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The Distance between a Point and a Bezier Curve on a Bezier Surface

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***Abstract*—**The distance between two objects is an important problem in CAGD, CAD and CG etc. It will be presented in this paper that a simple and quick method to estimate the distance between a point and a Bezier curve on a Bezier surface.

***Keywords*—**Geodesic-like curve, distance, projection, Bezier.

I. INTRODUCTION

COMPUTATION of the distance between two objects is an important problem in computer aided geometric design. However, most of the methods about this problem investigate the distance problem in the space R^3, and only few methods study the distance problem on a curved space. On the other hand, the distance problem in R^3 has some different cases. For the case of the distance between two points in R^3, one can estimate easily the distance by Pythagoras' theorem. Computing the distance between a point and a curve or between a point and a surface is equivalent to the projection problem in R^3. Ma and Hewitt[4] investigate the point inversion and projection for nurbs curve and surface in 2003. Hu[2] presented an accuracy algorithm for improving this orthogonal projection problem in 2005. In 2006, Selimovic also proposed a quick algorithm for estimating the projection of points on nurbs surfaces and curves. The most complicated distance problem in R^3 is to compute the distance between two surfaces. Kim[3], Lin et al.[5], Zhou et al.[8] and Thomas et al.[9] all presented some different elegant methods for improving this problem in their investigations.

The distance problem on a regular surface is more complicated than that in R^3. The key point of the distance problem on a regular surface is to compute the geodesics on a regular surface. There are a few methods for improving the distance problem in regular surfaces. In 1996, Maekawa investigates this problem on free-form parametric surfaces and proposed an algorithm for estimating the distance between a point and a curve on a parametric surface. Traditionally, the methods to compute geodesics are discrete since the solutions of these methods are discrete curves (a list of points). These methods are accuracy on the geodesic problem between two points. However they are not easy to extend to improve the projection problem on surfaces. In 2010, a new method of the geodesic problem via the geodesic-like curves is proposed in [1]. The geodesic-like curve is a mathematical curve, like as a Bezier curve or a B-spline curve. The geodesic-like method is a powerful tool for improving the geodesic problems. This note will present that the projection problem on a Bezier surface can be improved by the Bezier geodesic-like curves.

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II. THE GEODESIC AND THE BEZIER GEODESIC-LIKE CURVES

Let S be a regular surface with a parameterization b, $b: U \subset R^2 \to S$, where (U,b) is a system of coordinate on S. Let $\gamma(t) = (x_1(t), x_2(t))$ be a curve from $[0,1]$ to U. Then γ is a geodesic curve in (U,b) on S if it satisfies the following system of geodesic equation.

$$\frac{d^2}{dt^2} x_k + \sum_{i,j=1}^{2} \Gamma_{i,j}^2 \frac{dx_i}{dt} \frac{d_j}{dt} = 0 \qquad (1)$$

where $k = 1,2$ and $\Gamma_{i,j}^2$ is the Christoffel symbol. From calculus of variation, a geodesic curve is equivalent to be a critical point of the energy variations. Then the Bezier geodesic-like curves will be defined by this notion.

A proper variation of γ is a differential map $h:[0,1]\times[-\varepsilon,\varepsilon] \to U$ such that

$$\begin{cases} h(s,0) = \gamma(s) & s \in [0,1] \\ h(0,t) = \gamma(0) & t \in [-\varepsilon,\varepsilon] \\ h(1,t) = \gamma(1) & t \in [-\varepsilon,\varepsilon] \end{cases} \qquad (2)$$

Intuitively, $h_t(\cdot) = h(\cdot,t)$ is a family of curves with the same terminal points $\gamma(0)$ and $\gamma(1)$. The energy of h_t is represented by

$$E(t) = \int_0^1 \left\| \frac{\partial (b \circ h)}{\partial s} \right\|^2 ds \quad t \in [-\varepsilon,\varepsilon]. \qquad (3)$$

Definition 1

Let S be a regular surface with parameterization b. A curve $\alpha:[0,1] \to U$ is a geodesic on S if there is a proper variation h of α such that $E'(0) = 0$ where h and E is defined in equations (1) and (2), respectively.

In 2010, a new method for estimating the geodesic was presented by the equation (2). The proper variation by the Bezier curves can be constructed and the critical point of energy function is called the Bezier geodesic-like curve.

