

# Stochastic Resonance in Nonlinear Signal Detection

Youguo Wang, and Lenan Wu

**Abstract**—Stochastic resonance (SR) is a phenomenon whereby the signal transmission or signal processing through certain nonlinear systems can be improved by adding noise. This paper discusses SR in nonlinear signal detection by a simple test statistic, which can be computed from multiple noisy data in a binary decision problem based on a maximum a posteriori probability criterion. The performance of detection is assessed by the probability of detection error  $P_{er}$ . When the input signal is subthreshold signal, we establish that benefit from noise can be gained for different noises and confirm further that the subthreshold SR exists in nonlinear signal detection. The efficacy of SR is significantly improved and the minimum of  $P_{er}$  can dramatically approach to zero as the sample number increases. These results show the robustness of SR in signal detection and extend the applicability of SR in signal processing.

**Keywords**—Probability of detection error, signal detection, stochastic resonance.

## I. INTRODUCTION

STOCHASTIC resonance (SR) is a phenomenon whereby the signal transmission or signal processing through certain nonlinear systems can be improved by adding noise. Most occurrences of SR involve a signal, which is subthreshold (weak signal) and is too weak to elicit a strong response from a single nonlinear system. Addition of noise then brings assistance to the subthreshold signal in eliciting a stronger beneficial response from the single nonlinear system [1]-[3], SR has been observed in a diverse range of physical and biological systems, including neurons and neuron models. The study of SR in signal detection has also received some attentions [4]-[11]. This paper discuss SR by a simple test statistic, which can be computed from multiple noisy data in a binary decision problem based on a maximum a posteriori probability (MAP) criterion. The performance of detection is assessed by the probability of detection error  $P_{er}$ . Recently, an equivalent test statistic has been used to improve signal detection in non-Gaussian noise [12]. We calculate the detection error  $P_{er}$  for four representative noises. We establish that benefit from noise can be gained and confirm further that the subthreshold SR exists in nonlinear signal

detection. When the sample number is raised, the efficacy of SR is significantly improved and the minimum of  $P_{er}$  can dramatically approach to zero. These results show the robustness of SR in signal detection and extend the applicability of SR in signal processing.

## II. NONLINEAR SIGNAL DETECTION

Let  $x$  is a random signal which assumes values  $s_1$  (hypothesis  $H_1$ ) or  $s_0$  (hypothesis  $H_0$ ) with the prior probability  $P_1$  and  $P_0 = 1 - P_1$ ,  $\eta$  is threshold white noise,  $u$  is a fixed threshold level,  $y$  is a binary output:  $y = \begin{cases} 1 & \text{if } x + \eta > u \\ 0 & \text{if } x + \eta < u \end{cases}$ . The binary output  $y$  is sampled so as to yield the binary data set  $(y_1, y_2, \dots, y_N)$ . We want to use these data  $(y_1, y_2, \dots, y_N)$  to detect  $x$ . Recently, an important

and simple test statistic:  $\bar{y} = \frac{1}{N} \sum_{i=1}^N y_i$  has been used to improve signal detection in non-Gaussian noise [12]. Here, we use an equivalent test statistic  $y = \sum_{i=1}^N y_i$ , which assumes integer values between 0 and  $N$  ( $y \in R = \{0, 1, \dots, N\}$ ). A given detector will decide hypothesis  $H_0$  whenever the observation  $y$  falls in region  $R_0$  ( $\subset R$ ) or decide  $H_1$  when  $y$  falls in the complementary region  $R_1$  ( $R_0 \cup R_1 = R$ ). In doing so, the detector achieves the probability of detection error  $P_{er}$  given by:

$$P_{er} = P_1 \sum_{n \in R_0} P_r(y = n | H_1) + P_0 \sum_{n \in R_1} P_r(y = n | H_0). \quad (1)$$

Where  $\Pr(y = n | H_1)$  (respectively  $\Pr(y = n | H_0)$ ) is the conditional probability for observing  $y$  when  $H_1$  (respectively  $H_0$ ) holds. Since  $R_0$  and  $R_1$  are complementary, so we have

$$\sum_{n \in R_0} \Pr(y = n | H_1) = 1 - \sum_{n \in R_1} \Pr(y = n | H_1). \quad (2)$$

Substituted in (1) yields

$$P_{er} = P_1 + \sum_{n \in R_1} [P_0 \Pr(y = n | H_0) - P_1 \Pr(y = n | H_1)]. \quad (3)$$

