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Stochastic Programming Model for Power Generation

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Abstract—We consider power system expansion planning under uncertainty. In our approach, integer programming and stochastic programming provide a basic framework. We develop a multistage stochastic programming model in which some of the variables are restricted to integer values. By utilizing the special property of the problem, called block separable recourse, the problem is transformed into a two-stage stochastic program with recourse. The electric power capacity expansion problem is reformulated as the problem with first stage integer variables and continuous second stage variables. The L-shaped algorithm to solve the problem is proposed.

Keywords—electric power capacity expansion problem, integer programming, L-shaped method, stochastic programming

I. Introduction

THIS paper is concerned with the capacity expansion planning of power systems under uncertainty. The basic objective of the capacity expansion planning is to determine an investment schedule for the installation of new generation plants and economic operations which ensure a reliable supply to the electricity demand.

Stochastic programming (Birge and Louveaux [3]) deals with optimization under uncertainty. Birge [2] is a state-of-the-art survey in this field. A stochastic programming problem with recourse is referred to as a two-stage stochastic problem. In the first stage, a decision has to be made without complete information on random factors. After the value of random variables are known, recourse action can be taken in the second stage. For the continuous stochastic programming problem with recourse, an L-shaped method (Van Slyke and Wets [10]) is well-known. It came from the shape of the non-zeros in the constraint matrix. The L-shaped method was used to solve the stochastic concentrator location problems (Shiina [8], [9]). For a multistage stochastic programming with recourse, nested decomposition methods have been studied by Birge [1] in the linear case and Louveaux [6] in the quadratic case. Louveaux [7] introduced the concept of block-separable recourse. This property is essential for capacity expansion as described later. Birge et al. [4] explored a parallel implementation of the nested decomposition algorithm.

An important problem has been left unsolved in modeling the capacity expansion problem. The capacity expansion models we mentioned above are based on a stochastic linear programming problem with continuous decision variables. However, decision variables are restricted to integer values in some real problems. For example, the decision to build new plants or not is represented by a binary variable. To evaluate the exact investment and operation cost of power generation, we develop a multistage stochastic programming problem in which some of the decisions are binary variables

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A stochastic integer programming problem is a difficult problem to solve. If integer variables are involved in a recourse problem, optimality cuts do not provide facets of the epigraph of recourse function. It is difficult to approximate the recourse function of a multistage problem, since it is necessary to consider the nesting of integer programming problems. In the capacity expansion problem, integer variables are involved only in the decisions about the investment of new technology. By utilizing the special property called block separable recourse (Louveaux [7]), the decision variables are classified into two categories, the aggregate level decision and the detailed level decision. Binary decision variables are involved only in aggregate level decision. The algorithm we develop exploits this property and solves the problem efficiently.

II. Multistage Stochastic Programming Problem

We consider a multistage stochastic linear programming problem with stages $t = 0, 1, \ldots, H$ as

(Multistage Stochastic Linear Programming Problem)

$$\min c^0 x^0 + E_{\xi^1}[\min \boldsymbol{c}^1 \boldsymbol{x}^1 + \cdots + E_{\xi^H|\xi^1\cdots\xi^{H-1}}[\min \boldsymbol{c}^H \boldsymbol{x}^H]\cdots]$$
subject to
$$W^0 x^0 = h^0$$
$$\boldsymbol{T}^0 x^0 + W^1 \boldsymbol{x}^1 = \boldsymbol{h}^1, \text{ a.s.}$$
$$\vdots$$
$$\boldsymbol{T}^{H-1} \boldsymbol{x}^{H-1} + W^H \boldsymbol{x}^H = \boldsymbol{h}^H, \text{ a.s.}$$
$$x^0 \geq 0, \boldsymbol{x}^t \geq 0, t = 1, \ldots, H, \text{ a.s.},$$

where c^0 is a known vector in $\Re^{n_0}$, h^0 is a known vector in $\Re^{m_0}$, and each W^t is a known $m_t \times n_t$ matrix. Bold face vectors and matrices are possibly stochastic, where $\boldsymbol{c^t}, \boldsymbol{h^t}, \boldsymbol{T^t}$ are in $\Re^{n_t}, \Re^{m_t}, \Re^{m_t} \times \Re^{n_t}$, respectively. Let $\boldsymbol{x^t}$ denote decision vector in $\Re^{n_t}$ for stage $t, t = 1, \ldots, H$. They are chosen so that the constraints hold almost surely (denoted a. s.).