Definition 2

Let $b(u,v)$, $b:U\rightarrow R^3$ where U is an open set in R^2, be a parameterization of a regular surface S. The Bezier geodesic-like curve $c(s)$ of degree n, $c:[0,1]\rightarrow U$, between (u_0,v_0) and (u_n,v_n) is a Bezier curve with terminal points (u_0,v_0) and (u_n,v_n) on U whose energy attend minimal. That is, $c(s)=\sum_{i=0}^{n}B_i^n(s)(\tilde{u}_i,\tilde{v}_i)$ is a critical point of equation (4).

$$\begin{aligned}&E(u_1,\cdots,u_{n-1},v_1,\cdots,v_{n-1})\\&=\int_0^1\left\|\frac{\partial}{\partial s}b(\sum_{i=0}^{n}B_i^n(s)(u_i,v_i))\right\|^2 ds\end{aligned} \tag{4}$$

Since, by the Weierstrass theorem, any piecewise differential curve can be approximate the B-spline curves, a Bezier geodesic-like curve approaches a geodesic on surface when its degree tends to infinity.

The set of control points of geodesic-like curve is a critical points of $E(u_i,v_j)$, that is the solution of the system of

$$\begin{cases}E_{u_i}=\dfrac{\partial}{\partial u_i}E=0 & \text{for each } i=1\cdots n-1\\ E_{v_j}=\dfrac{\partial}{\partial v_j}E=0 & \text{for each } j=1\cdots n-1\end{cases} \tag{5}$$

After some computations, the system of equations (5) can be rewritten as

$$\begin{cases}\int_0^1<(b_{uu}u'+b_{uv}v')B_i^n(s)+b_u(\frac{d}{dt}B_i^n(s)),b_uu'+b_vv'>dt=0\\ \int_0^1<(b_{uv}u'+b_{vv}v')B_i^n(s)+b_v(\frac{d}{dt}B_i^n(s)),b_uu'+b_vv'>dt=0\end{cases} \tag{6}$$

where $u=u(t)=\sum_{i=0}^{n}B_i^n(t)u_i$ and $v=v(t)=\sum_{i=0}^{n}B_i^n(t)v_i$.

Obviously, the solutions of the system of equations (5) are the Bezier geodesic-like curves on surface. The system of equations (6) is called the system of geodesic-like equations.

III. The Projection Problem on Bezier Surfaces

Consider the distance problem between a point and a Bezier curve on Bezier surfaces in this section. First, there are some notations. Consider $b(u,v)$, $b:U\rightarrow R^3$, is a Bezier surface, p is a point on U and $c:[0,1]\rightarrow U$ is a Bezier curve on U, $c(t)=\sum_{i=0}^{m}B_i^n(t)c_i$. Hence, the system of geodesic-like equations is a system of polynomial equations. For simplicity, the point p in U with $b(p)$ in S is identified with the curve $c(t)$ in U with $b(c(t))$ on S in this note. The distance between two points p,q on a surface is denoted by $d(p,q)$. Hence, the distance between the point p and the curve $c(t)$ is the solution of

$$\min_{t\in[0,1]} \quad d(p,c(t)). \tag{7}$$

Since the distance between two points on surface is the length of minimal geodesic between these points, the equation (7) can be rewrite as

$$\min_{\gamma\in\Gamma} \quad \|\gamma\| \tag{8}$$

where Γ is the set of geodesic from the point p to the curve $c(t)$. From that the geodesic is the solution of equation (3), the solution of equation (8) is equivalent to the solution of

$$\min_{t\in[0,1]}\min_{\tau\in[-\varepsilon,\varepsilon]} \quad E(s,t)=\int_0^1\left\|\frac{\partial}{\partial s}h(s,\tau)\right\|^2 ds \tag{9}$$

where $h(s,\tau)$ is the proper variation of the curve from p to $c(t)$ for some $t\in[0,1]$. It can be estimated that the distance from p to the curve $c(t)$ by the discretization of $c(t)$ and discrete geodesic methods. This method is the simplest algorithm for this problem. However, it is also the most expensive algorithm.

