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Systems with Queueing and their Simulation

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***Abstract*—**In the queueing theory, it is assumed that customer arrivals correspond to a Poisson process and service time has the exponential distribution. Using these assumptions, the behaviour of the queueing system can be described by means of Markov chains and it is possible to derive the characteristics of the system. In the paper, these theoretical approaches are presented on several types of systems and it is also shown how to compute the characteristics in a situation when these assumptions are not satisfied

***Keywords*—**Queueing theory, Poisson process, Markov chains.

I. Introduction

The fundamentals of the queueing theory were laid by the Danish mathematician A. K. Erlang, who worked for a telecommunication company in Copenhagen and in 1909 described an application of the probabilistic theory in telephony. Further development of the theory is mainly associated with the Russian mathematician A. N. Kolmogorov. The classification of queueing systems, as we use it today, was introduced in the 1950's by English mathematician D. G. Kendall. Today, the queueing theory belongs to the classic part of logistics and it is described in several monographs such as [1], [2], [3], [4], [5], [6], [8], [9].
Generally, at random moments, customers (demands) enter the system and require servicing. Service options may be limited, e.g. the number of service lines (or channel operator). If at least one serving line is empty, the demand arriving at the system is immediately processed. However, the service time is also random in nature because performance requirements may vary. If all service lines are busy, then the requirements (customers) must wait for their turn in a queue for the processing of previous requests. However, not all requests are handled or queued for later use immediately. For example, a telephone call is not connected because the phone number is busy.

Service lines are frequently arranged in parallel, e.g. at the hairdresser's, where customers waiting for a haircut are served by several stylists, or at a gas station, where motorists are calling at several stands of fuel. However, there are also serial configurations of queueing systems.

II. Classification of Queueing Systems

There are many deterministic techniques to solve the problem under investigation. These include:

The queue is usually understood in the usual sense **FIFO** – *first in*, *first out*), but a **LIFO** operation (*last in*, *first out*) is also possible. Sometimes, **LCFS** (*last come*, *first served*) is used to refer to a LIFO strategy [7].
Besides the FIFO and LIFO services, we may also encounter services consisting in requirements being randomly selected from the queue by the service system (**SIRO** - selection in random order) and service managed by priority requirements (**PRI** - Priority).

The *queue length* may be *limited*, after achieving a certain (predefined) number of requests in a queue, subsequent requirements are rejected such as the number of reservations for a library book currently checked out or *unlimited*, this virtually means that the limiting number is prohibitive.

The requirements in the queue may have *limited* or *unlimited patience*. In the case of infinite patience, requests are waiting for their turn in a queue while in a system with limited patience whether a request enters the queue depends to a large extent on the queue length. The queue length is sometimes also referred to as system *capacity* denoting the maximum number of requests that may be present in the system.

In 1951, Kendall proposed a classification based on three main aspects in the form A/B/C, where

- A characterises the probability distribution of the random period (interval) between the subsequent requirement arrivals,
- B characterises the probability distribution of the *service time of a requirement*,
- C is the number of parallel service lines (or channels), if this number is "unrestricted" (i.e. very large), C is usually expressed as infinity (∞).

As already mentioned, the system can be characterised by a larger number of features so that the Kendall classification was further extended to

A/B/C/D/E/F,

with the symbols D, E, and F having the following meaning:

- D an integer indicating the maximum number of requests in the system (i.e. the capacity of the system). Unless explicitly restricted, expressed by ∞,
- E an integer indicating the maximum number of requests in the input stream (or in resource requirements). If it is unlimited, ∞ is used,

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F the queue type (FIFO/LIFO/SIRO/PRI).

Parameter A can have the following values:

M intervals between the arrivals of requests are mutually stochastically independent and have an exponential distribution, which means that the input stream is a Poisson (Markov) process; for details, see below,

E_k Erlang distribution with parameters λ and k,

K_n χ^2 distribution with n degrees of freedom,

N normal (Gaussian) distribution,

U uniform distribution,

G general case, the time between the arrivals of requests is given by its distribution function,

D intervals between the arrivals of demands are constant (they are deterministic in nature).

