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The Baer Radical of Rings in Term of Prime and Semiprime Generalized Bi-ideals

Rattiya Boonruang and Aiyared Iampan

Abstract—Using the idea of prime and semiprime bi-ideals of rings, the concept of prime and semiprime generalized bi-ideals of rings is introduced, which is an extension of the concept of prime and semiprime bi-ideals of rings and some interesting characterizations of prime and semiprime generalized bi-ideals are obtained. Also, we give the relationship between the Baer radical and prime and semiprime generalized bi-ideals of rings in the same way as of bi-ideals of rings which was studied by Roux.

Keywords—ring, prime and semiprime (generalized) bi-ideal, Baer radical.

I. Introduction and Preliminaries

The notion of generalized bi-ideals which is a generalization of bi-ideals of rings introduced by Szász [5], [6] in 1970. In 1971, Lajos and Szász [3] studied bi-ideals in associative rings. In 1983, Walt [7] studied prime and semiprime bi-ideals of associative rings with unity. In 1995, Roux [4] extended the results of prime and semiprime bi-ideals of associative rings with unity to associative rings without unity. Moreover, Roux proved that the Baer radical of rings is the intersection of all semiprime bi-ideals. The concept of bi-ideals play an important role in studying the structure of rings. Now, the notion of generalized bi-ideals is an important and useful generalization of bi-ideals of rings. Therefore, we will study generalized bi-ideals of rings in the same way as of bi-ideals of rings which was studied by Roux.

Our aim in this paper is threefold.

1) To introduce the concept of prime and semiprime generalized bi-ideals of rings.
2) To characterize the properties of prime and semiprime generalized bi-ideals of rings.
3) To characterize the relationship between the Baer radical and prime and semiprime generalized bi-ideals of rings.

To present the main results we discuss some elementary definitions that we use later. Throughout this paper, A will represent a ring. A subset I of A is called a *left(right) ideal* of A if

(1) I is a subgroup of $\langle A, +\rangle$,
(2) $ax \in I(xa \in I)$ for all $a \in A$ and $x \in I$.

A subset I of A is called an *ideal* of A if it is both a left and a right ideal of A. Let X be a subset of A and support that $\{A_j \mid j \in J\}$ be a family of all (left, right) ideals of A containing X. Then $\bigcap_{j\in J} A_j$ is called the *(left, right) ideal of A generated by X* [2] and denoted by $((X)_l, (X)_r)(X)$. If $X = \{x\}$, then $((X)_l, (X)_r)$ (X) is usually denoted by (x) $((x)_l, (x)_r)$. From [2], we have

$$(x)_r = \{nx + \sum_{i=1}^{m} xs_i \mid s_i \in A, m \in \mathbb{Z}^+, n \in \mathbb{Z}\}$$

and

$$(x)_l = \{nx + \sum_{i=1}^{m} s_ix \mid s_i \in A, m \in \mathbb{Z}^+, n \in \mathbb{Z}\}.$$

Let I be an ideal of A. Then

(1) I is called a *prime ideal* of A if
$$XY \subseteq I \text{ implies } X \subseteq I \text{ or } Y \subseteq I$$
for any ideals X and Y of A. Equivalently,
$$xAy \subseteq I \text{ implies } x \in I \text{ or } y \in I$$
for any $x, y \in A$ [1].

(2) I is called a *semiprime ideal* of A if
$$X^2 \subseteq I \text{ implies } X \subseteq I$$
for any ideal X of A. Equivalently,
$$xAx \subseteq I \text{ implies } x \in I$$
for any $x \in A$ [1].

From [1], a semiprime ideal of A is an intersection of prime ideals of A. If I is a left(right) ideal of A, then I is a subgroup of $\langle A, +\rangle$. Since $II \subseteq AI \subseteq I$, we have I is a subsemigroup of $\langle A, \cdot\rangle$. Hence I is a subring of A. A subset B of A is called a *bi-ideal* [4] of A if

(1) B is a subring of A,
(2) $b_1ab_2 \in B$ for all $b_1, b_2 \in B$ and $a \in A$.

We can easily prove that bi-ideals are a generalization of left(right) ideals. A subset B of A is called a *generalized bi-ideal* [5] of A if

(1) B is a subgroup of $\langle A, +\rangle$,
(2) $b_1ab_2 \in B$ for all $b_1, b_2 \in B$ and $a \in A$.

