

# Structure of covering-based rough sets

Shiping Wang, Peiyong Zhu, William Zhu

**Abstract**—Rough set theory is a very effective tool to deal with granularity and vagueness in information systems. Covering-based rough set theory is an extension of classical rough set theory. In this paper, firstly we present the characteristics of the reducible element and the minimal description covering-based rough sets through down-sets. Then we establish lattices and topological spaces in covering-based rough sets through down-sets and up-sets. In this way, one can investigate covering-based rough sets from algebraic and topological points of view.

**Keywords**—Covering, poset, down-set, lattice, topological space, topological base.

## I. INTRODUCTION

**I**N completeness and vagueness of knowledge is widespread phenomena in information systems. The modeling and reasoning with incompleteness and uncertainty is an important research topic in data mining. It is necessary to deal with the incomplete and vague information in concept formulation, classification and data analysis.

In order to achieve this goal, many theories and methods have been proposed, for example, fuzzy set theory [1], [2], computing with words [3], [4], rough set theory [5], [6] and granular computing [7], [8], [9], [10]. From the structures of these theories, two structures are mainly used, that is, algebraic structure [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], and topological structure [27], [28], [29], [30].

This paper focuses on establishing algebraic and topological structures of covering-based rough sets through down-sets and up-sets. Firstly, we bridge posets with covering-based rough sets, then covering-based rough sets can be investigated in posets. Down-sets and up-sets are defined in the poset environment. Secondly, we describe the reducible element and the minimal description with down-sets, and give a sufficient and necessary condition of the reducible element. Moreover, we con-

struct lattices and topological spaces in covering-based rough sets through down-sets and up-sets. Therefore, topological structure and algebraic structure are established in covering-based rough sets.

The rest of this paper is arranged as follows. Section II recalls some fundamental definitions used in this paper. Section III characterizes reducible elements and minimal descriptions in a covering through down-sets. Section IV establishes lattices and topological spaces through down-sets and up-sets in covering-based rough sets. In this way, algebraic and topological structures are established in covering-based rough sets. Section V concludes this paper.

## II. BASIC DEFINITIONS

The classical rough set theory is based on equivalence relations. An equivalence relation corresponds to a partition, while a covering is an extension of a partition.

**Definition 1:** (Covering [17]) Let  $U$  be a domain of discourse,  $\mathbf{C}$  a family of subsets of  $U$ .  $\mathbf{C}$  is called a covering of  $U$ , if none of subsets in  $\mathbf{C}$  is empty, and  $\bigcup \mathbf{C} = U$ .

In this paper, we assume  $U$  is a finite set.

The reducible element is one of core concepts in covering-based rough sets.

**Definition 2:** (Reducible element [17]) Let  $\mathbf{C}$  be a covering of domain  $U$  and  $K \in \mathbf{C}$ . If  $K$  is a union of some sets in  $\mathbf{C} - \{K\}$ , we say  $K$  is reducible in  $\mathbf{C}$ , otherwise  $K$  is irreducible.

In an information system or a decision system, some knowledge is redundancy. The purpose of the reduct concept is to abandon the redundancy knowledge.

**Definition 3:** (Minimal description [21]) Let  $\mathbf{C}$  be a covering of domain  $U$  and  $x \in U$ .

$$Md(x) = \{K \in \mathbf{C} | x \in K \wedge (\forall S \in \mathbf{C} \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\}$$

is called the minimal description of  $x$ .

In some situations, we can describe precisely an object or a concept if we master the essential characteristics related to this object or this concept.

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The concept of minimal description is proposed to achieve this goal.

Posets and lattices are basic concepts in computer science. They are used in this paper to describe algebraic structure in covering-based rough sets.

Partial relation is a special binary relation, while a poset is based on a partial relation.

**Definition 4:** (Poset [31]) A relation  $R$  on a set  $A$  is called a partial order if  $R$  is reflective, anti-symmetric, and transitive. The set  $A$  together with the partial order  $R$  is called a partially ordered set, or simply a poset, and we will denote this poset by  $(A, R)$ . If there is no possibility of confusion about the partial order, we may refer to the poset simply as  $A$ , rather than  $(A, R)$ .

Lattices have many applications in computer science.

**Definition 5:** (Lattice [31]) A lattice is a poset  $(L, \leq)$  in which every subset  $\{a, b\}$  consisting of two elements has a least upper bound and a greatest lower bound. We denote  $LUB(\{a, b\})$  by  $a \vee b$  and call it the join of  $a$  and  $b$ . Similarly, we denote  $GLB(\{a, b\})$  by  $a \wedge b$  and call it the meet of  $a$  and  $b$ . More generally, an upper-semilattice (or join-semilattice) is a poset  $(L, \leq)$  in which every subset  $\{a, b\}$  consisting of two elements has a least upper bound; a lower-semilattice (or meet-semilattice) is a poset  $(L, \leq)$  in which every subset  $\{a, b\}$  consisting of two elements has a greatest lower bound.