We assume the stochastic elements are defined over a finite discrete probability space $(\Xi, \sigma(\Xi), P)$, where $\Xi = \Xi^1 \times \cdots \times \Xi^H$ is the support of the random data in each stage with $\Xi^t = \{\xi^t_s = (T^t_s, h^t_s, c^t_s), s = 1, \ldots, k_t\}$ and (T^t_s, h^t_s, c^t_s) is a realization of $(\boldsymbol{T^t}, \boldsymbol{h^t}, \boldsymbol{c^t})$. The possible sequences of the realization of random variables $(\xi^1, \ldots, \xi^H)$ are called scenario. The scenarios are often described using a scenario tree as shown in Fig. 1. In stages $t \leq H$, we have limited

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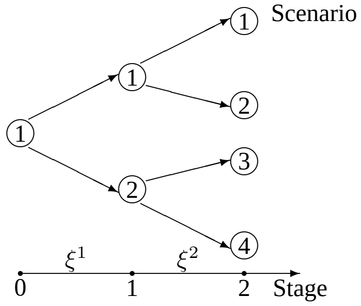

Fig. 1. Scenario tree

number of possible realizations which we call the stage t scenarios.

In a scenario tree, the stage t scenario connected to the stage $t-1$ scenario s is referred to as a successor of stage $t-1$ scenario s. The set of all successors of stage $t-1$ scenario s is denoted by $D^t(s)$. Similarly, the predecessor of stage t scenario s is denoted by $\alpha(s,t)$. These relationships are illustrated in Fig. 2.

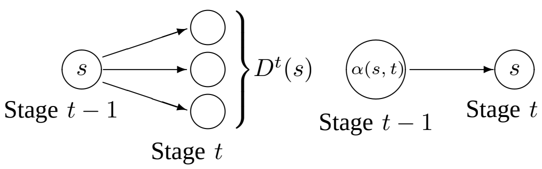

Fig. 2. Successor and predecessor

We can formulate a deterministic equivalent problem for the multistage stochastic linear programming problem by replicating the constraints for each possible event in $\Xi^t, t=1,\ldots,H$ since the probability space is finite and discrete. To solve the deterministic equivalent problem, the nested decomposition method (Birge [1], Birge et al. [4]) can be used. But this problem becomes large as the number of stages or the number of realizations increase. However in some problems, we can avoid difficulty in solving the problem if the problem has a special structure. The concept of block separable recourse was introduced by Louveaux [7].

Definition 2.1: A multistage stochastic linear program has *block separable recourse* if for all stages $t=1,\ldots,H$ the decision vectors $\boldsymbol{x}^t$ can be written as $\boldsymbol{x}^t=(\boldsymbol{w}^t,\boldsymbol{y}^t)$ where $\boldsymbol{w}^t$ represents aggregate level decisions and $\boldsymbol{y}^t$ represents detailed level decisions. The constraints also follow these partitions:

1) The stage t objective contribution is $\boldsymbol{c}^t\boldsymbol{x}^t = \boldsymbol{r}^t\boldsymbol{w}^t+\boldsymbol{q}^t\boldsymbol{y}^t$.
2) The constraint matrix W^t is block diagonal: $W^t = \begin{pmatrix} A^t & 0 \\ 0 & B^t \end{pmatrix}$, where A^t is associated to the vector $\boldsymbol{w}^t$ and B^t to the vector $\boldsymbol{y}^t$.
3) The other components of the constraints are random but we assume that $\boldsymbol{T}^t$ and $\boldsymbol{h}^t$ can be written: $\boldsymbol{T}^t = \begin{pmatrix} \boldsymbol{R}^t & 0 \\ \boldsymbol{S}^t & 0 \end{pmatrix}$ and $\boldsymbol{h}^t = \begin{pmatrix} \boldsymbol{b}^t \\ \boldsymbol{d}^t \end{pmatrix}$ to conform with the $(\boldsymbol{w}^t,\boldsymbol{y}^t)$ separation.

If the problem has block separable recourse, the problem can be rewritten as follows.

