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Stress Distribution in Axisymmetric Indentation of an Elastic Layer-Substrate Body

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***Abstract*—**We focus on internal stress and displacement of an elastic axisymmetric contact problem for indentation of a layer-substrate body. An elastic layer is assumed to be perfectly bonded to an elastic semi-infinite substrate. The elastic layer is smoothly indented with a flat-ended cylindrical indenter. The analytical and exact solutions were obtained by solving an infinite system of simultaneous equations using the method to express a normal contact stress at the upper surface of the elastic layer as an appropriate series. This paper presented the numerical results of internal stress and displacement distributions for hard-coating system with constant values of Poisson's ratio and the thickness of elastic layer.

***Keywords*—**Indentation, contact problem, stress distribution, coating materials, layer-substrate body.

I. Introduction

THE coating technology has been used to reduce friction and improve surface materials. Fracture of the coating materials occur due to delamination and spalling at the boundary between coating layer and the substrate. Therefore, the analysis of internal stress distributions of contact problem is very important.

The contact problem of elastic layer-substrate body has been studied by many researches [1]-[5]. However, they focused on applied load of rigid indenter and the distribution of contact stress and not internal stress distribution.

In the present study, the axisymmetric elastic contact problem for a layer-substrate composite consisting of an elastic layer perfectly bonded to an elastic semi-infinite substrate indented by a rigid flat-ended cylindrical indenter is considered. Instead of using the Fredholm integral equation [1], [2], this study obtained an analytical solution by solving an infinite system of simultaneous equations using a method to express the normal contact stress at the upper surface of the elastic layer as an appropriate series [6], [7]. Numerical results of internal stress and displacement distributions were given for hard-coating system, and Poisson's ratio and the thickness of elastic layer were constants.

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II. Formulation of the Problem

Consider the problem where a composite material consisting of an elastic layer perfectly bonded to an elastic semi-infinite substrate is indented by a rigid flat-ended cylindrical indenter, as shown in Fig. 1. A cylindrical coordinate system (r, θ, z) is used in this study. Displacement components along r, θ and z are denoted by u_r, v_θ and w_z, respectively. Components of the stress tensor are expressed by σ_r, σ_θ, σ_z, τ_{rz}, $\tau_{r\theta}$ and $\tau_{\theta z}$. A general solution of the equilibrium equations for the elastic layer and substrate without torsion can be derived using harmonic stress functions ${\varphi^{(i)}}_0$ and ${\varphi^{(i)}}_3$ (i = 1, 2), i.e. [8], where superscripts 1 and 2 represent the layer and substrate, respectively.

$$
\begin{aligned}
&2G_i {u_r}^{(i)} = \frac{\partial {\varphi_0}^{(i)}}{\partial r} + z\frac{\partial {\varphi_3}^{(i)}}{\partial r}\\
&{v_\theta}^{(i)} = 0\\
&2G_i {w_z}^{(i)} = \frac{\partial {\varphi_0}^{(i)}}{\partial z} + z\frac{\partial {\varphi_3}^{(i)}}{\partial z} - (3-4\nu_i){\varphi_3}^{(i)}\\
&{\sigma_r}^{(i)} = \frac{\partial^2 {\varphi_0}^{(i)}}{\partial r^2} + z\frac{\partial^2 {\varphi_3}^{(i)}}{\partial r^2} - 2\nu_i \frac{\partial {\varphi_3}^{(i)}}{\partial z}\\
&{\sigma_\theta}^{(i)} = \frac{\partial {\varphi_0}^{(i)}}{r\partial r} + z\frac{\partial {\varphi_3}^{(i)}}{r\partial r} - 2\nu_i \frac{\partial {\varphi_3}^{(i)}}{\partial z}\\
&{\sigma_z}^{(i)} = \frac{\partial^2 {\varphi_0}^{(i)}}{\partial z^2} + z\frac{\partial^2 {\varphi_3}^{(i)}}{\partial z^2} - 2(1-\nu_i)\frac{\partial {\varphi_3}^{(i)}}{\partial z}\\
&{\tau_{rz}}^{(i)} = \frac{\partial^2 {\varphi_0}^{(i)}}{\partial r\partial z} + z\frac{\partial^2 {\varphi_3}^{(i)}}{\partial r\partial z} - (1-2\nu_i)\frac{\partial {\varphi_3}^{(i)}}{\partial r}\\
&{\tau_{r\theta}}^{(i)} = {\tau_{\theta z}}^{(i)} = 0\\
&(i = 1, 2)
\end{aligned}
\tag{1}
$$