Hence generalized bi-ideals are a generalization of bi-ideals. Let B be a generalized bi-ideal of A. Then

(1) B is called a *prime generalized bi-ideal* of A if
$$xAy \subseteq B \text{ implies } x \in B \text{ or } y \in B$$
for any $x, y \in A$.

(2) B is called a *semiprime generalized bi-ideal* of A if
$$xAx \subseteq B \text{ implies } x \in B$$
for any $x \in A$.

For any generalized bi-ideal B of A, let

$$L(B) = \{x \in B \mid Ax \subseteq B\}$$

and

R. Boonruang and A. Iampan* are with the Department of Mathematics, School of Science, University of Phayao, Phayao 56000, THAILAND.
*Corresponding author. e-mail: aiyared.ia@up.ac.th.
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$$H(B) = \{y \in L(B) \mid yA \subseteq L(B)\}.$$

Let $\{P_i \mid i \in I\}$ be a family of all prime ideals of A. Then $\bigcap_{i \in I} P_i$ is called the *Baer radical* [1] of A and denoted by $\beta(A)$. From [1], we have $\beta(A)$ is the smallest semiprime ideal of A. A ring A is called *regular* [4] if for any $a \in A$, there exists $x \in A$ such that $a = axa$.

II. Lemmas

Before the characterizations of prime and semiprime generalized bi-ideals of rings for the main results, we give some auxiliary results which are necessary in what follows. The following two lemmas are easy to verify.

Lemma II.1. *For all $x \in A$, xA is a right ideal and Ax is a left ideal of A.*

Lemma II.2. *For all $x \in A$, xAx is a bi-ideal of A.*

Lemma II.3. *Let B be a generalized bi-ideal of A. Then $L(B)$ is a left ideal of A such that $L(B) \subseteq B$.*

Proof: By definition, it is clear that $\emptyset \neq L(B) \subseteq B$. Let $x, y \in L(B)$. Then $x, y \in B$ and $Ax \subseteq B$ and $Ay \subseteq B$, so $x - y \in B$ and $A(x-y) \subseteq Ax - Ay \subseteq B$. Thus $x - y \in L(B)$, so $L(B)$ is a subgroup of $\langle A, + \rangle$. Let $x \in L(B)$ and $z \in A$. Since $zx \in Ax \subseteq B$, we have $zx \in B$ and $Azx \subseteq AAx \subseteq Ax \subseteq B$. Hence $zx \in L(B)$, so $L(B)$ is a left ideal of A and $L(B) \subseteq B$. ■

Lemma II.4. *Let B be a generalized bi-ideal of A. Then $H(B)$ is a subgroup of $\langle A, + \rangle$.*

Proof: Let $x, y \in H(B)$. Then $x, y \in L(B), xA \subseteq L(B)$ and $yA \subseteq L(B)$. Since $x \in L(B)$, $x \in B$ and $Ax \subseteq B$. Since $y \in L(B)$, $y \in B$ and $Ay \subseteq B$. Since $x, y \in B$ and B is a subgroup of $\langle A, + \rangle$, we have $x - y \in B$. Thus $A(x - y) \subseteq Ax - Ay \subseteq B$, so $x - y \in L(B)$. Now, $(x-y)A \subseteq xA - yA \subseteq L(B) - L(B) \subseteq L(B)$, so $x - y \in H(B)$. Hence $H(B)$ is a subgroup of $\langle A, + \rangle$. ■

Lemma II.5. *Let B be a left ideal of A. Then $L(B) = B$.*

Proof: Clearly, $L(B) \subseteq B$. Conversely, let $x \in B$. Since B is a left ideal A, we have $Ax \subseteq B$. Thus $x \in L(B)$, so $L(B) = B$. ■

III. Main Results

In this section, give some characterizations of prime and semiprime generalized bi-ideals of rings. Finally, we prove that the Baer radical of rings is the intersection of all prime and semiprime bi-ideals.

Proposition III.1. *Let B be a generalized bi-ideal of A. Then B is a prime generalized bi-ideal of A if and only if for any right ideal R and left ideal L of A, $RL \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof: Assume that B is a prime generalized bi-ideal of A. Let R be a right ideal of A and L a left ideal of A such that $RL \subseteq B$. Suppose that $R \nsubseteq B$, let $x \in L$ and $r \in R \backslash B$. Then $rAx \subseteq RL \subseteq B$. Since B is a prime generalized bi-ideal of A and $r \notin B$, we have $x \in B$. Hence $L \subseteq B$.