(Multistage Stochastic Linear Programming Problem with Block Separable Recourse)

$$\begin{aligned}
\min\quad & r^0w^0+q^0y^0 \\
& +E_{\xi^1}[\min(\boldsymbol{r}^1\boldsymbol{w}^1+\boldsymbol{q}^1\boldsymbol{y}^1) \\
& \vdots \\
& +E_{\xi^H|\xi^1\cdots\xi^{H-1}}[\min(\boldsymbol{r}^H\boldsymbol{w}^H+\boldsymbol{q}^H\boldsymbol{y}^H)]\cdots] \\
\text{subject to}\quad & A^0w^0=b^0,\ B^0y^0=d^0 \\
& \boldsymbol{R}^0w^0+A^1\boldsymbol{w}^1=\boldsymbol{b}^1,\ \text{a.s.} \\
& \boldsymbol{S}^0w^0+B^1\boldsymbol{y}^1=\boldsymbol{d}^1,\ \text{a.s.} \\
& \vdots \\
& \boldsymbol{R}^{H-1}\boldsymbol{w}^{H-1}+A^H\boldsymbol{w}^H=\boldsymbol{b}^H,\ \text{a.s.} \\
& \boldsymbol{S}^{H-1}\boldsymbol{w}^{H-1}+B^H\boldsymbol{y}^H=\boldsymbol{d}^H,\ \text{a.s.} \\
& w^0\ge 0, \boldsymbol{w}^t\ge 0, t=1,\ldots,H,\ \text{a.s.} \\
& y^0\ge 0, \boldsymbol{y}^t\ge 0, t=1,\ldots,H,\ \text{a.s.}
\end{aligned}$$

The equivalence of multistage programs with block-separable recourse and two-stage programs was shown in Louveaux [7].

Proposition 2.1 (Louveaux [7]): A multistage stochastic program with block separable recourse is equivalent to a two stage stochastic program, where the first-stage is the extensive form of the aggregate level problems, and the value function of the second stage is the sum (weighted by the appropriate probabilities) of the detailed level recourse functions for all stage t scenarios, $t=1,\ldots,H$.

From the proposition, the multistage stochastic program with block separable recourse can be transformed into the two stage stochastic program with recourse. The electric power capacity expansion problem turns out to be the problem that has first stage integer variables and continuous second stage variables.

The deterministic equivalent problem for the multistage stochastic programming problem with block-separable recourse can be written as

(Deterministic Equivalent for Multistage Stochastic LP Problem with Block Separable Recourse)

$$\begin{aligned}
& \min\ r^{1,0}w^{1,0}+q^{1,0}y^{1,0} \\
& +\sum_{s=1}^{K_1}p_s^1r^{s1}w^{s1}+\ldots+\sum_{s=1}^{K_H}p_s^Hr^{sH}w^{sH} \\
& +\sum_{s=1}^{K_1}p_s^1Q_s^1(w^{1,0})+\ldots+\sum_{s=1}^{K_H}p_s^HQ_s^H(w^{\alpha(s,H),H-1}) \\
& \text{subject to} \\
& A^0w^{1,0}=b^0, B^0y^{1,0}=d^0 \\
& R^{\alpha(s,1),0}w^{s0}+A^1w^{s1}=b^{s1}, s=1,\ldots,K_1 \\
& \vdots \\
& R^{\alpha(s,H),H-1}w^{s,H-1}+A^Hw^{sH}=b^{sH}, s=1,\ldots,K_H \\
& w^0\ge 0,\ w^{st}\ge 0, s=1,\ldots,K_t, t=1,\ldots,H \\
& Q_s^t(w^{\alpha(s,t),t-1}) \\
& =\min\{q^{st}y^{st}|S^{\alpha(s,t),t-1}w^{\alpha(s,t),t-1}+B^ty^{st}=d^{st}, y^{st}\ge 0\}, \\
& \qquad s=1,\ldots,K_t, t=1,\ldots,H,
\end{aligned}$$