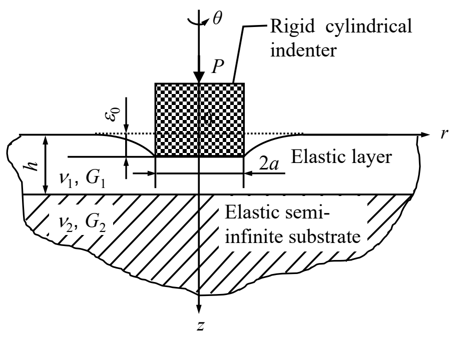

Fig. 1 An elastic layer perfectly bonded to an elastic semi-infinite substrate indented by a rigid cylindrical indenter

G_i and ν_i (i =1, 2) denote the shear modulus and Poisson's ratios of the layer and substrate, respectively. The harmonic functions $\varphi^{(i)}{}_0$ and $\varphi^{(i)}{}_3$ (i = 1, 2) can be written as

$$\begin{aligned}\varphi_0^{(1)} &= \int_0^\infty \left\{D^{(1)}(\lambda)\cosh\lambda z + A^{(1)}(\lambda)\sinh\lambda z\right\}J_0(\lambda r)\mathrm{d}\lambda \\ \varphi_3^{(1)} &= \int_0^\infty \left\{B^{(1)}(\lambda)\sinh\lambda z + C^{(1)}(\lambda)\cosh\lambda z\right\}J_0(\lambda r)\mathrm{d}\lambda\end{aligned} \tag{2}$$

$$\begin{aligned}\varphi_0^{(2)} &= \int_0^\infty A^{(2)}(\lambda)J_0(\lambda r)e^{-\lambda z}\mathrm{d}\lambda \\ \varphi_3^{(2)} &= \int_0^\infty \{B^{(2)}(\lambda)J_0(\lambda r)e^{-\lambda z}\mathrm{d}\lambda\end{aligned} \tag{3}$$

where $J_n(\lambda r)$ is the Bessel function of the first kind of order n and, $A^{(1)}(\lambda)$, $B^{(1)}(\lambda)$, $C^{(1)}(\lambda)$, $D^{(1)}(\lambda)$, $A^{(2)}(\lambda)$ and $B^{(2)}(\lambda)$ are unknown functions that can be determined from the boundary conditions.

If shear tractions between the indenter and layer are assumed to be negligible, then the boundary conditions of the upper surface of the layer indented by cylindrical indenter with constant penetration depth can be described by:

$$(w_z)_{z=0}^{(1)} = \varepsilon_0, \quad (0 \le r \le a) \tag{4}$$

$$(\sigma_z)_{z=0}^{(1)} = 0, \quad (a < r < \infty) \tag{5}$$

$$(\tau_{rz})_{z=0}^{(1)} = 0, \quad (0 \le r < \infty) \tag{6}$$

The elastic layer is perfectly bonded to the semi-infinite substrate; therefore, the continuity conditions of the components of displacement and traction at the interface $z = h$ can be written in the following form:

$$\left[\{S^{(1)}\} = \{S^{(2)}\}\right]_{z=h} \tag{7}$$

where

$$\left\{S^{(i)}\right\} = \left[u_r^{(i)} \quad w_z^{(i)} \quad \sigma_z^{(i)} \quad \tau_{rz}^{(i)}\right]^T \tag{8}$$