Conversely, assume that for any right ideal R and left ideal L of A, $RL \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$. Let $x, y \in A$ be such that $xAy \subseteq B$. Then

$$(xA)(Ay) \subseteq xA^2y \subseteq xAy \subseteq B.$$

By Lemma II.1, we have xA is a right ideal and Ay is a left ideal of A. By assumption, we have $xA \subseteq B$ or $Ay \subseteq B$. Suppose $xA \subseteq B$. Then $x^2 \in B$. Let $z \in (x)_r(x)_l$. Then, by I and I, we get

$$z = \sum_{i=1}^{n} (m_i x + x a_i)(k_i x + b_i x)$$

for some $a_i, b_i \in A$ and $m_i, k_i, n \in \mathbb{Z}^+$, so

$$z = \sum_{i=1}^{n} m_i k_i x^2 + m_i x b_i x + k_i x a_i x + x a_i b_i x.$$

Since $x^2 \in B, b_i x, a_i x, a_i b_i x \in A$ and $xA \subseteq B$, we have $z \in B$. Hence $(x)_r(x)_l \subseteq B$. By assumption, we have

$$(x)_r \subseteq B \text{ or } (x)_l \subseteq B.$$

Hence $x \in B$. We can prove in a similar manner that $y \in B$. Therefore B is a prime generalized bi-ideal of A. ■

Proposition III.2. *Let B be a prime generalized bi-ideal of A. Then B is a prime one-sided ideal of A.*

Proof: We have to show that B is a one-sided ideal of A. Now,

$$(BA)(AB) \subseteq BAB \subseteq B.$$

Since BA is a right ideal and AB is a left ideal of A and by Proposition III.1, we have $BA \subseteq B$ or $AB \subseteq B$. Hence B is a right ideal or a left ideal of A. ■

Proposition III.3. *Let B be a generalized bi-ideal of A. Then $H(B)$ is the largest ideal of A such that $H(B) \subseteq B$.*

Proof: Since $H(B) \subseteq L(B)$ and $L(B) \subseteq B$, $H(B) \subseteq B$. By Lemma II.4, we have $H(B)$ is a subgroup of $\langle A, + \rangle$. Let $x \in H(B)$ and $y \in A$. Then $x \in L(B)$, so $Ax \subseteq B$ and $xA \subseteq L(B)$. Thus $yx \in Ax \subseteq B$. Since $Ayx \subseteq Ax \subseteq B$, we have $yx \in L(B)$. By Lemma II.3, we have $yxA \subseteq AxA \subseteq AL(B) \subseteq L(B)$. Thus $yx \in H(B)$. Hence $H(B)$ is a left ideal of A. Similarly, $xy \in xA \subseteq L(B)$. Thus $xyA \subseteq xA \subseteq L(B)$, so $xy \in H(B)$. Hence $H(B)$ is a right ideal of A. Therefore $H(B)$ is an ideal A such that $H(B) \subseteq B$. Assume that S is an ideal of A such that $S \subseteq B$ and let $s \in S$. Then $s \in B$ and $As \subseteq AS \subseteq S \subseteq B$, so $s \in L(B)$. Hence $S \subseteq L(B)$. Now, $sA \subseteq SA \subseteq S \subseteq L(B)$, so $s \in H(B)$. Hence $S \subseteq H(B)$. Therefore $H(B)$ is the largest ideal of A such that $H(B) \subseteq B$. ■

Proposition III.4. *Let B be a generalized bi-ideal of A. Then $H(B)$ is a prime ideal of A.*

Proof: Let X and Y be ideals of A such that $XY \subseteq H(B)$. Since $H(B) \subseteq B$, $XY \subseteq B$. By Proposition III.1, we have $X \subseteq B$ or $Y \subseteq B$. By Proposition III.3, we have $H(B)$ is the largest ideal of A such that $H(B) \subseteq B$. Thus

$X \subseteq H(B)$ or $Y \subseteq H(B)$. Hence $H(B)$ is a prime ideal of A. ■

Corollary III.5. *The Baer radical $\beta(A)$ is the intersection of all prime generalized bi-ideals of A.*

Proof: Let

$$\mathscr{B} = \{B \mid B \text{ is a prime generalized bi-ideal of } A\},$$
$$\mathscr{H} = \{H(B) \mid B \text{ is a prime generalized bi-ideal of } A\},$$
$$\mathscr{P} = \{P \mid P \text{ is a prime ideal of } A\}.$$