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where:
$K_t =$ Cumulative number of realizations through stage t,
$K_t = k_0 \times k_1 \times \cdots \times k_t, t = 1, \ldots, H, k_0 = 1$,
$(r^{st}, q^{st}) =$ Realization of random objective coefficents $(\boldsymbol{r}^t, \boldsymbol{q}^t)$ for stage t scenario $s, s = 1, \ldots, K_t, t = 0, \ldots, H$, $(r^{1,0}, q^{1,0}) \equiv (r^0, q^0)$ for the sake of simplicity,
$(R^{st}, S^{st}) =$ Realization of random technology matrix $(\boldsymbol{R}^t, \boldsymbol{S}^t)$ for stage t scenario s, $s = 1, \ldots, K_t, t = 0, 1, \ldots, H-1$,
$w^{st} =$ Aggregate decision vector to take in stage t for stage t scenario $s, s = 1, \ldots, K_t, t = 0, 1, \ldots, H$,
$y^{st} =$ Detailed level decision vector to take in stage t for stage t scenario $s, s = 1, \ldots, K_t, t = 0, 1, \ldots, H$,
$p_s^t =$ Probability that stage t scenario s occurs, $s = 1, \ldots, K_t, t = 1, \ldots, H$,
$\alpha(s, t) =$ The predecessor or parent of stage t scenario s, $s = 1, \ldots, K_t, t = 1, \ldots, H$.

III. Electric Power Generation

We consider the application of the multistage stochastic programming problem to the electric power generation problem. The basic objective of the problem is to determine an investment schedule of new technology and to operate power plants to ensure an economic and reliable supply to electricity demand.

The load patterns are modeled by load cycles or load duration curves. For long-range planning, block approximations of load duration curves are used. A load duration curve represents the number of hours in which the load equals or exceeds the given load value.

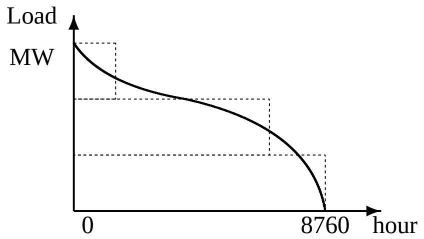

Fig. 3. Yearly load duration curve

In this problem, the investment cost, the operation cost or the electricity demand are regarded as random. The problem is formulated as a multistage stochastic programming problem in which the decision to install new technology or not is represented by a binary variable. Let

- $t = 0, 1, \ldots, H$ index the period of stages;
- $i = 1, \ldots, n$ index the available types of plants;
- $j = 1, \ldots, m$ index the operating modes in the load duration curve.

We also define the following:

- $a_i =$ availability factor of plant i;
- $g_i^t =$ existing capacity of plant i at stage t, decided before $t = 0$;
- $C_i^t =$ maximum capacity of plant i that can be installed at stage t;
- $\boldsymbol{r}_i^t =$ unit investment cost of plant i at stage t;
- $\boldsymbol{f}_i^t =$ fixed investment cost of plant i at stage t;
- $\boldsymbol{q}_i^t =$ unit production cost of plant i at stage t;
- $\boldsymbol{d}_j^t =$ maximal power demanded in mode j at stage t;
- $\boldsymbol{\tau}_j^t =$ duration of mode j at stage t.

We consider the set of decisions as:

- $\boldsymbol{x}_i^t =$ new capacity made available for plant i at stage t;
- $\boldsymbol{w}_i^t =$ total capacity of plant i available at stage t;
- $\boldsymbol{v}_i^t = \begin{cases} 1, & \text{if new capacity type } i \text{ is installed at stage } t; \\ 0, & \text{otherwise;} \end{cases}$
- $\boldsymbol{y}_{ij}^t =$ generation level of plant i at stage t in mode j.

The multistage stochastic electric power capacity expansion problem is formulated as follows.

(Multistage Stochastic Electric Power Generation Problem)

$$
\begin{aligned}
\min \; & \sum_{i=1}^{n} (r_i^0 w_i^0 + f_i^0 v_i^0) \\
& + E_{\xi^1} [\min \sum_{i=1}^{n} (\boldsymbol{r}_i^1 \boldsymbol{w}_i^1 + \boldsymbol{q}_i^1 \sum_{j=1}^{m} \boldsymbol{\tau}_j^1 \boldsymbol{y}_{ij}^1 + \boldsymbol{f}_i^1 \boldsymbol{v}_i^1) \\
& \vdots \\
& + E_{\xi^H | \xi^1 \cdots \xi^{H-1}} [\min \sum_{i=1}^{n} (\boldsymbol{r}_i^H \boldsymbol{w}_i^H + \boldsymbol{q}_i^H \sum_{j=1}^{m} \boldsymbol{\tau}_j^H \boldsymbol{y}_{ij}^H + \boldsymbol{f}_i^H \boldsymbol{v}_i^H)] \cdots]
\end{aligned}
$$

