International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:6, No:11, 2012

Systematic Unit-Memory Binary Convolutional Codes from Linear Block Codes over $\mathbb{F}_{2^r} + v\mathbb{F}_{2^r}$

John Mark Lampos and Virgilio Sison

Abstract—Two constructions of unit-memory binary convolutional codes from linear block codes over the finite semi-local ring $\mathbb{F}_{2^r} + v\mathbb{F}_{2^r}$, where $v^2 = v$, are presented. In both cases, if the linear block code is systematic, then the resulting convolutional encoder is systematic, minimal, basic and non-catastrophic. The Hamming free distance of the convolutional code is bounded below by the minimum Hamming distance of the block code. New examples of binary convolutional codes that meet the Heller upper bound for systematic codes are given.

Keywords—Convolutional codes, semi-local ring, free distance, Heller bound.

I. Introduction

Most of the conventional communication systems use rate-$1/n$ convolutional codes mainly because of the high performance-complexity ratio of these codes which make them well-suited for practical applications. An alternative to these codes would be the so-called unit-memory convolutional codes (UMC) which have a fully connected trellis that requires a higher implementation complexity but generally possess better distance properties. UMC's are interesting in the sense that their block length can be chosen to agree with the word length of computers or microprocessors that are used in the coding and decoding process. Lee [6] first studied binary UMC's and found a number of interesting examples. It was shown that binary UMC's always achieve the largest free distance among all codes of the same rate and number of encoder states. Lee also asserted that the byte-oriented nature of short UMC's make them attractive for use, with Viterbi decoding, as the inner coding component of a concatenated system. Thomessen and Justesen [13] derived bounds on the distance profile and free distance of binary unit-memory codes. Their results suggested that UMC's may be expected to have superior properties. There are already a number of methods of searching for UMC's such as the use of combinatorial optimization and circulant submatrices. For a more thorough discussion of these methods, we refer the reader to [1].

Convolutional codes can be considered as block codes. In [10], it was shown that the free distance of a field convolutional code is lower bounded by its minimum block distance. The paper [7] offered methods of constructing binary convolutional codes from certain cyclic block codes and showed that the lower bound for the Hamming free distance is a function of the minimum distances of the block code and its dual. For the ring case, the paper [12] constructed quaternary convolutional codes from linear block codes over the Galois ring $GR(4, m)$. Here a systematic block encoder gives rise to a convolutional encoder that is systematic, basic, non-catastrophic and minimal, while the squared Euclidean free distance is bounded below by twice the Hamming distance of the block code. Further, a quick extension of the binary Heller bounds to convolutional codes over finite Frobenius rings endowed with a homogenous weight was given in [11], both for the systematic and non-systematic cases.

Very little research has been done so far to study convolutional codes with large free distances over some special rings. In the present work, we make an investigation of unit-memory convolutional codes over the finite semi-local ring $R = \mathbb{F}_{2^r} + v\mathbb{F}_{2^r}$ where $v^2 = v$, and show that, in many examples, the Heller upper bound for systematic codes can be achieved. Two techniques of constructing systematic unit-memory binary convolutional codes from linear block codes over R are adopted. The advantage of considering systematic convolutional encoders is that they always possess trivial right inverses and are minimal, thus enabling simpler implementation and useful application. Systematic encoders allow for the recovery of the encoded sequence easily. This type of encoders provide good estimates of the information digits with the same level of reliability as that for hard-decisioned received sequences without employing a lengthy decoding process. Such convenience is lost in non-systematic encoders.

We prove that the Hamming free distance of the convolutional code is bounded below by the minimum Hamming distance of the block code. Thus a UMC with large free distance can be attained by a proper choice of block code. The material is organized as follows: Section II gives the preliminaries and basic definitions, while Section III explains the two constructions. Section IV proves the lower bound for the free distance, and Section V gives new examples of low-complexity unit-memory binary convolutional codes with free distances that meet the Heller upper bound for systematic codes. These codes are among the best in the class of systematic binary UMC's of certain rate and number of encoder states.