By substituting (2) and (3) into the equilibrium equations (1) and using the boundary conditions of (4)-(8), the unknown functions can be reduced to one and we obtain following dual integral equations:

$$(w_z)_{z=0}^{(1)} = -\frac{1-\nu_1}{G_1}\int_0^\infty C^{(1)}(\lambda)J_0(\lambda r)\mathrm{d}\lambda = \varepsilon_0 - f(r), \quad (0 \le r < a) \tag{9}$$

$$(\sigma_z)_{z=0}^{(1)} = \int_0^\infty \frac{e_3(\lambda)}{e_0(\lambda)}C^{(1)}(\lambda)\lambda J_0(\lambda r)\mathrm{d}\lambda = 0, \quad (a < r < \infty) \tag{10}$$

where

$$\begin{aligned}
&e_0(\lambda) = a(\lambda)\Gamma^2 + b(\lambda)\Gamma + c(\lambda)\\
&e_3(\lambda) = d(\lambda)\Gamma^2 + e(\lambda)\Gamma + f(\lambda)\\
&a(\lambda) = (4\nu_2 - 3)e^{4\beta} + (16\nu_2 - 12)\beta e^{2\beta} - 4\nu_2 + 3\\
&b(\lambda) = \{(12 - 16\nu_1)\nu_2 + 12\nu_1 - 10\}e^{4\beta} + (8 - 16\nu_2)\beta e^{2\beta}\\
&\qquad + 4\nu_2 + 4\nu_1 - 6\\
&c(\lambda) = (4\nu_1 - 3)e^{4\beta} + 4\beta e^{2\beta} - 4\nu_1 + 3\\
&d(\lambda) = (4\nu_2 - 3)e^{4\beta} + \{(12 - 16\nu_2)\beta^2 - 8\nu_2 + 6\}e^{2\beta}\\
&\qquad + 4\nu_2 - 3\\
&e(\lambda) = \{(12 - 16\nu_1)\nu_2 + 12\nu_1 - 10\}e^{4\beta}\\
&\qquad + \{(16\nu_2 - 8)\beta^2 + (16\nu_1 - 8)\nu_2 - 8\nu_1 + 4\}e^{2\beta} - 4\nu_2 - 4\nu_1 + 6\\
&f(\lambda) = (4\nu_1 - 3)e^{4\beta} + (-4\beta^2 - 16\nu_1^2 + 24\nu_1 - 10)e^{2\beta} + 4\nu_1 - 3
\end{aligned} \tag{11}$$

The dual integral equations (9) and (10) are usually transformed into a Fredholm equation of the second kind [1], [2]. In this study, the normal contact stress between the indenter and the layer surface was expressed as an appropriate series function that contains Tchebycheff polynomials $T_n(x)$, and a Hankel inversion was applied so that the problem finally reduced to following an infinite system of simultaneous equations:

$$\sum_{n=0}^{\infty} b_n A_{mn} = \delta_{0m}, \quad (m = 0, 1, 2, \ldots) \tag{12}$$

where b_n (n = 0, 1, 2, …), δ_{0m} are unknown coefficients matrix and the Kronecker delta function, and A_{mn} is given by:

$$A_{mn} = \int_0^\infty p(\lambda)X_m(\lambda)Z_n(\lambda)\mathrm{d}\lambda \tag{13}$$

$$p(\lambda) = \frac{e_0(\lambda)}{e_3(\lambda)} \tag{14}$$

$$X_m(\lambda) = J_m^2(\lambda a/2), \quad (m = 0, 1, 2, \ldots) \tag{15}$$

$$Z_n(\lambda) = J_{n+1/2}\left(\frac{\lambda a}{2}\right)J_{-n-1/2}\left(\frac{\lambda a}{2}\right), \quad (n = 0, 1, 2, \ldots) \tag{16}$$

Detailed techniques of numerical calculation of (12) are summarized in [7].

By calculating the infinite system of simultaneous equations (12), the unknown functions $A^{(1)}(\lambda)$, $B^{(1)}(\lambda)$, $C^{(1)}(\lambda)$, $D^{(1)}(\lambda)$, $A^{(2)}(\lambda)$ and $B^{(2)}(\lambda)$ can be determined. Therefore, it is possible to estimate the stress and displacement components at any position in the layer-substrate body.