subject to

$$
\begin{aligned}
& w_i^0 = x_i^0, i = 1, \ldots, n \\
& \boldsymbol{w}_i^1 = w_i^0 + \boldsymbol{x}_i^1, i = 1, \ldots, n, \text{a.s.} \\
& \boldsymbol{w}_i^t = \boldsymbol{w}_i^{t-1} + \boldsymbol{x}_i^t, i = 1, \ldots, n, t = 2, \ldots, H, \text{a.s.} \\
& x_i^0 \le C_i^0 v_i^0, i = 1, \ldots, n \\
& \boldsymbol{x}_i^t \le C_i^t \boldsymbol{v}_i^t, i = 1, \ldots, n, t = 1, \ldots, H, \text{a.s.} \\
& \sum_{i=1}^{n} \boldsymbol{y}_{ij}^t = \boldsymbol{d}_j^t, j = 1, \ldots, m, t = 1, \ldots, H, \text{a.s.} \\
& \sum_{j=1}^{m} \boldsymbol{y}_{ij}^t \le a_i (g_i^t + \boldsymbol{w}_i^{t-1}), i = 1, \ldots, n, t = 1, \ldots, H, \text{a.s.} \\
& v_i^0 \in \{0, 1\}, \boldsymbol{v}_i^t \in \{0, 1\}, i = 1, \ldots, n, t = 1, \ldots, H, \text{a.s.} \\
& w_i^0 \ge 0, \boldsymbol{w}_i^t \ge 0, i = 1, \ldots, n, t = 1, \ldots, H, \text{a.s.} \\
& x_i^0 \ge 0, \boldsymbol{x}_i^t \ge 0, i = 1, \ldots, n, t = 1, \ldots, H, \text{a.s.} \\
& \boldsymbol{y}_{ij}^t \ge 0, i = 1, \ldots, n, j = 1, \ldots, m, t = 1, \ldots, H, \text{a.s.}
\end{aligned}
$$

Since the problem has the property of block separable recourse from Proposition 1, the decision variables $v_i^0, \boldsymbol{v}_i^t, w_i^0, \boldsymbol{w}_i^t, x_i^0, \boldsymbol{x}_i^t, i = 1, \ldots, n, t = 1, \ldots, H$ and $\boldsymbol{y}_{ij}^t \ge 0, i = 1, \ldots, n, j = 1, \ldots, m, t = 1, \ldots, H$ correspond to the aggregate level decisions and the detailed level decisions, respectively.

We assume the stochastic elements in bold face are defined over a finite discrete probability space $(\Xi, \sigma(\Xi), P)$, where $\Xi = \Xi^1 \times \cdots \times \Xi^H$ is the support of the random data in each period with $\Xi^t = \{\xi_s^t = (r_i^t, f_i^t, q_i^t, d_j^t, \tau_j^t), s = 1, \ldots, k_t\}$ and $(r_i^t, f_i^t, q_i^t, d_j^t, \tau_j^t)$ is a realization of $(\boldsymbol{r}_i^t, \boldsymbol{f}_i^t, \boldsymbol{q}_i^t, \boldsymbol{d}_j^t, \boldsymbol{\tau}_j^t)$.

The deterministic equivalent problem for the multistage stochastic electric power capacity expansion problem can be written as

(Deterministic Equivalent for Multistage Stochastic Electric Power Generation Problem)

$$\begin{aligned}
\min & \sum_{i=1}^{n}(r_i^{1,0}w_i^{1,0}+f_i^{1,0}v_i^{1,0}) \\
& +\sum_{s=1}^{K_1}p_s^1\sum_{i=1}^{n}(r_i^{s1}w_i^{s1}+f_i^{s1}v_i^{s1}) \\
& +\ldots+\sum_{s=1}^{K_H}p_s^H\sum_{i=1}^{n}(r_i^{sH}w_i^{sH}+f_i^{sH}v_i^{sH}) \\
& +\sum_{s=1}^{K_1}p_s^1Q_s^1(w^{1,0})+\ldots+\sum_{s=1}^{K_H}p_s^HQ_s^H(w^{\alpha(s,H),H-1})
\end{aligned}$$

