 1, 2, 3,\ldots, M-1$

$$\begin{aligned}&(u_m^1)^3 - 3\left[u_m^0 - kf_2(x)\right](u_m^1)^2 \\ &+\left(\frac{4}{\alpha} - 3\left[(u_m^0)^2 - 2u_m^0 kf_2(x) + (kf_2(x))^2\right]\right)u_m^1 \\ &= \frac{2}{\alpha}(2mp+p)u_{m+1}^0 + \frac{2}{\alpha}(4-4mp)u_m^0 + \frac{2}{\alpha}(2mp-p)u_{m-1}^0 \\ &+\left[(u_m^0)^2 - 2u_m^0 kf_2(x) + (kf_2(x))^2 - \frac{4}{\alpha}\right]\left(u_m^0 - kf_2(x)\right).\end{aligned} \tag{6}$$

**Case 2:** $n > 0$ and $m = 1, 2, 3,\ldots, M-1$

$$\begin{aligned}&(u_m^{n+1})^3 - 3u_m^{n-1}(u_m^{n+1})^2 + \left(\frac{4}{\alpha} - 3(u_m^{n-1})^2\right)u_m^{n+1} \\ &= \frac{2}{\alpha}(2mp+p)u_{m+1}^n + \frac{2}{\alpha}(4-4mp)u_m^n \\ &+\frac{2}{\alpha}(2mp-p)u_{m-1}^n + \left((u_m^{n-1})^2 - \frac{4}{\alpha}\right)u_m^{n-1}.\end{aligned} \tag{7}$$

**Case 3:** $n = 0$ and $m = M$

$$\begin{aligned}&(u_M^1)^3 - 3\left[u_M^0 - kf_2(x)\right](u_M^1)^2 \\ &+\left(\frac{4}{\alpha} - 3\left[(u_M^0)^2 - 2u_M^0 kf_2(x) + (kf_2(x))^2\right]\right)u_M^1 \\ &= \frac{2}{\alpha}(4Mp)u_{M-1}^0 + \frac{2}{\alpha}(4-4Mp)u_M^0 \\ &+\left[(u_M^0)^2 - 2u_M^0 kf_2(x) + (kf_2(x))^2 - \frac{4}{\alpha}\right]\left(u_M^0 - kf_2(x)\right)\end{aligned} \tag{8}$$

**Case 4:** $n > 0$ and $m = M$

$$\begin{aligned}&(u_M^{n+1})^3 - 3u_M^{n-1}(u_M^{n+1})^2 + \left(\frac{4}{\alpha} - 3(u_M^{n-1})^2\right)u_M^{n+1} \\ &= \frac{2}{\alpha}(4Mp)u_{M-1}^n + \frac{2}{\alpha}(4-4Mp)u_M^n + \left((u_m^{n-1})^2 - \frac{4}{\alpha}\right)u_M^{n-1}\end{aligned} \tag{9}$$

The finite difference schemes (6)-(9) has been programmed in MATLA and the numerical solutions are shown graphically as to be discussed in the next section.

## III. Results and Discussion

The numerical simulation of the vibrating suspended string equation accounting for the second and third power nonlinear damping cases under the same initial shape; i.e. $sin(7x)$, and the various values of $\alpha$; i.e. $\alpha = 0.5$, 1.3, 2 are illustrated in Figs. 1 and 2, respectively. We have found that the amplitude of vibration decreases rapidly in case of the second power nonlinear damping accounted for especially when alpha is less than 1.2. It can be seen from Fig. 1 that the frequency of the vibration is the same as the frequency of the initial shape. In addition, the amplitude of vibration is greater than an interval of -1 to 1 when alpha is larger than 1.3.

The solution of the vibration equation is quite different if the third power nonlinear damping is considered as shown in Fig. 2. That is, the amplitude of vibration increases gradually with time for every value of alpha, while the frequency of oscillation demonstrates both decrease and increase when compared with the frequency of an initial shape. Moreover, the amplitude of vibration decreases when alpha increase as shown in Fig. 2. To investigate how initial velocity affects the solution, the initial velocity has been varied and assigned to equal to its position; i.e. $\psi(x) = 0$, 1 and $x$. The numerical solution shows that the vibration shape of the suspended string with the third power nonlinear damping barely change as illustrated in Figs. 2-5.

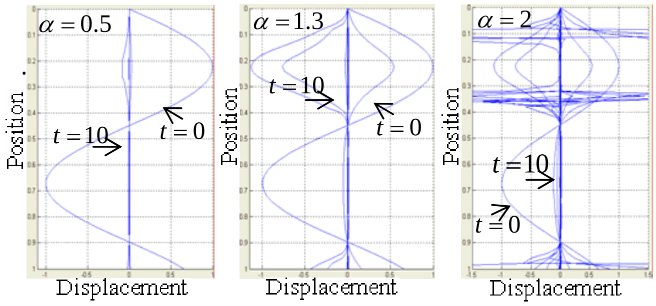


Fig. 1 Graphical comparison of the vibration displacements with the second power nonlinear damping term for different values of $\alpha$ and time (without the initial velocity; $\psi(x) = 0$ m/s)

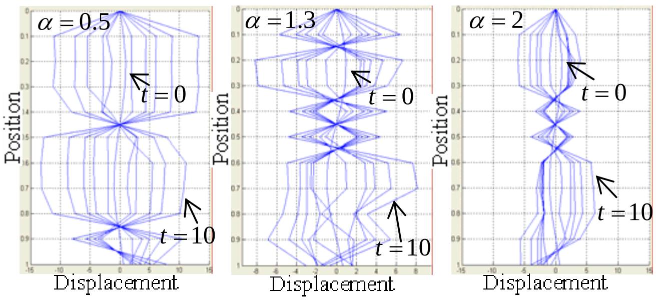


Fig. 2 Graphical comparison of the vibration displacements with the third power nonlinear damping term for different values of $\alpha$ and time (without the initial velocity; $\psi(x) = 0$ m/s)