To minimize  $P_{er}$ , we let all and only those  $y$ , which make  $P_0 \Pr(y = n | H_0) - P_1 \Pr(y = n | H_1)$  be negative, compose  $R_1$ . This gives the optimal detector, also known as the maximum a posterior probability (MAP) detector, which implements the

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test  $L(y = n) = \frac{P_1}{P_0} \frac{\Pr(y = n | H_1)}{\Pr(y = n | H_0)}$ , by use of the likelihood ratio  $L(y = n) = \frac{\Pr(y = n | H_1)}{\Pr(y = n | H_0)}$ . The minimal *Per* reached by

the MAP detector of (3) is expressed as:  
 $P_{er} = \sum_{n \in R} \min[P_0 \Pr(y = n | H_0), P_1 \Pr(y = n | H_1)]$   
 $= \frac{1}{2} - \frac{1}{2} \sum_{n \in R} |P_0 \Pr(y = n | H_0) - P_1 \Pr(y = n | H_1)|$ . (4)

The second equality is by  $\min(a, b) = \frac{1}{2}(a + b) - \frac{1}{2}|a - b|$ .

### III. SR AND NOISE IMPROVE SIGNAL DETECTION

Given the white noise  $\eta$  is with probability density function  $f_\eta(x)$  and cumulative distribution function  $F_\eta(x)$ . Note that  $y_i$  is a Bernoulli random variable, taking on values zero and one with Bernoulli probabilities, and  $y$  is binomially distributed. The Bernoulli probabilities depend on the hypothesis  $H_0$  or  $H_1$ . We have the conditional probabilities  $\Pr(y_i = 0 | H_0) = \Pr(\eta + s_0 < u) = F_\eta(u - s_0) = q_0$  and  $\Pr(y_i = 1 | H_0) = 1 - q_0$ ,  $\Pr(y_i = 0 | H_1) = \Pr(\eta + s_1 < u) = F_\eta(u - s_1) = q_1$  and  $\Pr(y_i = 1 | H_1) = 1 - q_1$ . The conditional probability  $\Pr(y = n | H_0)$  and  $\Pr(y = n | H_1)$  then follow, according to the binomial distribution, as

$$\Pr(y = n | H_0) = \binom{N}{n} (1 - q_0)^n q_0^{N-n}, \text{ and} \quad (5)$$

$$\Pr(y = n | H_1) = \binom{N}{n} (1 - q_1)^n q_1^{N-n}. \quad (6)$$

Where  $\binom{N}{n}$  is the binomial coefficient. The probability of error

*Per* of (4) follows directly from (5) and (6) for a specific noise with density function  $f_\eta(x)$  and cumulative distribution  $F_\eta(x)$ . Here, we discuss four representative noises whose density function is with zero symmetry axes: Uniform noise (finite support PDF), Gaussian noise (thin-tailed PDF), Laplace noise (heavy-tailed PDF), and Cauchy noise (impulsive PDF without mean and variance).

#### A. Uniform Noise

The Uniform PDF with zero mean and variance  $\sigma^2$  has the form  $f(x) = \begin{cases} \frac{1}{\sqrt{12}\sigma} & -\frac{\sqrt{12}\sigma}{2} < x < \frac{\sqrt{12}\sigma}{2} \\ 0 & \text{otherwise} \end{cases}$ . Then

$$q_0 = F_\eta(u - s_0) = \int_{-\infty}^{u-s_0} f(x)dx = \begin{cases} 0 & u - s_0 < -\sqrt{3}\sigma \\ \frac{u - s_0 + \sqrt{3}\sigma}{\sqrt{12}\sigma} & -\sqrt{3}\sigma < u - s_0 < \sqrt{3}\sigma, \\ 1 & \sqrt{3}\sigma < u - s_0 \end{cases}, \quad (7)$$

$$q_1 = F_\eta(u - s_1) = \int_{-\infty}^{u-s_1} f(x)dx = \begin{cases} 0 & u - s_1 < -\sqrt{3}\sigma \\ \frac{u - s_1 + \sqrt{3}\sigma}{\sqrt{12}\sigma} & -\sqrt{3}\sigma < u - s_1 < \sqrt{3}\sigma. \\ 1 & \sqrt{3}\sigma < u - s_1 \end{cases}. \quad (8)$$