subject to

$$\begin{aligned}
& w_i^{1,0}=x_i^{1,0}, i=1,\ldots,n \\
& w_i^{st}=w_i^{\alpha(s,t),t-1}+x_i^{st}, i=1,\ldots,n, s=1,\ldots,K_t, \\
& t=1,\ldots,H \\
& x_i^{st}\le C_i^tv_i^{st}, i=1,\ldots,n, s=1,\ldots,K_t, t=0,1,\ldots,H \\
& v_i^{st}\in\{0,1\}, i=1,\ldots,n, s=1,\ldots,K_t, t=0,1,\ldots,H \\
& w_i^{st}\ge 0, i=1,\ldots,n, s=1,\ldots,K_t, t=0,1,\ldots,H \\
& x_i^{st}\ge 0, i=1,\ldots,n, s=1,\ldots,K_t, t=0,1,\ldots,H \\
& Q_s^t(w^{\alpha(s,t),t-1}) \\
& =\min\ \{\sum_{i=1}^{n}\sum_{j=1}^{m}q_i^{st}\tau_j^{st}y_{ij}^{st}|\ \sum_{i=1}^{n}y_{ij}^{st}=d_j^{st}, j=1,\ldots,m \\
& \sum_{j=1}^{m}y_{ij}^{st}\le a_i(g_i^t+w_i^{\alpha(s,t),t-1}), i=1,\ldots,n \\
& y_{ij}^{st}\ge 0, i=1,\ldots,n, j=1,\ldots,m\}, \\
& s=1,\ldots,K_t, t=1,\ldots,H,
\end{aligned}$$

where:

- $K_t=$ Cumulative number of realizations through stage t, $K_t=k_0\times k_1\times\cdots\times k_t, t=1,\ldots,H, k_0=1$,
- $(r^{st},q^{st},f^{st})=$ Realization of random objective coeffecent vectors $(\boldsymbol{r}^t,\boldsymbol{q}^t,\boldsymbol{f}^t)$ for stage t scenario s, $s=1,\ldots,K_t, t=0,1,\ldots,H$, $(r^{1,0},f^{1,0})\equiv(r^0,f^0)$ for the sake of simplicity,
- $(\tau^{st},d^{st})=$ Realization of random duration and demand vectors $(\boldsymbol{\tau}^{st},\boldsymbol{d}^t)$ for stage t scenario s, $s=1,\ldots,K_t, t=1,\ldots,H$,
- $w^{st}=$ Total capacity decision vector to take in stage t for stage t scenario s, $s=1,\ldots,K_t, t=0,1,\ldots,H$,
- $x^{st}=$ New capacity decision vector to take in stage t for stage t scenario s, $s=1,\ldots,K_t, t=0,1,\ldots,H$,
- $v^{st}=$ Binary decision vector to take in stage t for stage t scenario s, $s=1,\ldots,K_t, t=0,1,\ldots,H$,
- $y^{st}=$ Generation level decision vector to take in stage t for stage t scenario s, $s=1,\ldots,K_t, t=1,\ldots,H$,
- $p_s^t=$ Probability that stage t scenario s occurs, $s=1,\ldots,K_t, t=1,\ldots,H$,
- $\alpha(s,t)=$ The predecessor or parent of stage t scenario s, $s=1,\ldots,K_t, t=1,\ldots,H$.

The electric power capacity expansion problem can be transformed to the problem that has first stage integer variables and continuous second stage variables. To solve the problem, the following master problem is formulated.

(Master Problem for Multistage Stochastic Electric Power Generation Problem)

$$\begin{aligned}
\min & \sum_{i=1}^{n}(r_i^{1,0}w_i^{1,0}+f_i^{1,0}v_i^{1,0}) \\
& +\sum_{s=1}^{K_1}p_s^1\sum_{i=1}^{n}(r_i^{s1}w_i^{s1}+f_i^{s1}v_i^{s1}) \\
& \vdots \\
& +\sum_{s=1}^{K_H}p_s^H\sum_{i=1}^{n}(r_i^{sH}w_i^{sH}+f_i^{sH}v_i^{sH}) \\
& +\sum_{s=1}^{K_1}p_s^1\theta_s^1+\ldots+\sum_{s=1}^{K_H}p_s^H\theta_s^H
\end{aligned}$$