Here  $\vee, \wedge$  are binary operations, and  $(L, \vee, \wedge)$  is an algebraic system induced by the lattice  $(L, \leq)$ . Sometimes, we also call  $(L, \vee, \wedge)$  a lattice.

In this paper, down-sets and up-sets play an important role in studying covering-based rough sets.

**Definition 6:** (Down-set and up-set) Let  $(P, \prec)$  be a poset. For all  $A \subseteq P$ , one can define,

$$\downarrow A = \{x \in P \mid \exists a \in A, x \prec a\},$$

$$\uparrow A = \{x \in P \mid \exists a \in A, a \prec x\}.$$

$\downarrow A$  is called a down-set of  $A$  on the poset  $(P, \prec)$ ;  $\uparrow A$  is called an up-set  $A$  on the poset  $(P, \prec)$ . When there is no confusion, we say  $\downarrow A$  is a down-set of  $A$ , and  $\uparrow A$  an up-set of  $A$ .

Topological space and topological base are used to describe topological structure in covering-based rough sets.

**Definition 7:** (Topological space) Let  $X$  be a nonempty set and  $\tau$  a family of subsets of  $X$ .  $(X, \tau)$  is called a topological space, if  $\tau$  satisfies the following three conditions:

- (1)  $\emptyset, X \in \tau$ ;
- (2)  $\forall A, B \in \tau \implies A \cap B \in \tau$ ;
- (3)  $\forall \tau' \subseteq \tau \implies \bigcup \tau' \in \tau$ .

**Definition 8:** (Topological base) Let  $(X, \tau)$  be a topological space and  $\beta \subseteq \tau$ .  $\beta$  is called a topological base of  $X$ , if  $\forall G \in \tau$ , there exists  $\{B_\lambda \mid \lambda \in \Lambda\} \subseteq \beta$  such that  $G = \bigcup_{\lambda \in \Lambda} B_\lambda$ , where  $\Lambda$  is an index set.

### III. CHARACTERISTICS OF REDUCIBLE ELEMENTS AND MINIMAL DESCRIPTIONS THROUGH DOWN-SETS

The following proposition shows the connection between posets and covering-based rough sets.

**Proposition 1:** If  $\mathbf{C}$  is a covering of domain  $U$ , then  $(\mathbf{C}, \subseteq)$  is a poset.

The above proposition constructs posets in covering-based rough sets. These foundations are crucial for the establishment of the structures for covering-based rough sets.

The reducible element is an important concept in covering-based rough sets. Its characteristic can be described with down-sets precisely.

**Proposition 2:** Let  $\mathbf{C}$  be a covering of domain  $U$  and  $K \in \mathbf{C}$ .  $K$  is reducible in  $\mathbf{C}$  if and only if  $K = \bigcup(\downarrow \{K\} - \{K\})$ .

*Proof:* ( $\implies$ ): For all  $x \in K$ , since  $K$  is reducible, then there exists  $\{C_1, \dots, C_m\} \subseteq \mathbf{C}$ , such that  $K = \bigcup_{i=1}^m C_i$ . So there exists  $i_x \in \{1, \dots, m\}$ , such that  $x \in C_{i_x}$ . Since  $C_{i_x} \subseteq K$  and  $C_{i_x} \neq K$ , then  $C_{i_x} \in \downarrow \{K\} - \{K\}$ , so  $x \in C_{i_x} \subseteq \bigcup(\downarrow \{K\} - \{K\})$ .  $K = \bigcup_{x \in K} \{x\} \subseteq \bigcup_{x \in K} C_{i_x} \subseteq \bigcup(\downarrow \{K\} - \{K\})$ . Conversely, it is obvious that  $\bigcup(\downarrow \{K\} - \{K\}) \subseteq K$ . In a word,  $K = \bigcup(\downarrow \{K\} - \{K\})$ .

( $\impliedby$ ): If  $K = \bigcup(\downarrow \{K\} - \{K\})$ , then  $K$  is reducible since  $\downarrow \{K\} - \{K\} \subseteq \mathbf{C} - \{K\}$ , in other words,  $K$  is a union of some sets in  $\mathbf{C} - \{K\}$ . ■

Unless otherwise stated, down-sets and up-sets in covering-based rough sets are defined on the poset  $(\mathbf{C}, \subseteq)$ , where  $\mathbf{C}$  is a covering of domain  $U$ .

In the following, the minimal description is presented by down-sets.