#### B. Gaussian Noise

The Gaussian PDF with zero mean and variance  $\sigma^2$  has the form  $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{x^2}{2\sigma^2})$ . Then

$$q_0 = F_\eta(u - s_0) = \int_{-\infty}^{u-s_0} f(x)dx = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{u - s_0}{\sqrt{2}\sigma}\right), \quad (9)$$

$$q_1 = F_\eta(u - s_1) = \int_{-\infty}^{u-s_1} f(x)dx = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{u - s_1}{\sqrt{2}\sigma}\right). \quad (10)$$

#### C. Laplace Noise

The Laplace PDF with zero mean and variance  $\sigma^2$  has the form

$$f(x) = \frac{1}{\sqrt{2}\sigma} \exp\left(-\frac{\sqrt{2}|x|}{\sigma}\right). \text{ Then}$$

$$q_0 = F_\eta(u - s_0) = \int_{-\infty}^{u-s_0} f(x)dx = \begin{cases} \frac{1}{2} \exp\left(\frac{\sqrt{2}(u - s_0)}{\sigma}\right) & u - s_0 < 0 \\ 1 - \frac{1}{2} \exp\left(-\frac{\sqrt{2}(u - s_0)}{\sigma}\right) & u - s_0 > 0 \end{cases}, \quad (11)$$

$$q_1 = F_\eta(u - s_1) = \int_{-\infty}^{u-s_1} f(x)dx = \begin{cases} \frac{1}{2} \exp\left(\frac{\sqrt{2}(u - s_1)}{\sigma}\right) & u - s_1 < 0 \\ 1 - \frac{1}{2} \exp\left(-\frac{\sqrt{2}(u - s_1)}{\sigma}\right) & u - s_1 > 0 \end{cases}. \quad (12)$$

D. Cauchy Noise

The Cauchy PDF with zero location and finite dispersion  $\sigma^2$  (but infinite variance) has the form

$$f(x) = \frac{\sigma}{\pi(x^2 + \sigma^2)}. \text{ Then}$$

$$q_0 = F_\eta(u - s_0) = \int_{-\infty}^{u-s_0} f(x)dx$$

$$= \frac{1}{2} + \frac{1}{\pi} \arctan \frac{u - s_0}{\sigma}, \tag{13}$$

$$q_1 = F_\eta(u - s_1) = \int_{-\infty}^{u-s_1} f(x)dx$$

$$= \frac{1}{2} + \frac{1}{\pi} \arctan \frac{u - s_1}{\sigma}. \tag{14}$$

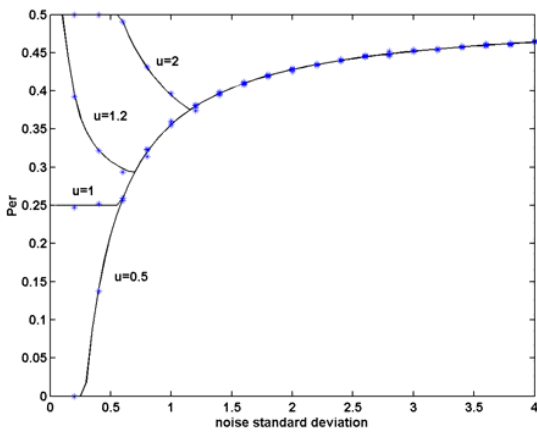


Fig. 1 *Per* is a function of standard deviation for different threshold levels for Uniform noise with zero mean

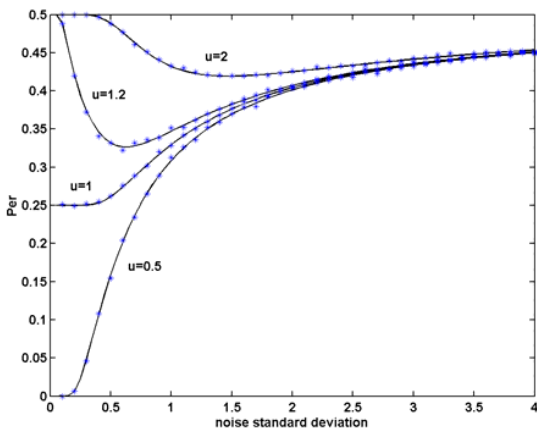


Fig. 2 *Per* is a function of standard deviation for different threshold levels for Gaussian noise with zero mean