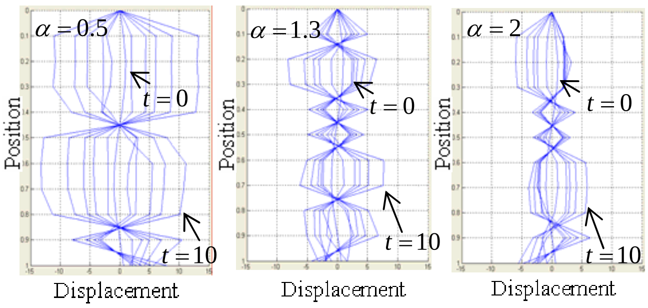


Fig. 3 Graphical comparison of the vibration displacements with the third power nonlinear damping term for different values of $\alpha$ and time (with the initial velocity $\psi(x) = 1$ m/s)

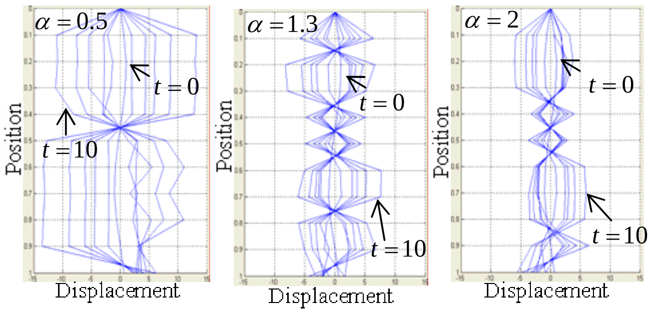


Fig. 4 Graphical comparison of the vibration displacements with the third power nonlinear damping term for different values of $\alpha$ and time (with the initial velocity $\psi(x) = x$ m/s)

Fig. 5 shows the vibration when the third power nonlinear damping term is taken into account. Three different values of $\alpha$, i.e. $\alpha$ = 18, 1,000 and 10,000 have been chosen and it is found that the frequency of oscillation decrease as $\alpha$ increases. In our experiment, several values of $\alpha$ have been assumed and the vibration amplitudes are always within the interval of -4 to 4. In addition, the vibrations are almost the same when $\alpha$ is large as illustrated in Fig. 5 when $\alpha$ = 1,000 and $\alpha$ = 10,000.

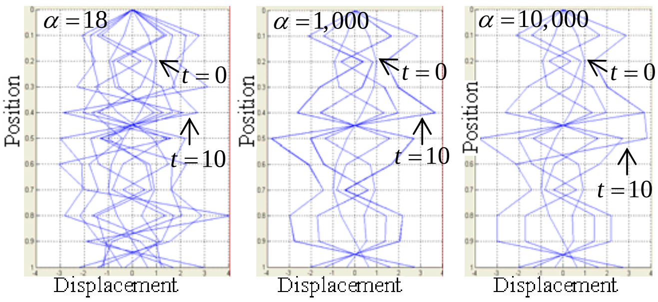


Fig. 5 Graphical comparison of the vibration displacements with the third power nonlinear damping term for different values of $\alpha$ (without the initial velocity; $\psi(x) = 0$ m/s)

Unlike the second power nonlinear damping term, the third power results in a increase in the amplitude of vibration. As a consequence, the resonance can not be minimized by adding the third power nonlinear damping force while the second power nonlinear damping term prevents resonance from occurring under the defined coefficient of the damping term.

The stability condition of the finite difference scheme is given by $2mp < 1$ where $p = k^2/h$, $h = 0.1$ and $k = 0.05$. The study can be further carried out by investigating various nonlinear damping cases i.e. $c > 3$ and the external forces are added to the string equation.

## IV. Conclusion

The numerical solution based on the finite difference method for the suspended string equation with the third power nonlinear damping term is shown. The coefficient and the power of the nonlinear damping term play a important role in dictating the amplitude of vibration.

For the third power nonlinear damping case, the amplitude of vibration increases which is opposite to the second power nonlinear damping case which demonstrates the decrease in the amplitude.

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## References

[1] N. S. Koshlyakov, E. V. Gliner and M. M. Smirnov, Differential Equations of Mathematical Physics, Moscow, 1962 (in Russian). English Translation : North-Holland Publ. Co, 1964.

[2] M. Yamaguchi, T. Nagai and K. Matsukane, *Forced oscillations of nonlinear damped equation of suspended string*, Journal of Mathematical Analysis and Applications. **342**, No.1 (2008), 89-107.

[3] J. Kasemsuwan, *Exponential decay for nonlinear damped equation of suspended string*, Proceedings of 2009 International Symposium on Computing, Communication, and Control, 2009, 308-312.

[4] J. Kasemsuwan, P. Chitsakul and P. Chaisanit, *Simulation of suspended string equation*, The 3rd Thai-Japan International Academic Conference, 2010, 60-61.

[5] K. Subklay and J. Kasemsuwan, *Numerical simulation of suspended string Vibration*, The 8th Kasetsart University Kamphaeng Saen Campus Conference**,** 2011, 1486-1491.

[6] J. Kasemsuwan, *Numerical Solution of the Damped Vibration of Suspended String*, University of the Thai Chamber of Commerce Journal, 2012, peer-review.