II. Preliminaries and Definitions

A. Structure of the Ring $\mathbb{F}_{2^r} + v\mathbb{F}_{2^r}$

Let $\mathbb{F}_{2^r}$ be the Galois field with 2^r elements. We denote by R the finite unitary commutative ring $\mathbb{F}_{2^r} + v\mathbb{F}_{2^r} = \{a + bv \mid a, b \in \mathbb{F}_{2^r},\ v^2 = v\}$ with 4^r elements. It has two maximal ideals, namely (v) and $(1 + v)$, thus R is a semi-local ring

J.M. Lampos and V. Sison are with the Institute of Mathematical Sciences and Physics, University of the Philippines Los Baños, College, Laguna, Philippines. Emails: {jmtlampos, vpsison}@uplb.edu.ph.

with $|(v)| = |(1+v)| = 2^r$, and is neither a finite chain ring nor a Galois ring. A nonzero element of R is a zero divisor if and only if it is an $\mathbb{F}_{2^r}$-multiple of v or $1+v$ and there are $2(2^r-1)$ of them. Thus, we can write $(v) = \{xv | x \in \mathbb{F}_{2^r}\}$ and $(1+v) = \{x(1+v) | x \in \mathbb{F}_{2^r}\}$. Apparently, the zero divisors are contained in the two maximal ideals. A nonzero element is a unit if and only if it is not an $\mathbb{F}_{2^r}$-multiple of v or $1+v$, and there are $(2^r-1)^2$ of them. Using the Chinese Remainder Theorem, we can show that R is isomorphic to the product $\mathbb{F}_{2^r} \times \mathbb{F}_{2^r}$. It can also be shown that R is isomorphic to the quotient ring $\mathbb{F}_{2^r}[x]/(x^2+x)$ through the map $a+bv \mapsto a+bx+(x^2+x)$. In addition, R is Frobenius with generating character $\chi : R \to \mathbb{T},\ \ \chi(x+vy) = e^{i\pi \cdot tr(y)}$, where tr denotes the trace map on $\mathbb{F}_{2^r}$ and $\mathbb{T}$ is the multiplicative group of unit complex numbers.

B. Linear Block Codes over $R = \mathbb{F}_{2^r} + v\mathbb{F}_{2^r}$

A rate-k/n *linear block code* of length n over R spanned by the rows of a matrix $G \in R^{k \times n}$ is the R-submodule given by the set $B = \{v \in R^n \mid v = uG, u \in R^k\}$. If no proper subset of the rows of G generates B, then G is called a *generator matrix* for B. If the columns of G contain the columns of the $k \times k$ identity matrix, then G is said to be *systematic*. The code B is *systematic* if it has a systematic generator matrix. The matrix G is in *standard form* if, after a suitable permutation of the coordinates, G takes the form

$$\begin{pmatrix} I_{k_1} & A & C & D \\ 0 & vI_{k_2} & vE & vF \\ 0 & 0 & (1+v)I_{k_3} & (1+v)H \end{pmatrix} \tag{1}$$

where I_{k_i} is the $k_i \times k_i$ identity matrix, $k = k_1 + k_2 + k_3$, A, C, and D are matrices over R and E, F, and H are binary matrices. In this case, we say that B is of type $\{k_1, k_2, k_3\}$ and $|B| = (p^{2r})^{k_1} \cdot (p^r)^{k_2} \cdot (p^r)^{k_3}$. The code B is a free R-module if and only if $k_2 = k_3 = 0$, and a code over R is called *free* if it is a free R-module.