The stress and displacement components of elastic layer given by:

$$\begin{aligned}
(\sigma_z)^{(1)} &= \int_0^\infty \left[\eta_z^{(1)}(\lambda, z)\right]\lambda J_0(\lambda r)\mathrm{d}\lambda\\
(\tau_{rz})^{(1)} &= \int_0^\infty \left[\eta_{rz}^{(1)}(\lambda, z)\right]\lambda J_1(\lambda r)\mathrm{d}\lambda\\
2G_1(w_z)^{(1)} &= \int_0^\infty \left[\zeta_z^{(1)}(\lambda, z)\right]J_0(\lambda r)\mathrm{d}\lambda\\
2G_1(u_r)^{(1)} &= \int_0^\infty \left[\zeta_r^{(1)}(\lambda, z)\right]J_1(\lambda r)\mathrm{d}\lambda
\end{aligned} \tag{17}$$

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where

$$
\begin{aligned}
\eta_z^{(1)} &= \{z\lambda \sinh \lambda z - 2(1-\nu_1)\cosh \lambda z\} B^{(1)}(\lambda) \\
&\quad + \{z\lambda \cosh \lambda z - \sinh \lambda z\} C^{(1)}(\lambda) + \lambda D^{(1)}(\lambda)\cosh \lambda z \\
\eta_{rz}^{(1)} &= \{-z\lambda \cosh \lambda z + (1-2\nu_1)\sinh \lambda z\} B^{(1)}(\lambda) \\
&\quad + \{-z\lambda \sinh \lambda z\} C^{(1)}(\lambda) - \lambda D^{(1)}(\lambda)\sinh \lambda z \\
\zeta_z^{(1)} &= \{z\lambda \cosh \lambda z - (3-4\nu_1)\sinh \lambda z\} B^{(1)}(\lambda) \\
&\quad + \{z\lambda \sinh \lambda z - 2(1-\nu_1)\cosh \lambda z\} C^{(1)}(\lambda) + \lambda D^{(1)}(\lambda)\sinh \lambda z \\
\zeta_r^{(1)} &= z\lambda B^{(1)}(\lambda)\sinh \lambda z - \{z\lambda \cosh \lambda z + (1-2\nu_1)\sinh \lambda z\} z\lambda C^{(1)}(\lambda) \\
&\quad - \lambda D^{(1)}(\lambda)\cosh \lambda z
\end{aligned}
\tag{18}
$$

For the elastic substrate, the stress and displacement are given by:

$$
\begin{aligned}
(\sigma_z)^{(2)} &= \int_0^\infty \left[\eta_z^{(2)}(\lambda, z)\right] e^{-\lambda z} \lambda J_0(\lambda r) \mathrm{d}\lambda \\
(\tau_{rz})^{(2)} &= \int_0^\infty \left[\eta_{rz}^{(2)}(\lambda, z)\right] e^{-\lambda z} \lambda J_1(\lambda r) \mathrm{d}\lambda \\
2G_2 (w_z)^{(2)} &= \int_0^\infty \left[\zeta_z^{(2)}(\lambda, z)\right] e^{-\lambda z} J_0(\lambda r) \mathrm{d}\lambda \\
2G_2 (u_r)^{(2)} &= \int_0^\infty \left[\zeta_r^{(2)}(\lambda, z)\right] e^{-\lambda z} J_1(\lambda r) \mathrm{d}\lambda
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
\eta_z^{(2)} &= \lambda A^{(2)}(\lambda) + \{z\lambda + 2(1-\nu_2)\} B^{(2)}(\lambda) \\
\eta_{rz}^{(2)} &= \lambda A^{(2)}(\lambda) + \{z\lambda + (1-2\nu_2)\} B^{(2)}(\lambda) \\
\zeta_z^{(2)} &= -\lambda A^{(2)}(\lambda) - \{z\lambda + (3-4\nu_2)\} B^{(2)}(\lambda) \\
\zeta_r^{(2)} &= -\lambda A^{(2)}(\lambda) - z\lambda B^{(2)}(\lambda)
\end{aligned}
\tag{20}
$$

In this study, the unknown functions reduced into $C^{(1)}(\lambda)$ in (18) and (20) by using computer algebra software, which is wxMaxima (MIT). After that, numerical calculations of (17) and (19) are conducted to obtain internal distributions of stress and displacement.