Parameter B can have the same values as the parameter A, but referring to a random requirement service time.

Since most of the queueing systems assume that the requirements in the input stream can be characterised by a *Poisson* (*Markov*) process, it will be further described. A Poisson process is a stream of events that satisfies the following properties:

1. *Stationarity* (*homogeneity over time*) - number of events in equally long time intervals is the same.
2. *Regularity* - the probability of more than one event occurring during an interval of a sufficiently small length Δt is negligibly small. This means that, in this interval $(t, t+\Delta t)$, either exactly one event occurs with probability $\lambda\,\Delta t$ or no event will occur with a probability of $1-\lambda\,\Delta t$. In other words, in a Poisson process, a system may only pass to the next "higher" state or remain in the same condition.
3. *Independence of increases* - the number of events that occur in one time interval does not depend on the number of events in other intervals,

III. THE M/M/1/∞/∞/FIFO SYSTEM

Consider first the situation at the input *separately* from the service process and introduce the random variable *number of requests that enter the system during the interval* $\langle t_0, t_0+\Delta t\rangle$ where $\Delta t \in (0,\infty)$. Due to the stationarity of the Poisson process, the number of requests does not depend on the choice of the initial time t_0 and important is only the length of the interval Δt in question.

Let $p_k(t)$ denote the probability that, at time t, just k requirements are in the system. The regularity of the Poisson process implies that the probability that, at time $t+\Delta t$, k requirements will be in the system is equal to the probability that, at time t, $k-1$ requirements were in the system and, during Δt, one requirement arrived with a probability of $\lambda\,\Delta t$ or, at time t, k requirements were in the system and, during Δt, with probability $1-\lambda\,\Delta t$, no new requirement came. From the rules for calculating the probabilities of the conjunction and disjunction of independent events, we get the equation:

$$p_k(t+\Delta t) = p_{k-1}(t).\,\lambda\,\Delta t + p_k(t).(1-\lambda\,\Delta t),\ k = 1,2,\ \ldots \tag{1}$$

The probability that, at time $t+\Delta t$, no requirement is in the system is given by the probability that there was not any and neither had any come during the interval Δt:

$$p_0(t+\Delta t) = p_0(t).(1-\lambda\,\Delta t) \tag{2}$$

After some simplification of equations (1) and (2), we get equations (3) and (4).

$$\frac{p_k(t+\Delta t) - p_k(t)}{\Delta t} = \lambda\, p_{k-1}(t) - \lambda\, p_k(t), k = 1,2,\ldots \tag{3}$$

$$\frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = -\lambda\, p_0(t) \tag{4}$$

Let us now calculate the limit for $\Delta t \to 0$ in equations (3) and (4):

$$\lim_{\Delta t\to 0} \frac{p_k(t+\Delta t) - p_k(t)}{\Delta t} = \lim_{\Delta t\to 0} \lambda\, p_{k-1}(t) - \lambda\, p_k(t),$$
$$k = 1,2,\ldots$$

$$\lim_{\Delta t\to 0} \frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = \lim_{\Delta t\to 0} (-\lambda\, p_0(t))$$

The expressions on the left-hand sides of the previous two equations are derivatives of the functions $p_k(t)$ and $p_0(t)$ at t denoted by $p_k'(t)$ and $p_0'(t)$, with the limit having no effect on the right-hand sides of the equations. Hence, we obtain recurrence equations (5), (6)

$$p_k'(t) = \lambda\, p_{k-1}(t) - \lambda\, p_k(t), \quad k = 1,2,\ldots \tag{5}$$

$$p_0'(t) = -\lambda\, p_0(t) \tag{6}$$

These recurrence equations form a system infinitely many first-order ordinary differential equations. To solve this system, we need to know the initial conditions. However, it is clear that, at time 0, no requirements are in the system, and therefore

$$p_k(0) = 0, \quad k = 1,2,\ldots \tag{7}$$

$$p_0(0) = 1 \tag{8}$$

From the theory of ordinary differential equations, it is known that the solution to the system of equations (5) and (6) with initial conditions (7) and (8) is a system of functions

$$p_k(t) = e^{-\lambda t}\,\frac{(\lambda t)^k}{k!}, \quad k = 0,1,2,\ldots \tag{9}$$

Specially, for k=0, we get

$$p_0(t) = e^{-\lambda t} \tag{10}$$

From equation (9) we can see that, in the M/M/1 system, the random number of requests that enter the system during a time interval of length t has a Poisson distribution with parameter $\lambda\, t$.