C. Binary Convolutional Codes

Let $\mathbb{F}_2[D]$ be the ring of polynomials in the delay operator D with coefficients from $\mathbb{F}_2$. We shall consider a *rate-k/n binary convolutional code* $\mathcal{C}$ to be an $\mathbb{F}_2[D]$-submodule of $\mathbb{F}_2[D]^n$ obtained as the $\mathbb{F}_2[D]$-rowspan of a matrix $G(D) \in \mathbb{F}_2[D]^{k \times n}$. The rows of $G(D)$ are assumed to be linearly independent. The polynomial matrix $G(D)$ is called a *generator matrix* or a *convolutional encoder* of $\mathcal{C}$. Polynomial encoders are feedback-free (or non-recursive), they do not re-enter part of the output into the encoder as part of the next input. If we denote an information sequence by the k-vector $u(D) = [u_1(D), u_2(D), \ldots, u_k(D)]$, the corresponding code sequence (or codeword) is the n-vector $[v_1(D), v_2(D), \ldots, v_n(D)]$ which results from the product $v(D) = u(D)G(D)$.

Two generator matrices are said to be *equivalent* if they generate the same code, they only differ in the way a code sequence is obtained from the information space. The encoder $G(D)$ for $\mathcal{C}$ is *systematic* if it causes the information symbols to appear unchanged among the code symbols, or equivalently, if some k of its columns form the $k \times k$ identity matrix. A generator matrix for a binary convolutional code can always be chosen to be systematic. Moreover, $G(D)$ is said to be *basic* if it has a polynomial right inverse. The matrix $G(D)$ is said to be *minimal* if there exists a realization of $G(D)$ that uses the least number of encoder states required to generate the code. The encoder state at a given instant is the contents of the memory cells at that instant. It is known that a systematic generator matrix is minimal.

The maximum degree of the components in the ith row of $G(D)$ is called the *ith constraint length*. The sum of the k constraint lengths, denoted by ν, is called the *overall constraint length* or the *state complexity* of the code. A basic generator matrix is said to be *minimal-basic* if the overall constraint length ν is minimal over all equivalent basic generator matrices. Minimal-basicity implies minimality, and ν gives the smallest number of encoder states used, which is equal to 2^ν for the controller canonical realization of $G(D)$.

The maximum among the k constraint lengths is the *memory* of $\mathcal{C}$ denoted by μ. If $\mu = 1$, then $\mathcal{C}$ is referred to as a *unit-memory convolutional code* (UMC). Hence, for UMC's, the state complexity is merely the number of rows in the convolutional encoder, which is k.

We equip $\mathbb{F}_2[D]^n$ with the Hamming weight on $\mathbb{F}_2$. Given $x(D) = [x_1(D), x_2(D), \ldots, x_n(D)] \in \mathbb{F}_2[D]^n$, the *serial weight of* $x_i(D)$ is the number of nonzero coefficients of $x_i(D)$. The *serial weight of* $x(D)$ is the sum of the serial weights of the $x_i(D)$'s. The *free distance* of $\mathcal{C}$, denoted by d_f, is the minimum among the serial weights of the nonzero codewords of $\mathcal{C}$. A convolutional encoder is said to be *non-catastrophic* if it does not map an input sequence with infinitely many nonzero symbols into a code sequence of finite serial weight. Note that a basic generator matrix is always non-catastrophic, and a minimal generator matrix is non-catastrophic. The *block weight* of $x(D)$ is the number of nonzero components in $x(D)$. The *block distance* d_B of $\mathcal{C}$ is the minimum among the block weights of the nonzero codewords of $\mathcal{C}$. From [10] the block distance and the free distance of $\mathcal{C}$ satisfy

$$d_B \le d_f\ . \tag{2}$$

It was proved in [9] that, for a rate-k/n binary convolutional code generated by a minimal-basic encoder with overall constraint length ν, the (Hamming) free distance d_f satisfies

$$d_f \le (n-k)\left(\lfloor \nu/k \rfloor + 1\right) + \nu + 1. \tag{3}$$

which is referred to as the generalized Singleton bound.

On the other hand, the Heller upper bound for the free distance d_f of a rate $\rho = k/n$ binary convolutional code with a systematic encoder of memory μ is given by

$$d_f \le \min_{L \ge 1} \left\{ \left\lfloor \frac{(\mu(1-\rho)+L)n}{2(1-2^{-kL})} \right\rfloor \right\}. \tag{4}$$

It is worth noting that the Hamming weight on the Galois field $\mathbb{F}_{2^r}$, which is Frobenius, is homogeneous with average value $\Gamma = (2^r-1)/2^r$. For the binary case, the average value is $1/2$ which can be seen in (4).