Since every prime ideal of A is a prime generalized bi-ideal, we have $\mathscr{P} \subseteq \mathscr{B}$. Thus

$$\bigcap \mathscr{B} \subseteq \bigcap \mathscr{P} = \beta(A).$$

Since $H(B) \subseteq B$ and by Proposition III.4, we have

$$\beta(A) = \bigcap \mathscr{P} \subseteq \bigcap \mathscr{H} \subseteq \bigcap \mathscr{B}.$$

From III and III, we have $\beta(A) = \bigcap \mathscr{B}$. This completes the proof. ■

Proposition III.6. *Let B be a semiprime generalized bi-ideal and L (R) a left(right) ideal of A. If $L^2 \subseteq B (R^2 \subseteq B)$, then $L \subseteq B (R \subseteq B)$.*

Proof: Assume $L^2 \subseteq B$ and suppose that $L \nsubseteq B$. Then there exists $x \in L$ but $x \notin B$. Now, $xAx \subseteq LAL \subseteq LL \subseteq B$. Since B is a semiprime generalized bi-ideal of A, we have $x \in B$ that is a contradiction. Hence $L \subseteq B$. In a similar way, we can prove that if $R^2 \subseteq B$, then $R \subseteq B$. ■

Proposition III.7. *Let B be a semiprime generalized bi-ideal of A. Then $H(B)$ is a semiprime ideal of A.*

Proof: By Proposition III.3, we have $H(B)$ is an ideal of A. Let X be an ideal of A such that $X^2 \subseteq H(B)$. Since $H(B) \subseteq B$, $X^2 \subseteq B$. By Proposition III.6, we have $X \subseteq B$. By Proposition III.3 again, we have $X \subseteq H(B)$. Hence $H(B)$ is a semiprime ideal of A. ■

Corollary III.8. *The Baer radical $\beta(A)$ is the intersection of all semiprime generalized bi-ideals of A.*

Proof: Let

$$\mathscr{S} = \{S \mid S \text{ is a semiprime ideal of } A\},$$
$$\mathscr{C} = \{C \mid C \text{ is a semiprime generalized bi-ideal of } A\},$$
$$\mathscr{H} = \{H(C) \mid C \text{ is a semiprime generalized bi-ideal of } A\}.$$

Since every semiprime ideal of A is a semiprime generalized bi-ideal, we have $\mathscr{S} \subseteq \mathscr{C}$. Since $\beta(A)$ is the smallest semiprime ideal of A, we have

$$\bigcap \mathscr{C} \subseteq \bigcap \mathscr{S} = \beta(A).$$

By Proposition III.7, we have $H(C)$ is a semiprime ideal of A and $H(C) \subseteq C$. Thus

$$\beta(A) = \bigcap \mathscr{S} \subseteq \bigcap \mathscr{H} \subseteq \bigcap \mathscr{C}.$$

From III and III, we have $\beta(A) = \bigcap \mathscr{C}$. The proof is then completed. ■

Proposition III.9. *A ring A is regular if and only if every generalized bi-ideal of A is a semiprime generalized bi-ideal.*

Proof: Assume that A is regular and let B be a generalized bi-ideal of A. Let $a \in A$ be such that $aAa \subseteq B$. Since A is regular, there exists $x \in A$ such that $a = axa$. Thua $a = axa \in aAa \subseteq B$. Hence B is a semiprime generalized bi-ideal of A.

Conversely, assume that every generalized bi-ideal of A is a semiprime generalized bi-ideal. Let $a \in A$. Then, by Lemma II.2, we have aAa is a generalized bi-ideal of A. By assumption, we have aAa is a semiprime generalized bi-ideal of A. Now, $aAa \subseteq aAa$, we get $a \in aAa$. Thus $a = axa$ for some $x \in A$. Hence A is regular, and so the proof is completed. ■

IV. Conclusion

In comparison our above results with results of bi-ideals of rings, we see that the Baer Radical is the intersection of all prime and semiprime generalized bi-ideals of A which is an analogous result of bi-ideals of rings.

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Rattiya Boonruang was born in Lampang, Thailand, in 1989. She received her B.S. from University of Phayao, Thailand, under the independent study advisor of Dr. Aiyared Iampan.

Aiyared Iampan was born in Nakhon Sawan, Thailand, in 1979. He received his M.S. and Ph.D. from Naresuan University, Thailand, under the thesis advisor of Asst. Prof. Dr. Manoj Siripitukdet.