The mean of this random variable is $\lambda\, t$ and, specifically for t=1, the mean value of the random number of requests that enter the system per unit of time is equal to λ. We say that λ is the *mean intensity of the input* or shortly the *input intensity* and it expresses the average number of requests that enter the system per unit time.

Further we show that the random interval between the arrivals of requests has an exponential distribution. Denote this random variable by T. Then the probability that, after a requirement, no further requirement for the entire time interval t had entered the system is equal to $p_0(t)$ and, therefore, by equation (10)

$$P(T > t) = p_0(t) = e^{-\lambda t} \tag{11}$$

From here we obtain the *distribution function* $F(t)$ of the *exponential distribution* with parameter λ.

$$F(t) = P(T \le t) = 1 - P(T > t) = 1 - e^{-\lambda t} \tag{12}$$

The *mean value* of the random variable T representing the average time between two consecutive requests is

$$E(T)=1/\lambda \tag{13}$$

Analogously, we can now examine the service process. We assume that the random variable *service time of one requirement* (*service time* for short) has an exponential distribution. Denote by μ the distribution of this parameter, generally $\mu \neq \lambda$. The mean value of the random service time T_O is

$$E(T_O)=1/\mu \tag{14}$$

and the parameter μ indicates the mean number of requests served per time unit of channel work time, briefly *mean service intensity* or *service intensity*.

To derive the characteristics of the system, it is more convenient to describe the system activity by a *graph of system transitions*. The nodes of the graph represent the states and the directed edges the transitions from one state to another, and these edges are assigned the probability of the transition from one state to another. State S_n for fixed $t \in \langle 0, \infty)$, or more exactly $S_n(t)$, is a random variable and indicates that, at time t, n requests are in the system. If exactly n requirements, $n \ge 1$, are in a system of type M/M/1/∞/∞/FIFO, then one of them is operating in a single line system (service channel) with the remaining $n-1$ waiting in the queue. The transitions between states which differ by the number of requirements in a system can be thought of as a birth-and-death process with the birth request representing an entry into the system and the death-request corresponding to a request leaving the system after finishing its operation. For the given input assumptions, the Poisson stream of requests with a parameter λ and an exponential distribution of service time with parameter μ, the queueing system behaviour may be described by *Markov processes*.

Due to the regularity, only those transition probabilities $P(S_i \to S_j)$ make sense for which either $i=j$ or i and j differ by 1. For example, the transition probability $P(S_0 \to S_0)$ corresponds to the probability of the event that, during the time interval of length Δt, no requirement enters the system, the transition probability $P(S_k \to S_{k-1})$, $k \ge 1$, is the probability of the event that, during the time interval of length Δt, no requirement enters the system and, at the same time, one request will be served and leaves the system, the transition probability $P(S_k \to S_k)$, $k \ge 1$, is equal to the probability of the event that, during the time interval of length Δt, no requirement enters the system and also no requirement leaves the system or, during this interval, one requirement enters, one requirement will be served, and one requirement leaves the system.