The normal contact stress $(\sigma_z)^{(1)}{}_{z=0}$ and normal displacement $(w_z)^{(1)}{}_{z=0}$ can be written in the following form [7]:

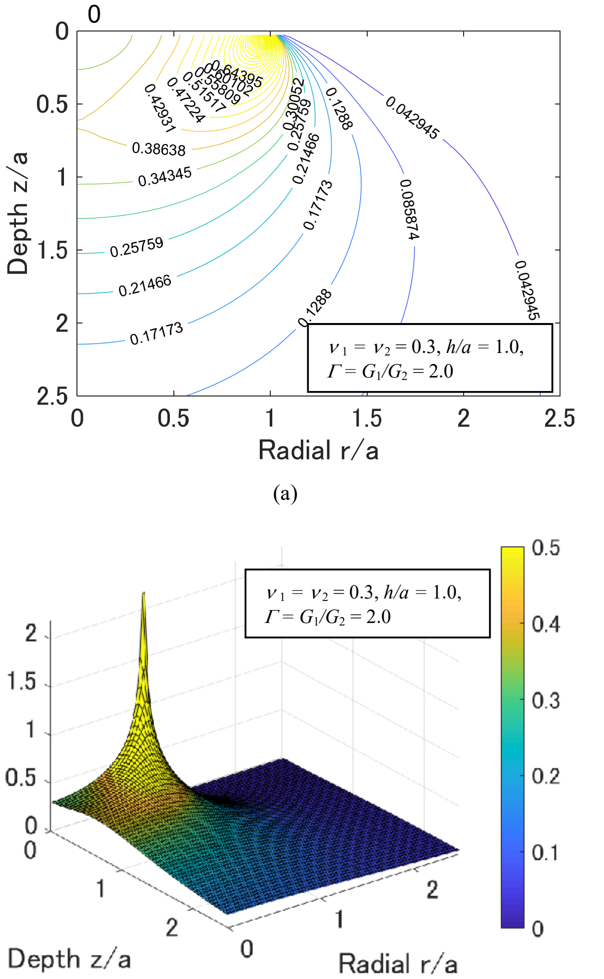
$$
(\sigma_z)_{z=0}^{(1)} = -\frac{2G_1\varepsilon_0}{(1-\nu_1)\pi r\sqrt{a^2-r^2}} \sum_{n=0}^{\infty} b_n T_{2n+1}\left(\frac{r}{a}\right), \quad (0 \le r < a), \tag{21}
$$

$$
\begin{aligned}
\frac{(w_z)_{z=0}^{(1)}}{\varepsilon_0} &= \sum_{n=0}^{\infty} b_n \left[\int_0^\infty \{p(\lambda)-1\} Z_n(\lambda) J_0(\lambda r) d\lambda \right. \\
&\quad + \int_0^\infty \left\{ Z_n(\lambda) - \frac{2}{\pi\lambda a}\sin\lambda a \right\} J_0(\lambda r) d\lambda \\
&\quad \left. + \frac{2}{\pi a}\left\{ H(a-r)\frac{\pi}{2} + H(r-a)\sin^{-1}\left(\frac{a}{r}\right) \right\} \right]
\end{aligned}
\tag{22}
$$

where $H(x)$ is the Heaviside unit step function.

III. Numerical Results and Discussion

The numerical results present internal distributions of stress and displacement. Numerical results were given for Poisson's ratio of the layer and substrate $\nu_1 = \nu_2 = 0.3$ and layer thickness ratio $h/a = 1.0$, and the ratio of the shear modulus of the layer and substrate $\Gamma = G_1/G_2 = 2.0$, which means hard-coating system.