D. The 2^r-ary Image

Let $\mathcal{B}_2 = \{v_1, v_2\}$ be a basis of R over $\mathbb{F}_{2^r}$ and $w = av_1 + bv_2 \in R$ where $a, b \in \mathbb{F}_{2^r}$. Define the mapping

$$\psi : R \to \mathbb{F}_{2^r}^2, \quad av_1 + bv_2 \mapsto (a, b). \tag{5}$$

Then ψ is an $\mathbb{F}_{2^r}$-module isomorphism. We extend ψ to R^n coordinate-wise. If $c = (c_1, c_2, \ldots, c_n) \in R^n$ and $c_i = a_i v_1 + b_i v_2$, then $\psi(c) = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n) \in \mathbb{F}_{2^r}^{2n}$. Let B be a linear block code of length n over R. The image set $\psi(B) = \{\psi(c) | \, c \in B\}$ will be called the 2^r*-ary image of* B *under* ψ *with respect to* $\mathcal{B}_2$. Clearly we have $|B| = |\psi(B)|$. Further, it can be shown that $\psi(B)$ is a linear block code over $\mathbb{F}_{2^r}$ with length $2n$.

III. Two Constructions

A. Construction I

Let B be a systematic linear block code over $R_2 = \mathbb{F}_2 + v\mathbb{F}_2$ with generator matrix G. We can view each $z \in R_2$ as a polynomial $p_z(v)$ in v with coefficients in $\mathbb{F}_2$ with degree at most 1. Let $G(D)$ be exactly the generator matrix G with every entry $z \in R_2$ in G replaced by $p_z(D)$. Clearly, the rows of $G(D)$ are still linearly independent, thus $G(D)$ is a systematic polynomial matrix over $\mathbb{F}_2[D]$ that generates a rate-k/n binary convolutional code, which we denote by $\mathcal{C}(B, G)$. We formalize this in a theorem.

Theorem 3.1: Let B be a linear block code over R_2 with a $k \times n$ generator matrix G. If G is systematic, then the resulting matrix $G(D)$ is a systematic generator matrix for $\mathcal{C}(B, G)$.

It follows immediately that $G(D)$ is minimal [8, Theorem 3], basic [4, Theorem 3], and non-catastrophic [5, Theorem 2.15 and Corollary 2.19].

B. Construction II

Consider a systematic linear block code B over R with a $k \times n$ generator matrix G. Let G_ψ denote the generator matrix of the 2^r-ary image of B under ψ with respect to the basis $\{1, v\}$. Hence, the entries of G_ψ are elements of $\mathbb{F}_{2^r}$, and it follows that $G_\psi \in \mathbb{F}_{2^r}^{2k \times 2n}$.

Consider the additive representation of each $z \in \mathbb{F}_{2^r}$. Hence, every $z \in \mathbb{F}_{2^r}$ can be viewed as polynomial $p_z(\omega)$ in ω with coefficients in $\mathbb{F}_2$ with degree at most $r - 1$ where ω is a root of a monic irreducible polynomial $h(x) \in \mathbb{F}_2[x]$. Now, let $G(D)$ be exactly the generator matrix G_ψ with every entry $z \in \mathbb{F}_{2^r}$ replaced by $p_z(D)$. After some permutation of the columns, we have $\psi(I_{k_1}) = (I_{k_1} \; 0)$. Hence, we are assured that $G(D)$ contains the $2k \times 2k$ identity matrix. Thus, $G(D)$ is a $2k \times 2n$ systematic polynomial matrix over $\mathbb{F}_2[D]$ that generates a binary convolutional code with memory at most $r - 1$, denoted by $\mathcal{C}(\psi(B), h, G_\psi)$. Thus, we have the following theorem.