From the regularity property and the method of calculating the total probability resulting from the partial probabilities of conjunction and disjunction of independent events, neglecting the powers of the interval length Δt higher than one, we get the following transition probabilities:

$$P(S_0 \to S_0) = 1 - \lambda \Delta t \tag{15}$$

$$P(S_0 \to S_1) = \lambda \Delta t \tag{16}$$

$$P(S_k \to S_{k-1}) = (1 - \lambda \Delta t)\, \mu \Delta t = \mu \Delta t - \lambda \mu \Delta t^2 \text{ Y } \mu \Delta t \tag{17}$$

$$P(S_k \to S_k) = (1 - \lambda \Delta t)(1 - \mu \Delta t) + \lambda \Delta t\, \mu \Delta t = 1 - \mu \Delta t - \lambda \Delta t + 2\lambda \mu \Delta t^2 \text{ Y } 1 - (\lambda + \mu) \Delta t \tag{18}$$

$$P(S_k \to S_{k+1}) = \lambda \Delta t (1 - \mu \Delta t) = \lambda \Delta t - \lambda \mu \Delta t^2 \text{ Y } \lambda \Delta t \tag{19}$$

Equations (17), (18) and (19) are satisfied for $k = 1,2, \ldots$

A graph of the M/M/1/∞/∞/FIFO system transitions is shown in Figure 1. For simplicity, nodes are indicated only by numbers rather than symbols S_i. Instead of the general denotations of the transition probabilities, we will write the expressions as determined by equations (15) – (19).

Using the transition probabilities between states we can determine the probabilities $p_k(t)$ indicating that, at time t, exactly k requirements are in the system, however, with entries and services not separated

$$p_0(t+\Delta t) = P(S_0 \to S_0) + P(S_1 \to S_0) = p_0(t).(1 - \lambda \Delta t) + p_1(t).\mu \Delta t \tag{20}$$

$$p_k(t+\Delta t) = P(S_{k-1} \to S_k) + P(S_k \to S_k) + P(S_{k+1} \to S_k) = p_{k-1}(t).\, \lambda \Delta t + p_k(t).[1 - (\lambda + \mu) \Delta t] + p_{k+1}(t).\, \mu \Delta t,\ k = 1,2,\ldots \tag{21}$$

After simplifying equations (20) and (21), we obtain equations (22) and (23)

$$\frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = -\lambda\, p_0(t) + \mu\, p_1(t) \tag{22}$$

$$\frac{p_k(t+\Delta t) - p_k(t)}{\Delta t} = \lambda\, p_{k-1}(t) - (\lambda + \mu)\, p_k(t) + + \mu\, p_{k+1}(t), \quad k = 1,2,\ldots \tag{23}$$

Let us now calculate the limit for $\Delta t \to 0$ in equations (22) and (23). We get:

$$\lim_{\Delta t \to 0} \frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = \lim_{\Delta t \to 0} [-\lambda\, p_0(t) + \mu\, p_1(t)]$$

$$\lim_{\Delta t \to 0} \frac{p_k(t+\Delta t) - p_k(t)}{\Delta t} = \lim_{\Delta t \to 0} [\lambda\, p_{k-1}(t) - - (\lambda + \mu)\, p_k(t) + \mu\, p_{k+1}(t)], \quad k = 1,2,\ldots$$

The expressions on the left-hand sides of the previous two equations are derivatives of the functions $p_0(t)$ and $p_k(t)$ at point t, i.e. $p_0'(t)$ and $p_k'(t)$ with the limit having no effect on their right-hand sides. Hence, we get recurrence equations (24), (25) as follows:

$$p_0'(t) = -\lambda\, p_0(t) + \mu\, p_1(t) \tag{24}$$

$$p_k'(t) = \lambda\, p_{k-1}(t) - (\lambda+\mu)\, p_k(t) + \mu\, p_{k+1}(t), \quad k = 1,2,... \tag{25}$$

These recurrence equations are a system of infinitely many first-order ordinary differential equations. To solve them, we need to know the initial conditions given by the state of a system at time t_0=0. If there are k_0 requirements in a system at time t_0=0, then the initial conditions are given by (26) and (27)

$$p_{k_0}(0) = 1 \tag{26}$$

$$p_k(0) = 0, \quad k \geq 1, k \neq k_0 \tag{27}$$

In the sequel, we will assume that $\lambda < \mu$, i.e. $\lambda/\mu < 1$. Denote the ratio λ/μ by the symbol ψ. We call it the *intensity of the system load*. Condition (28)

$$\psi = \frac{\lambda}{\mu} < 1 \tag{28}$$

is a necessary and sufficient condition for not the queue not to grow beyond all bounds. This condition also ensures that, after a sufficiently long period from the time the queueing system is opened, its situation stabilizes, i.e., there are limits

$$\lim_{t\to\infty} p_k(t) = p_k, \quad k = 0,1,\dots, \tag{29}$$

and then after a sufficiently long period from the opening of the queueing system, the probabilities $p_k(t)$ can be seen as constant, i.e.