subject to

$$\begin{aligned}
& w_i^{1,0}=x_i^{1,0}, i=1,\ldots,n \\
& w_i^{st}=w_i^{\alpha(s,t),t-1}+x_i^{st}, i=1,\ldots,n, s=1,\ldots,K_t, \\
& t=1,\ldots,H \\
& x_i^{st}\le C_i^tv_i^{st}, i=1,\ldots,n, s=1,\ldots,K_t, t=0,1,\ldots,H \\
& v_i^{st}\in\{0,1\}, i=1,\ldots,n, s=1,\ldots,K_t, t=0,1,\ldots,H \\
& w_i^{st}\ge 0, i=1,\ldots,n, s=1,\ldots,K_t, t=0,1,\ldots,H \\
& x_i^{st}\ge 0, i=1,\ldots,n, s=1,\ldots,K_t, t=0,1,\ldots,H \\
& \theta_s^t\ge Q_s^t(w^{\alpha(s,t),t-1}), s=1,\ldots,K_t, t=1,\ldots,H
\end{aligned}$$

In this formulation, the recourse functions $Q_s^t(w^{\alpha(s,t),t-1}), s=1,\ldots,K_t, t=1,\ldots,H$ are not known explicitly in advance. Therefore, the optimality cuts are added to approximate $\theta_s^t\ge Q_s^t(w^{\alpha(s,t),t-1})$. The optimal solution to the master problem is obtained by solving the mixed integer programming problem. Let $v_i^{*st}, w_i^{*st}, x_i^{*st}i=1,\ldots,n, s=0,1,\ldots,K_t, t=1,\ldots,H$, $\theta_s^{*t}, s=1,\ldots,K_t, t=1,\ldots,H$, be the optimal solution to the master problem. Then the recourse problem for stage t scenario s is solved at the optimal solution of the master problem, $w_i^{*\alpha(s,t),t-1}, i=1,\ldots,n$.

(Recourse Problem for Stage t Scenario s)

$$\begin{aligned}
& Q_s^t(w^{\alpha(s,t),t-1}) \\
& =\min\{\sum_{i=1}^{n}\sum_{j=1}^{m}q_i^{st}\tau_j^{st}y_{ij}^{st}|\sum_{i=1}^{n}y_{ij}^{st}=d_j^{st}, j=1,\ldots,m \\
& \sum_{j=1}^{m}y_{ij}^{st}\le a_i(g_i^t+w_i^{\alpha(s,t),t-1}), i=1,\ldots,n \\
& y_{ij}^{st}\ge 0, i=1,\ldots,n, j=1,\ldots,m\}, \\
& s=1,\ldots,K_t, t=1,\ldots,H
\end{aligned}$$

If the recourse problem is infeasible, the feasibility cut is added to the formulation of the master problem. If the solution of the master problem and the recourse problem do not satisfy the inequality $\theta_s^{*t}\ge Q_s^t(w^{*\alpha(s,t),t-1})$, the optimality cut which approximates $Q_s^t(w^{\alpha(s,t),t-1})$ is added to the master problem. The dual problem to the recourse problem for stage

t scenario s is described as follows.

(Dual to Recourse Problem for Stage t Scenario s)

$$\begin{array}{ll} \max & \sum_{j=1}^{m} d_j^{st}\lambda_j^{st} - \sum_{i=1}^{n} a_i(g_i^t + w_i^{\alpha(s,t),t-1})\mu_i^{st} \\ \text{subject to} & \lambda_j^{st} - \mu_i^{st} \le q_i^{st}\tau_j^{st}, j = 1,\ldots,m, i = 1,\ldots,n \\ & \mu_i \ge 0, i = 1,\ldots,n \end{array}$$