**Proposition 3:** Let  $\mathbf{C}$  be a covering of domain  $U$ . Then  $Md(x) = \{K \in \mathbf{C} \mid (x \in K) \wedge (\downarrow \{K\} = \{K\})\}$  for all  $x \in U$ .

*Proof:* According to Definition 3, it is straightforward. ■

Here  $\downarrow$  and  $\uparrow$  are operators. The following work is to explore their properties and structures in covering-based rough sets.

**Proposition 4:** Let  $C$  be a covering of domain  $U$ . For all  $C', C'' \subseteq C$ , the following properties about down-sets hold:

- (1)  $C' \subseteq \downarrow C'$ ;
- (2) If  $C' \subseteq C''$ , then  $\downarrow C' \subseteq \downarrow C''$ ;
- (3)  $\downarrow(\downarrow C') = \downarrow C'$ ;
- (4)  $\downarrow(C' \cup C'') = \downarrow C' \cup \downarrow C''$ .

*Proof:* (1) For all  $A \in C'$ ,  $A \in \downarrow C'$  since  $A \subseteq A$ . So  $C' \subseteq \downarrow C'$ .

(2) For all  $A \in \downarrow C'$ , there exists  $B \in C'$  such that  $A \subseteq B$ . Because  $C' \subseteq C''$ ,  $B \in C''$ . Thus  $A \in \downarrow C''$ . Therefore,  $\downarrow C' \subseteq \downarrow C''$ .

(3) On the one hand, according to (1), it is obvious that  $\downarrow C' \subseteq \downarrow(\downarrow C')$ . On the other hand, for all  $A \in \downarrow(\downarrow C')$ , there exists  $B \in \downarrow C'$ , such that  $A \subseteq B$ . Similarly, there exists  $C \in C'$  such that  $B \subseteq C$  because  $B \in \downarrow C'$ . Thus  $A \subseteq B \subseteq C$ . So one can obtain  $A \in \downarrow C'$  since  $A \subseteq C$ . Therefore,  $\downarrow(\downarrow C') \subseteq \downarrow C'$ . This proves  $\downarrow(\downarrow C') = \downarrow C'$ .

(4) We only need to prove  $\downarrow(C' \cup C'') \subseteq \downarrow C' \cup \downarrow C''$ . In fact, for all  $A \in \downarrow(C' \cup C'')$ , there exists  $B \in C' \cup C''$  such that  $B \subseteq A$ . If  $B \in C'$ , then  $A \in \downarrow C'$ . If  $B \in C''$ , then  $A \in \downarrow C''$ . To sum up,  $A \in \downarrow C' \cup \downarrow C''$ . This completes the proof. ■

Similarly, we list the properties about up-sets.

**Proposition 5:** Let  $C$  be a covering of domain  $U$ . For all  $C', C'' \subseteq C$ , the following properties about up-sets hold:

- (1)  $C' \subseteq \uparrow C'$ ;
- (2) If  $C' \subseteq C''$ , then  $\uparrow C' \subseteq \uparrow C''$ ;
- (3)  $\uparrow(\uparrow C') = \uparrow C'$ ;
- (4)  $\uparrow(C' \cup C'') = \uparrow C' \cup \uparrow C''$ .

#### IV. ALGEBRAIC AND TOPOLOGICAL STRUCTURES OF COVERING-BASED ROUGH SETS

Since a covering is a poset as shown in Proposition 1, then one can employ poset approaches to study covering-based rough sets.

**Definition 9:** Let  $C$  be a covering of domain  $U$ . One can define

$$\begin{aligned}\tau_* &= \{C' \subseteq C \mid \downarrow C' = C'\}, \\ \tau^* &= \{C' \subseteq C \mid \uparrow C' = C'\}.\end{aligned}$$

$\tau_*$  is the family of all of the fixed points of the operator  $\downarrow$  in  $C$ , and  $\tau^*$  the family of all of the fixed points of the operator  $\uparrow$  in  $C$ .

In the following, we establish an algebraic structure in covering-based rough sets. In fact, the family of all of the fixed points of the operator  $\downarrow$  is a lattice.

**Proposition 6:** Let  $C$  be a covering of domain  $U$ . Then  $(\tau_*, \cup, \cap)$ ,  $(\tau^*, \cup, \cap)$  are lattices.

*Proof:* For all  $C', C'' \in \tau_*$ , the following work is to prove  $C' \cup C'' \in \tau_*$  and  $C' \cap C'' \in \tau_*$ , respectively.