Now, this problem will be improved from the viewpoint of geodesic-like curves. Suppose that $p=(u_0,v_0)$ and $c(t)=(u_n(t),v_n(t))$. Let

$$\begin{aligned}&\gamma(t,u_1,\cdots,u_{n-1},v_1,\cdots,v_{n-1},s)\\&=\sum_{i=0}^{n}B_i^n(s)(u_i,v_i)+B_n^n(s)(u_n(s),v_n(s))\end{aligned}.$$

Hence, $\gamma(\cdot)=\gamma(t,u_1,\cdots,u_{n-1},v_1,\cdots,v_{n-1},\cdot)$ is a Bezier curve with terminal points p and $c(t)$. The energy of $\gamma(\cdot)$ forms as

$$E(t,u_1,\cdots,u_{n-1},v_1,\cdots,v_{n-1})=\int_0^1\left\|\frac{\partial}{\partial s}\gamma\right\|^2 ds. \tag{10}$$

From the property of geodesic-like curves, the minimal geodesic from the point p to the curve $c(t)$ can be approximated by the solution of

$$\min_{t,u_1,\cdots,u_{n-1},v_1,\cdots,v_{n-1}} E(t,u_1,\cdots,u_{n-1},v_1,\cdots,v_{n-1}). \qquad (11)$$

That is the solution of the equations

$$(E_t, E_{u_1}, \cdots, E_{u_{n-1}}, E_{v_1}, \cdots, E_{v_{n-1}}) = (0,\cdots,0) \qquad (12)$$

The system of equations (12) is called the system of geodesic-like equation about the projection problem. Because b is a Bezier surface and c is a Bezier curve, the system of equations (12) is still $2n-1$ nonlinear polynomial equations. This system can be solved by many elegant numerical algorithms, like as the iterative method, Newton method or other methods [6].

IV. Conclusion

In this section, the minimal geodesic-like curve between a point and a Bezier curve on different Bezier surfaces will be found. The Bezier surface in figures 1-2 is a 5×5 Bezier surfaces and the Bezier geodesic-like curves are Bezier curves of degree 4 in U. Figure 1 presents a minimal geodesic-like curve from the point p to the curve c. There are two minimal geodesic-like curves in figure 2. The difference of the lengths of these geodesic-like curves is less than the error of tolerance. Figure 3 shows the minimal geodesic-like curve between a point and an open curve on a part of a surface of revolution. In all of these simulations, the geodesic-like curves can be approximated the minimal geodesic between a point and a curve accuracy. Moreover, the computation of Bezier geodesic-like curve is not expansive and it only needs a low degree.

The method of geodesic-like curve can also improve the distance problem in $\mathbb{R}^3$. Since the minimal path in $\mathbb{R}^3$ is always a straight line, it only need the Bezier curve of degree 1to estimate. Hence, the energy function in equation (12) is a one parameter nonlinear polynomial equation. This equation can be solved accuracy and quickly.

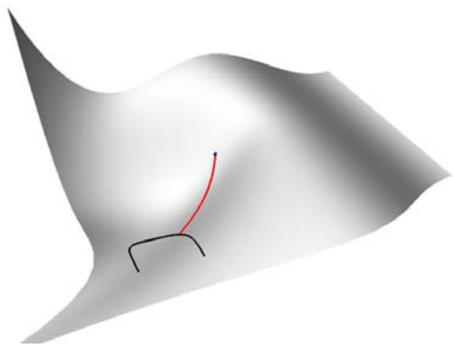

Fig. 1. The distance between a point and an open curve on a surface (I)

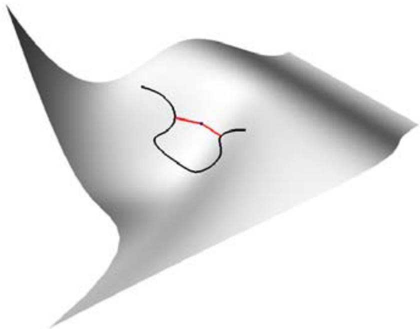

Fig. 2. The distance between a point and an open curve on a surface (II)

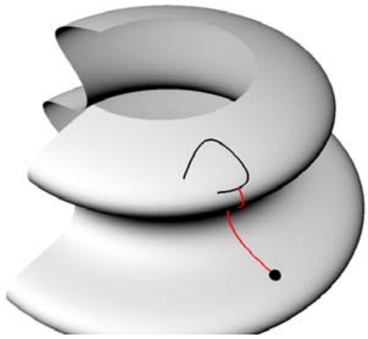

Fig. 3. The distance between a point and an open curve on a surface (III)

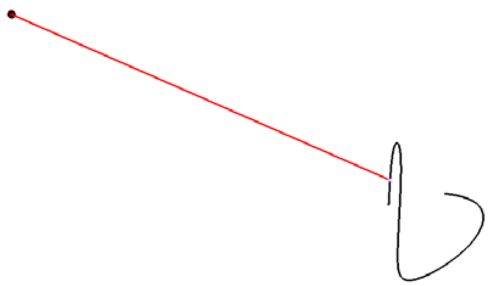

Fig. 4. The distance between a point and a curve in R^3.

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