$$p_k(t) = p_k = \text{const} \tag{30}$$

Since the derivatives of constants are zero and by equations (24) and (25), we obtain an infinite system of linear algebraic equations determined by (31) and (32).

$$0 = -\lambda\, p_0 + \mu\, p_1 \tag{31}$$

$$0 = \lambda\, p_{k-1} - (\lambda+\mu)\, p_k + \mu\, p_{k+1}, \quad k = 1,2,... \tag{32}$$

It is clear that (33) is satisfied

$$\sum_{k=0}^{\infty} p_k = 1 \tag{33}$$

Expressing p_1 from equation (31), we get

$$p_1 = \frac{\lambda}{\mu} p_0 = \psi\, p_0 \tag{34}$$

and, from (32), we express p_k for $k \geq 2$. For k=1, we get from (32)

$$\begin{aligned} p_2 &= \frac{1}{\mu}[-\lambda p_0 + (\lambda+\mu) p_1] = \frac{1}{\mu}[-\lambda p_0 + (\lambda+\mu)\psi p_0] = \\ &= \frac{1}{\mu}[-\lambda p_0 + (\lambda+\mu)\frac{\lambda}{\mu} p_0] = \\ &= \frac{\lambda}{\mu}[-p_0 + \frac{\lambda}{\mu} p_0 + p_0] = \left(\frac{\lambda}{\mu}\right)^2 p_0 = \psi^2\, p_0 \end{aligned} \tag{35}$$

and, generally for k=1,2, …, equation (36) is satisfied

$$p_k = \psi^k\, p_0 \tag{36}$$

Now p_0 remains to be determined. To do this, we use equations (33) and (36).

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} (\psi^k\, p_0) = p_0 \sum_{k=0}^{\infty} \psi^k = 1 \tag{37}$$

Since the sum in (37) is a geometric series with the quotient ψ, the first element ψ^0=1 and the sum $\frac{1}{1-\psi}$, we get $p_0 \frac{1}{1-\psi} = 1$ from (37), and thus

$$p_0 = 1 - \psi \tag{38}$$

Using (38), equation (36) can be expressed as

$$p_k = \psi^k (1-\psi), \quad k = 1,2,... \tag{39}$$

These equations make it possible to derive other important characteristics of the M/M/1/∞/∞/FIFO system, which include:

1. *Mean number of jobs in the system*:

$$\begin{aligned} E(N_s) &= \overline{n_s} = \sum_{k=0}^{\infty} k\, p_k = \sum_{k=1}^{\infty} [k\, \psi^k (1-\psi)] = \\ &= (1-\psi) \sum_{k=1}^{\infty} k\, \psi^k = (1-\psi)\,\psi \sum_{k=1}^{\infty} k\, \psi^{k-1} = \\ &= (1-\psi)\,\psi \frac{d}{d\psi} \int \sum_{k=1}^{\infty} k\, \psi^{k-1} d\psi = \\ & (1-\psi)\,\psi \frac{d}{d\psi} \sum_{k=1}^{\infty} \psi^k = (1-\psi)\,\psi \frac{d}{d\psi}\left(\frac{\psi}{1-\psi}\right) = \\ &= (1-\psi)\,\psi \frac{(1-\psi)+\psi}{(1-\psi)^2} = (1-\psi)\,\psi \frac{1}{(1-\psi)^2} = \\ &= \frac{\psi}{1-\psi} = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda} \end{aligned} \tag{40}$$