Theorem 3.2: Let B be a linear block code over R with a $k \times n$ generator matrix G. Moreover, let G_ψ denote the $2k \times 2n$ generator matrix of the 2^r-ary image of B under ψ with respect to the basis $\{1, v\}$. If G is systematic, then the resulting matrix $G(D)$ for $\mathcal{C}(\psi(B), h, G_\psi)$ is systematic.

As in Construction I, since $G(D)$ is systematic, it follows that $G(D)$ is also minimal, basic, and non-catastrophic.

IV. Free Distance Bounds

We can bound the free distance of the convolutional code from Construction I by the minimum distance of the block code.

Theorem 4.1: If d is the minimum Hamming distance of B and d_f is the free distance of $\mathcal{C}(B, G)$, then

$$d \leq d_f. \tag{6}$$

Proof: Let $z(D)$ be a nonzero codeword of $\mathcal{C}(B, G)$ such that $z(D) = [z_1(D), z_2(D), \ldots, z_n(D)] \in \mathcal{C}(B, G)$ and $z(D) = u(D)G(D)$ where $u(D) = [u_1(D), u_2(D), \ldots, u_k(D)] \in \mathbb{F}_2[D]^k$. Note that, we can also view $\mathbb{F}_2 + v\mathbb{F}_2$ as the quotient ring $\mathbb{F}_2[x]/(x^2 + x)$. Now, let $h(D) = D^2 + D$. Denote by $z(v)$ the codeword in B that resulted from reducing $z(D)$ mod $h(D)$ with D replaced by v. Suppose $z(v) \neq 0$. Denoting the Hamming weight by w_H, we have $w_H(z(v)) \geq d$. Since $\deg(r_i(D)) \leq \deg(z_i(D))$ and some $r_i(D)$ may be 0, then $w_H(z(D)) \geq w_H(z(v)) \geq d$. Thus, $d_f \geq d$. Now, suppose $z(v) = 0$. Thus, $z_i(D) = q_i(D)h(D)^{a_i}$ for each $1 \leq i \leq n$, where $a_i \geq 1$ are integers and $q_i(D) \in \mathbb{F}_2[D]$ are assumed to be no longer divisible by $h(D)$. Let $h(D)^{a_t}$ be the smallest power of $h(D)$ occurring in a $z_i(D)$. Dividing each $z_i(D)$ by $h(D)^{a_t}$, we get the codeword $z'(D) = [q_1(D)h(D)^{b_1}, q_2(D)h(D)^{b_2}, \ldots, q_n(D)h(D)^{b_n}]$, $b_i = a_i - a_t$ that results from the input vector $u'(D) = [q_{i_1}(D)h(D)^{b_{i_1}}, q_{i_2}(D)h(D)^{b_{i_2}}, \ldots, q_{i_n}(D)h(D)^{b_{i_k}}]$ where $i_1 < i_2 < \ldots < i_k$ indicate the k positions in $z'(D)$ where the components of the input vector occur. The input vector is part of the output vector since $\mathcal{C}(B, G)$ is systematic. Since $h(D)$ is not a zero divisor, then $w_H(z(D)) = w_H(z'(D))$. In addition, since at least one of the b_i's will be zero, then there will be at least one $z_i(D) = q_i(D)$. Since $q_i(D)$ are no longer divisible by $h(D)$, then reducing $z'(D)$ mod $h(D)$ will never be zero. Thus, we can now apply what we did in the case when $z(v) \neq 0$ and get the same result.

We can also bound the free distance of the convolutional code from Construction II by the minimum distance of the block code, as suggested by the next theorem.

Theorem 4.2: If δ is the minimum distance of $\psi(B)$ and d_f is the free distance of $\mathcal{C}(\psi(B), h, G_\psi)$, then

$$\delta \leq d_f. \tag{7}$$

Proof: Let $z(D)$ be a nonzero codeword such that

$$z(D) = [z_1(D), z_2(D), \ldots, z_n(D)] = u(D)G(D)$$

where $u(D) = [u_1(D), u_2(D), \ldots, u_k(D)] \in \mathbb{F}_2[D]^k$. We denote by $h(D)$ the monic irreducible polynomial $h(x)$ over $\mathbb{F}_2$ of degree r we used in the Galois extension of the field $\mathbb{F}_{2^r}$ where x is replaced by D. Also, denote by $z(\omega)$ the codeword in $\psi(B)$ that resulted from reducing $z(D)$ mod $h(D)$. The same technique in the proof of Theorem 4.1 is applied to prove the statement.