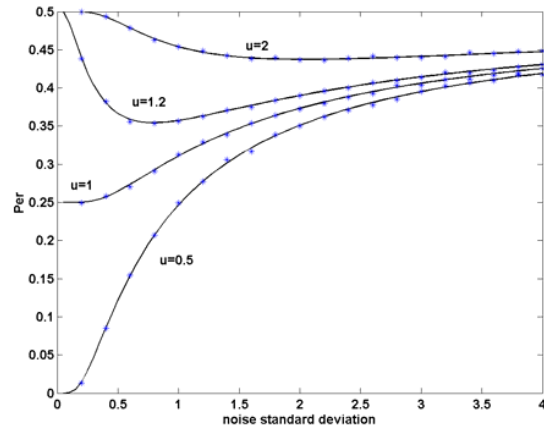


Fig. 3 *Per* is a function of standard deviation for different threshold levels for Laplace noise with zero mean

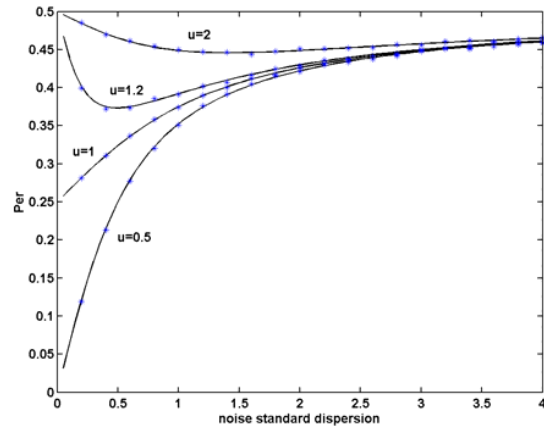


Fig. 4 *Per* is a function of standard dispersion for different threshold levels for Cauchy noise with zero mean

For different noise distributions, Figs. 1-4 show the variation of *Per* for different threshold levels and give the Monte Carlo computer simulation ( data points), where the parameters are  $s_0 = 0, s_1 = 1$  and  $P_0 = P_1 = 0.5$ . When the input signal is not subthreshold signal, *Per* increases monotonously from zero at zero noise intensity (standard deviation or dispersion). This show the performance of detection degenerates with the addition of noise. However, when the input signal is subthreshold signal (weak signal), SR is observed, *Per* decreases from initial value 0.5 (because  $P_0 = P_1 = 0.5$ ) at zero noise intensity to a minimum, where the noise intensity is optimal, and then increases. This show the performance of detection is improved with the addition of noise.

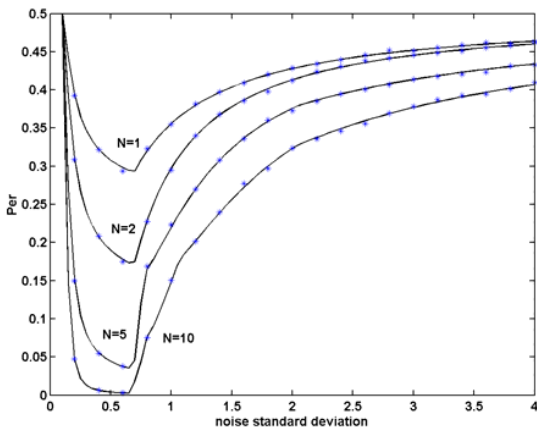


Fig. 5 *Per* is a function of standard deviation for different sample numbers for Uniform noise with zero mean

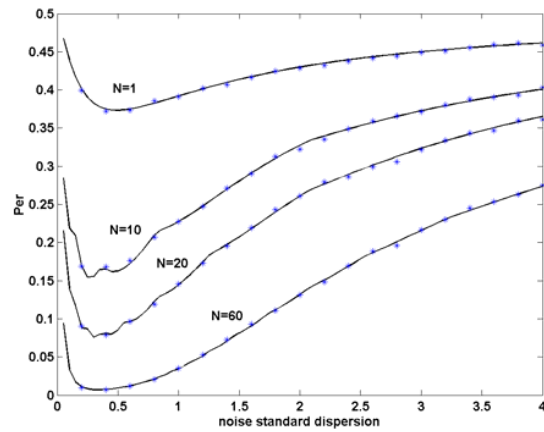


Fig. 8 *Per* is a function of standard deviation for different sample numbers for Cauchy noise with zero mean