2. *Mean number of jobs in the queue* (*mean queue length*):

$$E(N_f) = \overline{n_f} = \sum_{k=1}^{\infty}(k-1)p_k = \sum_{k=1}^{\infty} k\, p_k - \sum_{k=1}^{\infty} p_k =$$
$$= \overline{n_s} - (1-p_0) = \overline{n_s} - [1-(1-\psi)] = \overline{n_s} - \psi = \tag{41}$$
$$= \frac{\psi}{1-\psi} - [1-(1-\psi)] == \frac{\psi}{1-\psi} - \psi = \frac{\psi^2}{1-\psi} = \psi\, \overline{n_s}$$

3. *Mean time spent by a job in the system*:

$$E(T_s) = \overline{t_s} = \frac{\overline{n_s}}{\lambda} = \frac{\psi}{\lambda(1-\psi)} = \frac{\frac{\lambda}{\mu}}{\lambda\left(1-\frac{\lambda}{\mu}\right)} = \frac{1}{\mu-\lambda} \tag{42}$$

4. *Mean waiting time of a job in the queue*:

$$E(T_f) = \overline{t_f} = \frac{\overline{n_f}}{\lambda} = \frac{\psi^2}{\lambda(1-\psi)} = \frac{\psi}{\mu(1-\psi)} \tag{43}$$

5. *Mean service time*:

$$E(T_O) = \frac{1}{\mu} \tag{44}$$

6. *Service channel idle time*

$$K_0 = p_0 = 1-\psi \tag{45}$$

7. *Service channel load*

$$K_1 = 1-p_0 = 1-(1-\psi) = \psi \tag{46}$$

The equations (40)-(43) show that, in the system M/M/1/∞ /∞/FIFO, $\lambda = \mu$ or $\psi = 1$ cannot be true because this would result in the parameters growing beyond all limits.

IV. Simulation of Queueing System Processes

In practice, some assumptions may not be satisfied, which means that the formulas we derived are not entirely accurate. However, queueing systems can also be studied by Monte Carlo simulations generating random numbers representing the requirement entry moments and service times.

If these random variables are to be governed by a certain probability distribution, then it must be provided.

There are many methods of doing this such as the *elimination method* and *inverse function method*. The elimination method may be used to generate the values of continuous random variables whose probability density f is bounded in an interval $\langle a, b\rangle$ and zero outside this interval. This method is based on generating random points with coordinates (x, y) with uniform distribution in the rectangle $\langle a, b\rangle \times \langle 0, c\rangle$ where c is the maximum value of the probability density f in the interval $\langle a, b\rangle$.

A point generated is only taken to be a value of a random variable with the given distribution if $y \le f(x)$ otherwise it is eliminated from further calculations. When using the inverse function method, we first determine the probability distribution function F from the density function f using equation (47).

$$F(x) = \int_{-\infty}^{x} f(t)\, dt \tag{47}$$

We generate a random number r with uniform distribution on the interval $\langle 0,1\rangle$ that we consider to be the value of the distribution function at a yet unknown point x, i.e. $F(x) = r$. The point x here is obtained by the inverse function (48):

$x = F^{-1}(r)$ (48)

During the simulation experiments it is necessary to decide how to express the dynamic properties of the model, i.e., what strategy should be chosen time recording. There are two options - a *fixed-time-step method* and a *variable-time-step method*. In the first case, after each fixed interval of time, changes are monitored. In the variable-time-step method, bounds of time steps are given by just those moments when there is a change in the system such as a new requirement entering the system or requirement service being terminated and the requirement leaving the system.

Example:

Consider a queuing system with two service lines, unlimited source of patient requirements, the FIFO queue type and variable time step given in Table 1

Total waiting time for handling 15 requirements from Table 1 is 33 minutes. Hence, we statistically estimate the mean time of waiting requests in the queue

$$E(T_f) = \frac{33}{15} = 2{,}2\,\text{min}.$$

Now, we will determine from Table 1 the time intervals in which the number of requests does not change. The result is given in Table 2

We can see that, for 2+4=6 minutes from the total 70 minutes, there is no requirement in the system, hence, we estimate the probability p_0.