V. Examples

For the following examples, MAGMA routines are created to construct linear block codes over $\mathbb{F}_{2^r} + v\mathbb{F}_{2^r}$, and to compute for the distance d of the given block code, the block distance d_B and free distance d_f of the resulting binary convolutional code. The first four codes give free distances that meet the Heller bound for systematic codes. Observe that $d \leq d_B$ which can be proved generally. The distances and bounds of these codes are summarized in Table I. We also give the number of states S_t of $G(D)$.

Example 5.1: Consider the rate-$2/3$ systematic linear block code over $\mathbb{F}_2 + v\mathbb{F}_2$ generated by the matrix

$$G = \begin{pmatrix} 1 & 0 & 1+v \\ 0 & 1 & 1+v \end{pmatrix}$$

with Hamming distance $d = 1$. Construction I yields

$$G(D) = \begin{pmatrix} 1 & 0 & 1+D \\ 0 & 1 & 1+D \end{pmatrix}$$

which generates a rate-$2/3$ systematic UMC with $d_f = 2$ that meets the Heller bound in (4).

Example 5.2: Consider the rate-$2/4$ self-dual linear block code over $\mathbb{F}_2 + v\mathbb{F}_2$ generated by the matrix

$$G = \begin{pmatrix} 1 & 0 & v & 1+v \\ 0 & 1 & 1+v & v \end{pmatrix}$$

with Hamming distance $d = 2$. Construction I yields the matrix

$$G(D) = \begin{pmatrix} 1 & 0 & D & 1+D \\ 0 & 1 & 1+D & D \end{pmatrix}$$

which generates a rate-$2/4$ systematic self-dual binary UMC with $d_f = 4$ that meets the Heller bound in (4).

Example 5.3: Given a rate-$2/6$ 2-quasi cyclic systematic linear block code over $\mathbb{F}_2 + v\mathbb{F}_2$ whose generator matrix is given by

$$G = \begin{pmatrix} 1 & 0 & 1+v & v & 1 & v \\ 0 & 1 & v & 1+v & v & 1 \end{pmatrix}$$

with minimum Hamming distance $d = 3$. The resulting convolutional encoder from Contruction I gives

$$G(D) = \begin{pmatrix} 1 & 0 & 1+D & D & 1 & D \\ 0 & 1 & D & 1+D & D & 1 \end{pmatrix}$$

which is systematic and generates a Heller-optimal UMC with $d_f = 6$.

Example 5.4: Given the rate-$2/8$ systematic linear block code over $\mathbb{F}_2 + v\mathbb{F}_2$ generated by

$$G = \begin{pmatrix} 1 & 0 & v & v & 1 & 1+v & 1+v & v \\ 0 & 1 & 1 & 1+v & v & v & 1+v & 1+v \end{pmatrix}$$

with minimum Hamming distance $d = 4$. Using Construction I, we have

$$G(D) = \begin{pmatrix} 1 & 0 & D & D & 1 & 1+D & 1+D & D \\ 0 & 1 & 1 & 1+D & D & D & 1+D & 1+D \end{pmatrix}$$

which generates a systematic Heller-optimal UMC with $d_f = 9$.