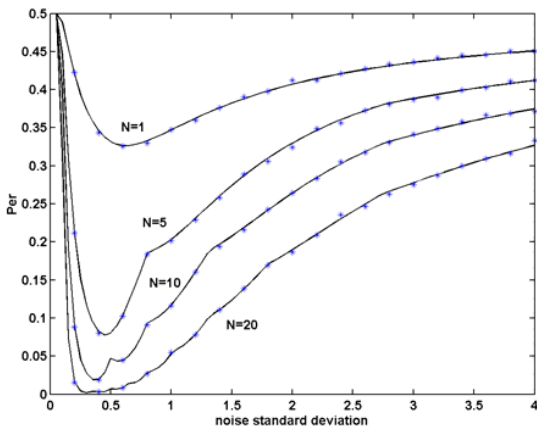


Fig. 6 *Per* is a function of standard deviation for different sample numbers for Gaussian noise with zero mean

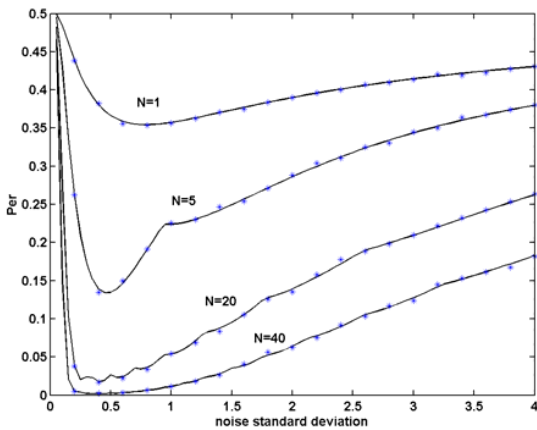


Fig. 7 *Per* is a function of standard deviation for different sample numbers for Laplace noise with zero mean

For a fixed threshold level  $u = 1.2$ , Figs. 5-8 give the theoretical results and the Monte Carlo computer simulation results of *Per* for different sample number  $N$ . The efficacy of SR is significantly enhanced and the minimum of *Per* can dramatically approach to zero as the sample number increases. The rate of approaching to zero is fastest for Uniform noise and is slowest for impulsive Cauchy noise.

#### IV. DISCUSSION

In this section, we shall simply discuss why SR is observed or not, and why the rate of approaching to zero is different for four different noises.

When  $N = 1$  and  $P_0 = P_1 = 0.5$ , equation (4) becomes

$$\begin{aligned}
 P_{er} &= \frac{1}{2} - \frac{1}{4} \sum_{n \in \{0,1\}} |\Pr(y = n | H_0) - \Pr(y = n | H_1)| \\
 &= \frac{1}{2} - \frac{1}{2} |\Pr(y = 0 | H_0) - \Pr(y = 0 | H_1)| \\
 &= \frac{1}{2} - \frac{1}{2} |q_0 - q_1| \\
 &= \frac{1}{2} - \frac{1}{2} F_{\eta}(u - s_1 < \eta < u - s_0). \tag{15}
 \end{aligned}$$

For fixed threshold level  $u = 0.5$  and  $u = 1.2$ , Figs. 9-12 present a similar change: the areas under the noise PDF between line  $a$  and line  $b$  decreases monotonously, which results that the *Per* increases monotonously and SR is not observed by (13). However, the areas under the noise PDF between line  $c$  and line  $d$  firstly increases and then decreases as the noise intensity increases, which result *Per* varies nonmonotonously and SR is observed by (13).

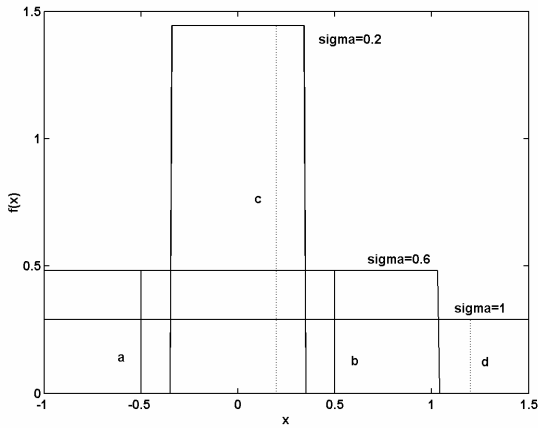


Fig. 9 Uniform noise PDF between  $a = -0.5$  and  $b = 0.5$ , and between  $c = 0.2$  and  $d = 1.2$  for different noise intensity (sigma)