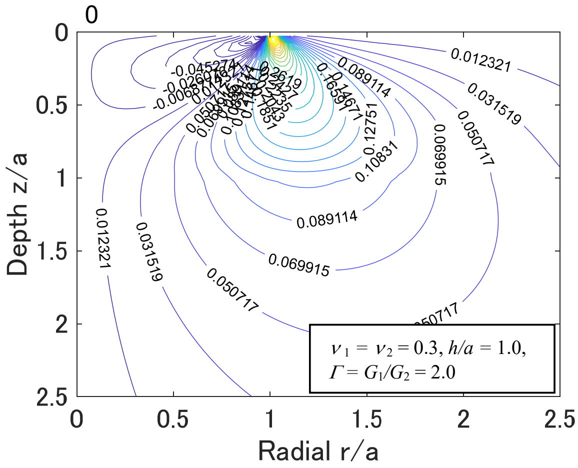


(a)

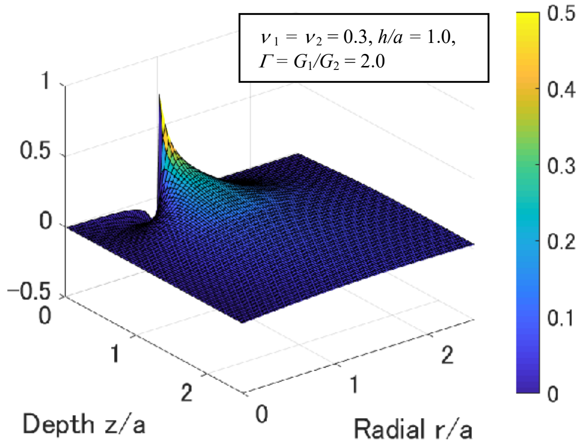
(b)

Fig. 2 (a) 2D contour and (b) 3D plot of $\bar{\sigma}_z = -\sigma_z \Big/ \left(\frac{P_1}{4a^2}\right)$

Figs. 2 and 3 show the internal distribution of σ_z and τ_{rz} normalized by $-P_1/4a^2$ in the layer-substrate body, respectively. In Fig. 2, σ_z has a singularity at the tip of cylindrical indenter. Furthermore, it can be seen that the results are reasonable because the value of σ_z is continuous at the boundary between layer and substrate, $z/a = 1.0$. As shown in Fig. 3, it seems that the discontinuity of the value of τ_{rz} at the boundary between layer and substrate, $z/a = 1.0$. It is caused by the continuous conditions between layer and substrate, which is not taking the continuous of the gradient of stress and displacement into account. Figs. 4 and 5 show internal distribution of w_z and u_r normalized by $-P_1/4G_1$ in the layer- substrate body, respectively. As shown in Fig. 4, the constant penetration depth on the surface of the layer is confirmed. Furthermore, it can be seen that the normalized u_r has a valley in the internal of layer-substrate body in Fig. 5. It is noted that the color bar in Figs. 2-5 truncate the value as appropriate.

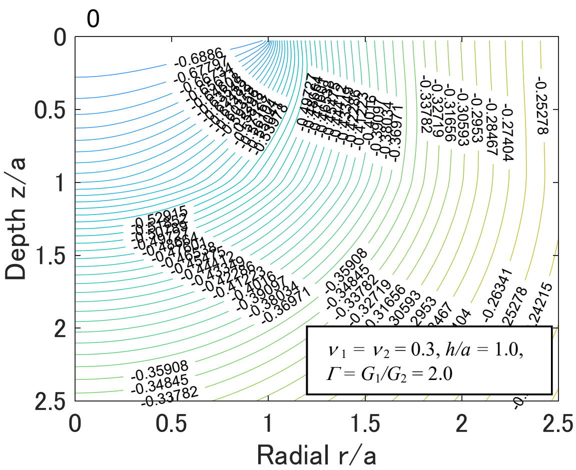


(a)