In fact, because  $C', C'' \in \tau_*$ ,  $C' = \downarrow C'$  and  $C'' = \downarrow C''$ . Obviously,  $C' \cup C'' \subseteq \downarrow(C' \cup C'')$ . Conversely, for all  $A \in \downarrow(C' \cup C'')$ , there exists  $B \in C' \cup C''$ , such that  $A \subseteq B$ . If  $B \in C'$ , then  $A \in \downarrow C'$ . If  $B \in C''$ , then  $A \in \downarrow C''$ . All in all,  $A \in (\downarrow C') \cup (\downarrow C'') = C' \cup C''$ . That is to say,  $\downarrow(C' \cup C'') \subseteq C' \cup C''$ . To sum up, this proves  $C' \cup C'' = \downarrow(C' \cup C'')$ . Thus  $C' \cup C'' \in \tau_*$ .

Generally, we can suppose  $C' \cap C'' \neq \emptyset$ . Since  $C' \cap C'' \subseteq C'$  and  $C' \cap C'' \subseteq C''$ ,  $\downarrow(C' \cap C'') \subseteq \downarrow C'$  and  $\downarrow(C' \cap C'') \subseteq \downarrow C''$ . Thus  $\downarrow(C' \cap C'') \subseteq (\downarrow C') \cap (\downarrow C'') = C' \cap C''$ . Conversely, obviously  $C' \cap C'' \subseteq \downarrow(C' \cap C'')$ . Thus,  $C' \cap C'' = \downarrow(C' \cap C'')$ , in other words,  $C' \cap C'' \in \tau_*$ . This proves  $(\tau_*, \cup, \cap)$  is a lattice.

Similarly, it is proved that  $(\tau^*, \cup, \cap)$  is a lattice. ■

Then we also establish topological structure in covering-based rough sets with down-sets and up-sets. In fact, the family of all of fixed points of the operator  $\downarrow$  is a topology.

**Proposition 7:** Let  $C$  be a covering of domain  $U$ .  $(C, \tau_*)$ ,  $(C, \tau^*)$  are topological spaces.

*Proof:*  $\emptyset \in \tau_*$  and  $C \in \tau_*$ ;  
For all  $C', C'' \in \tau_*$ ,  $C' \cap C'' \in \tau_*$ ;  
For all  $C', C'' \in \tau_*$ ,  $C' \cup C'' \in \tau_*$ . Then for all  $\tau' \subseteq \tau_*$ ,  $\cup \tau' \in \tau_*$  since  $\tau_*$  is a finite set. Thus  $(C, \tau_*)$  is a topological space.

Similarly, it is proved that  $(C, \tau^*)$  is a topological space. ■

In fact, in the above, we default  $\emptyset \in \tau_*$ .

In the following, we describe the structure of covering-based rough sets from the standpoint of topological base.

**Proposition 8:** Let  $C$  be a covering of domain  $U$ .  $\beta_* = \{\downarrow \{K\} \mid K \in C\}$

is a topological base of topological space  $(C, \tau_*)$ .

*Proof:* On the one hand, because  $\downarrow(\downarrow C') = \downarrow C'$  for all  $C' \subseteq C$ . Specially,  $\downarrow(\downarrow \{K\}) = \downarrow \{K\}$ . So  $\beta_* \subseteq \tau_*$ .

On the other hand, for all  $C' \in \tau_*$ ,  $C' = \downarrow C' = \cup_{K \in C'} \downarrow \{K\}$ . In fact,  $\forall K \in C'$ ,  $\downarrow \{K\} \subseteq \downarrow C'$  since  $\{K\} \subseteq C'$ . Thus  $\cup_{K \in C'} \downarrow \{K\} \subseteq \downarrow C'$ . Conversely, for all  $A \in \downarrow C'$ , there exists  $B \in C'$ , such that  $A \subseteq B$ . Then  $A \in \downarrow \{B\} \subseteq \cup_{K \in C'} \downarrow \{K\}$ . Therefore,  $\downarrow C' \subseteq \cup_{K \in C'} \downarrow \{K\}$ . To sum up, this proves  $\downarrow C' = \cup_{K \in C'} \downarrow \{K\}$ . Hence, any element

in  $\tau_*$  is a union of some elements in  $\beta_*$ , in other words,  $\beta_*$  is a topological base of  $(C, \tau_*)$ . ■

Similarly, we can get the following conclusion.

*Proposition 9:* Let  $C$  be a covering of domain  $U$ .

$$\beta^* = \{\uparrow \{K\} | K \in C\}$$

is a topological base of topological space  $(C, \tau^*)$ .

*Proof:* The proof is similar with that of Proposition 8. ■

## V. CONCLUSION

This paper employs down-sets and up-sets to study covering-based rough sets. we use them to characterize reducible elements and minimal descriptions, which are quite important concepts in covering-based rough sets. We also establish lattices and topological spaces in covering-based rough sets with down-sets and up-sets. Therefore, this work facilitates us to understand covering-based rough sets in algebraic and topological points of view.

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