$$p_0 = \frac{6}{70} = 0{,}0857$$

Similarly, we estimate p_1 , … , p_5.

$$p_1 = \frac{14}{70} = 0{,}2, \qquad p_2 = \frac{27}{70} = 0{,}3857, \qquad p_3 = \frac{14}{70} = 0{,}2,$$
$$p_4 = \frac{8}{70} = 0{,}1143, \quad p_5 = \frac{1}{70} = 0{,}0143$$

The mean number of requirements in the system is:

$$E(N_s) = \sum_{k=0}^{\infty} k\, p_k = 0.0{,}857 + 1.0{,}2 + 2.0{,}3857 + 3.0{,}2 +$$
$$+ 4.0{,}1143 + 5.0{,}0143 = 2{,}1$$

The mean number of requirements in the queue (mean queue length) is:

$$E(N_f) = \sum_{k=n+1}^{\infty}(k-n)p_k = \sum_{k=3}^{\infty}(k-2)\,p_k =$$
$$= p_3 + 2p_4 + 3p_5 = 0{,}2 + 2.0{,}1143 + 3.0{,}0143 = 0{,}4715$$

V. Conclusions

In this paper, an approach to modelling a queuing system based on the use of Markov processes was shown and, for *M*/*M*/1/∞/∞/FIFO, its characteristics have been derived in detail.

This approach can also be used for other systems such as *M*/*M*/1/1/∞ and *M*/*M*/*n*/*n*/∞. Since the assumptions that the input stream of requirements is a Poisson process and the service time has an exponential distribution may not be satisfied in practice, a simulation approach to solving the problem is shown.

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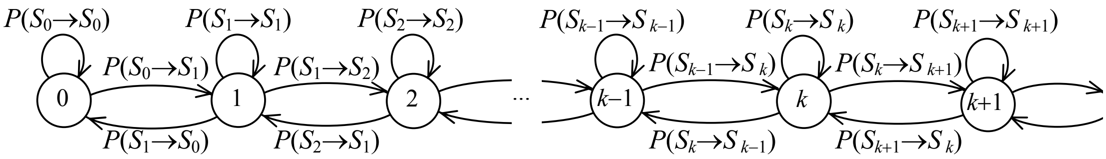

Fig. 1 Graph of M/M/1 / ∞ / ∞ / FIFO system transitions

TABLE I
SIMULATION OF QUEUEING SYSTEM

Time entry requirement [hh:min]	Service time [min]	1st service line		2nd service line		Idle time of lines [min]	Waiting time for service [min]
		start [hh:min]	end [hh:min]	start [hh:min]	end [hh:min]		
09:00	3	09:00	09:03				
09:05	9	09:05	09:14			2	
09:10	9			09:10	09:19		
09:11	9	09:14	09:23				3
09:14	9			09:19	09:28		5
09:24	6	09:24	09:30				
09:34	9	09:34	09:43			4	
09:37	9			09:37	09:46		
09:38	3	09:43	09:46				5
09:41	9	09:46	09:55				5
09:42	6			09:46	09:52		4
09:52	9			09:52	10:01		
09:53	6	09:55	10:01				2
09:56	9	10:01	10:10				5
09:57	9			10:01	10:10		4

TABLE II
SIMULATION RESULTS

Time interval	Time during which the number of requirements in the queuing system equals [min]					
	0	1	2	3	4	5
09:00 – 09:03		3				
09:03 – 09:05	2					
09:05 – 09:10		5				
09:10 – 09:11			1			
09:11 – 09:19				8		
09:19 – 09:23			4			
09:23 – 09:24		1				
09:24 – 09:28			4			
09:28 – 09:30		2				
09:30 – 09:34	4					
09:34 – 09:37		3				
09:37 – 09:38			1			
09:38 – 09:41				3		
09:41 – 09:42					1	
09:42 – 09:43						1
09:43 – 09:46					3	
09:46 – 09:53			7			
09:53 – 09:55				2		
09:55 – 09:56			1			
09:56 – 09:57				1		
09:57 – 10:01					4	
10:01 – 10:10			9			