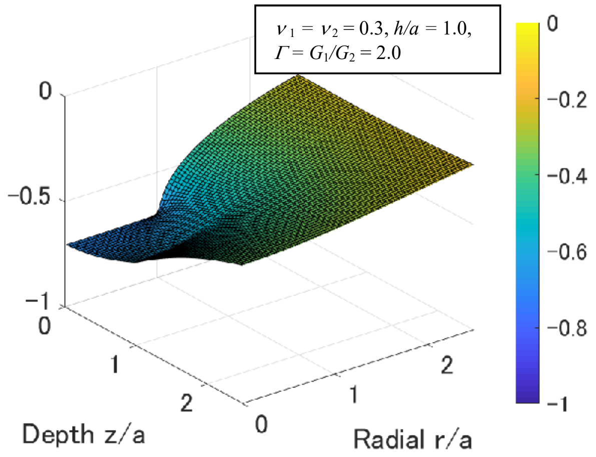


(b)

Fig. 3 (a) 2D contour and (b) 3D plot of $\bar{\tau}_{rz} = -\tau_{rz} \Big/ \left(\dfrac{P_1}{4a^2}\right)$

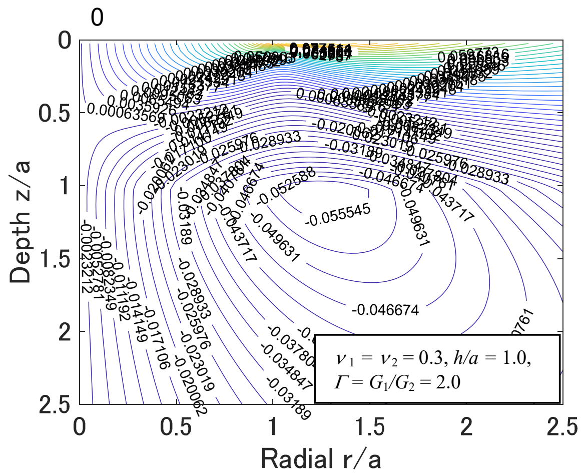


(a)

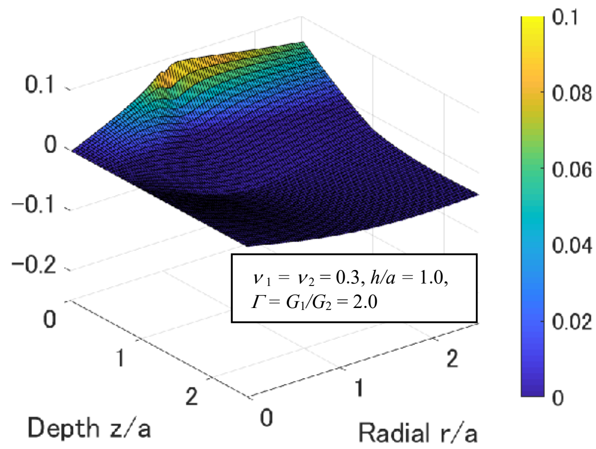


(b)

Fig. 4 (a) 2D contour and (b) 3D plot of $\bar{w}_z = -w_z \Big/ \left(\dfrac{P_1}{4G_1}\right)$



(a)



(b)

Fig. 5 (a) 2D contour and (b) 3D plot of $\bar{u}_r = -u_r \Big/ \left(\dfrac{P_1}{4G_1}\right)$

IV. Conclusion

The internal distributions of stress and displacement for an elastic layer-substrate body indented by the rigid cylindrical indenter have been calculated. The numerical results were based on analytical solution of an infinite system of simultaneous equations obtained by expressing the normal contact stress at the upper surface of the elastic layer as an appropriate series.

The numerical results of distributions of stress and displacement in this study were reasonable. Therefore, this study could provide basis understanding of the coating technology by investigating the effects of mechanical properties of elastic layer and substrate on the stress and displacement distributions. Especially, the distribution of shear stress τ_{rz} at the interface between layer and substrate is important for interfacial delamination and spalling.

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