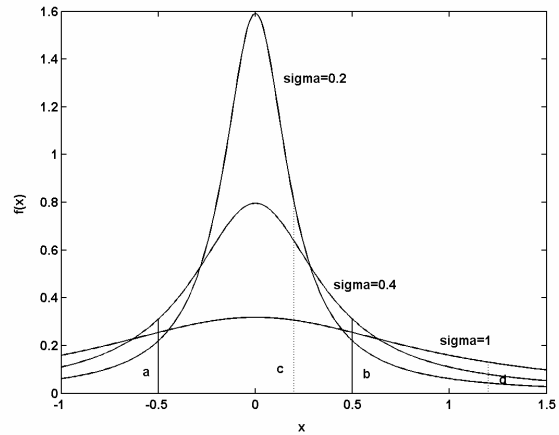


Fig. 12 Cauchy noise PDF between  $a = -0.5$  and  $b = 0.5$ , and between  $c = 0.2$  and  $d = 1.2$  for different noise intensity (sigma)

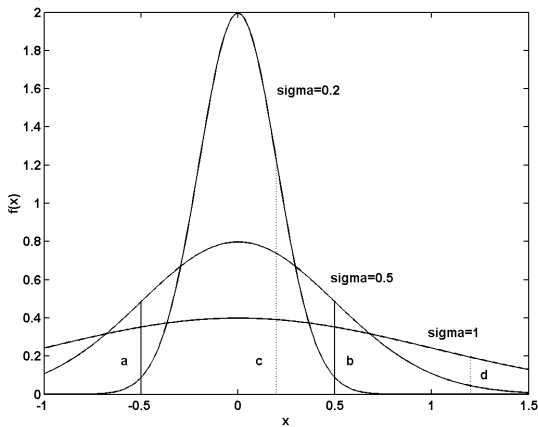


Fig. 10 Gaussian noise PDF between  $a = -0.5$  and  $b = 0.5$ , and between  $c = 0.2$  and  $d = 1.2$  for different noise intensity (sigma)

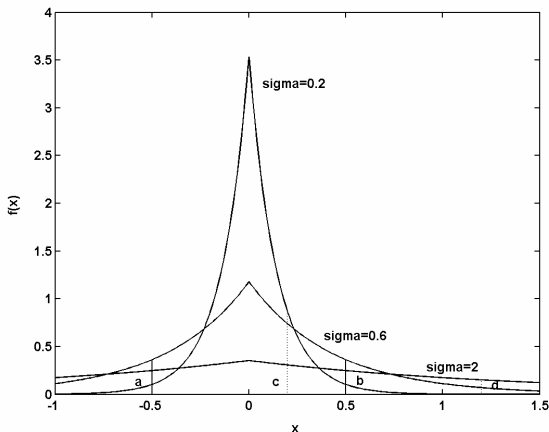


Fig. 11 Laplace noise PDF between  $a = -0.5$  and  $b = 0.5$ , and between  $c = 0.2$  and  $d = 1.2$  for different noise intensity (sigma)

For a fixed threshold level  $u = 1.2$  and noise intensity (standard deviation or dispersion)  $\sigma = 0.6$ , from Uniform noise, Gaussian noise, Laplace noise to Cauchy noise, Fig.13 presents that the areas under the noise PDF between line  $a$  and line  $b$  decreases monotonously. The area variation brings that the efficacy of SR decreases by (13). This result in the rate of approaching to zero is different. The area under the noise PDF between line  $a$  and line  $b$  is biggest for Uniform noise and is smallest for Cauchy noise, so the rate of approaching to zero is fastest for Uniform noise and is slowest for Cauchy noise.

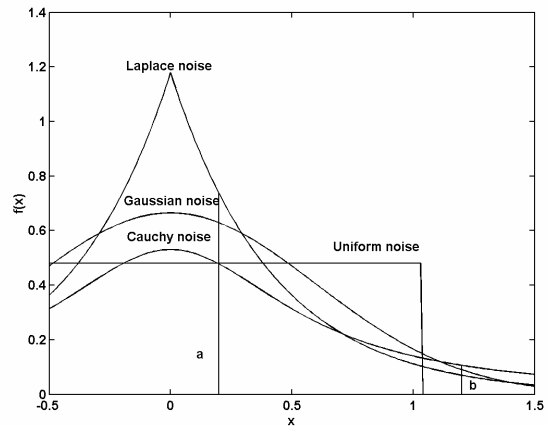


Fig. 13 PDFs between  $a = 0.2$  and  $b = 1.2$ , the noise intensity  $\sigma = 0.6$

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