Example 5.5: Suppose B is a rate-$2/4$ systematic linear block code over $\mathbb{F}_4 + v\mathbb{F}_4, v^2 = v$ whose generator matrix is given by

$$G = \begin{pmatrix} 1 & 0 & \omega & 1+v\omega^2 \\ 0 & 1 & 0 & \omega+v\omega^2 \end{pmatrix}$$

TABLE I
SYSTEMATIC HELLER-OPTIMAL UNIT-MEMORY BINARY CONVOLUTIONAL CODES

Code	n	k	d	S_t	d_B	d_f	Heller	Singleton
Example 5.1	3	2	1	4	2	2	2	5
Example 5.2	4	2	2	4	3	4	4	7
Example 5.3	6	2	3	4	5	6	6	11
Example 5.4	8	2	4	4	6	9	9	13

Then $\psi(B)$ is a rate-$4/8$ systematic linear block code over $\mathbb{F}_4$ whose generator matrix is given by

$$G_\psi = \begin{pmatrix} 1 & 0 & 0 & 0 & \omega & 0 & 1 & 1+\omega \\ 0 & 0 & 1 & 0 & 0 & 0 & \omega & 1+\omega \\ 0 & 1 & 0 & 0 & 0 & \omega & 0 & \omega \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with Hamming distance $\delta = 2$. The resulting systematic convolutional encoder is

$$G(D) = \begin{pmatrix} 1 & 0 & 0 & 0 & D & 0 & 1 & 1+D \\ 0 & 0 & 1 & 0 & 0 & 0 & D & 1+D \\ 0 & 1 & 0 & 0 & 0 & D & 0 & D \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which gives a free distance $d_f = 2$.

ACKNOWLEDGEMENT

The authors would like to thank Bernice Ruth C. Ilagan and Herbert S. Palines for helpful discussions. The first author acknowledges financial support from the DOST ASTHRDP.

REFERENCES

[1] W. Ebel, "A directed search approach for unit-memory convolutional codes," *IEEE Trans. Inform. Theory*, vol. 42, No.4, pp. 1290-1297, 1996.
[2] F. Fagnani, and S. Zampieri, "System-theoretic properties of convolutional codes over rings," *IEEE Trans. Inform. Theory*, vol. 47, no. 6, pp. 2256-2274, 2001.
[3] G.D. Forney, Jr., "Convolutional codes I: Algebraic structure," *IEEE Trans. Inform. Theory*, vol. IT-16, no. 6, pp. 720-738, 1970.
[4] R. Johannesson, Z. Wan, and E. Wittenmark, "Some structural properties of convolutional codes over rings," *IEEE Trans. Inf. Theory*, vol.44, pp. 839845, Mar. 1998.
[5] R. Johannesson, and K.Sh. Zigangirov, "Fundamentals of convolutional coding," *IEEE Press*, USA, 1999.
[6] L.-N. Lee, "Short unit-memory byte-oriented convolutional codes having maximal free distance," *IEEE Trans. Inform. Theory*, vol. IT-22, no. 3, pp. 349-352, May 1976.
[7] J.L. Massey, D.J. Costello, Jr., and J. Justesen, "Polynomial weights and code constructions," *IEEE Trans. Inform. Theory*, vol. IT-19, no. 1, pp. 101-110, January 1973.
[8] T. Mittelholzer, "Minimal encoders for convolutional codes over rings," in *Communications Theory and Applications*, London, U.K.:HW Comm. Ltd., pp. 3036, 1993.
[9] J. Rosenthal, and R. Smarandache, "Maximum Distance Separable Convolutional Codes," in *Appl. Algebra Engrg. Comm. Comput.*, vol. 10, no. 1, pp. 1532, 1999.
[10] V. Sidorenko, C. Medina, and M. Bossert, "From block to convolutional codes using block distances," *ISIT 2007,Nice, France*, pp. 2331-2335, 2007.
[11] V. Sison, "Heller-type bounds for the homogeneous free distance of convolutional codes over finite Frobenius rings," *Matimyás Matematika*, vol. 30, no. 1, pp. 23–30, 2007.
[12] P. Solé and V. Sison, "Quaternary convolutional codes from linear block codes over Galois rings," *IEEE Trans. Inform. Theory*, vol. 53, no. 6, pp. 2267–2270, June 2007.
[13] C. Thommesen and J. Justesen, "Bounds on distances and error exponents of unit-memory codes," *IEEE Trans. Inform. Theory*, vol. IT-29, no. 5, pp. 637-649, September 1983.