If $Q_s^t(w^{*\alpha(s,t),t-1}) = +\infty$, then $w^{*\alpha(s,t),t-1}$ is not feasible with respect to all constraints of the multistage stochastic electric power capacity expansion problem. By the duality theory, we have $\tilde{\lambda}_j^{st}, j = 1,\ldots,m, \tilde{\mu}_i^{st}(\ge 0), i = 1,\ldots,n$ so that $\sum_{j=1}^{m} d_j^{st}\tilde{\lambda}_j^{st} - \sum_{i=1}^{n} a_i(g_i^t + w_i^{*\alpha(s,t),t-1})\tilde{\mu}_i^{st} > 0$ and $\tilde{\lambda}_j^{st} - \tilde{\mu}_i^{st} \le 0$. For any feasible $w^{\alpha(s,t),t-1}$, there must exist a $y^{st} \ge 0$ such that $\sum_{i=1}^{n} y_{ij}^{st} = d_j^{st}, j = 1,\ldots,m, \sum_{j=1}^{m} y_{ij}^{st} \le a_i(g_i^t + w_i^{\alpha(s,t),t-1}), i = 1,\ldots,n$. Scalar multiplication of these constraints by $\tilde{\lambda}_j^{st}, j = 1,\ldots,m, \tilde{\mu}_i^{st}, i = 1,\ldots,n$ yields

$$\begin{aligned} & \sum_{j=1}^{m} d_j^{st}\tilde{\lambda}_j^{st} - \sum_{i=1}^{n} a_i(g_i^t + w_i^{\alpha(s,t),t-1})\tilde{\mu}_i^{st} \\ \le & \sum_{j=1}^{m}\sum_{i=1}^{n}\tilde{\lambda}_j^{st} y_{ij}^{st} - \sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mu}_i^{st} y_{ij}^{st} \\ = & \sum_{j=1}^{m}\sum_{i=1}^{n}(\tilde{\lambda}_j^{st} - \tilde{\mu}_i^{st}) y_{ij}^{st} \le 0 \end{aligned}$$

which cuts off $w^{*\alpha(s,t),t-1}$, since $\sum_{j=1}^{m} d_j^{st}\tilde{\lambda}_j^{st} - \sum_{i=1}^{n} a_i(g_i^t + w_i^{*\alpha(s,t),t-1})\tilde{\mu}_i^{st} > 0$.

Feasibility Cut: $$\sum_{j=1}^{m} d_j^{st}\tilde{\lambda}_j^{st} - \sum_{i=1}^{n} a_i(g_i^t + w_i^{\alpha(s,t),t-1})\tilde{\mu}_i^{st} \le 0 \tag{1}$$

If $Q_s^t(w^{*\alpha(s,t),t-1})$ is finite, we have the optimal primal solution $y_{ij}^{*st}, i = 1,\ldots,n, j = 1,\ldots,m$ and the optimal dual solution $\lambda_j^{*st}, j = 1,\ldots,m, \mu^{*st}, i = 1,\ldots,n$. The following inequality holds.

$$\begin{aligned} \theta_s^t \ \ge\ & Q_s^t(w^{*\alpha(s,t),t-1}) \\ =\ & \max\{\sum_{j=1}^{m} d_j^{st}\lambda_j^{st} - \sum_{i=1}^{n} a_i(g_i^t + w_i^{*\alpha(s,t),t-1})\mu_i^{st} | \\ & \quad \lambda_j^{st} - \mu_i^{st} \le q_i^{st}\tau_j^{st}, \quad j = 1,\ldots,m, \\ & \quad\quad i = 1,\ldots,n \\ & \quad \mu_i \ge 0, \quad i = 1,\ldots,n\} \\ =\ & \sum_{j=1}^{m} d_j^{st}\lambda_j^{*st} - \sum_{i=1}^{n} a_i(g_i^t + w_i^{*\alpha(s,t),t-1})\mu_i^{*st} \end{aligned}$$

To approximate the recourse function, the optimality cut which cuts off $(w^{*\alpha(s,t),t-1}, \theta_s^{*t})$ so that $\theta_s^{*t} < Q_s^t(w^{*\alpha(s,t),t-1})$ is added to the formulation of the master problem.

Optimality Cut: $$\theta_s^t \ge \sum_{j=1}^{m} d_j^{st}\lambda_j^{*st} - \sum_{i=1}^{n} a_i(g_i^t + w_i^{\alpha(s,t),t-1})\mu_i^{*st} \tag{2}$$

The algorithm of the L-shaped method for multistage stochastic electric power capacity expansion problem is shown as follows.

- **Step 1. Solve Master Problem**
  Solve the mixed integer programming master problem by branch and bound method. Let $w_i^{*st}, i = 1,\ldots,n, s = 0,1,\ldots,K_t, t = 1,\ldots,H$, $\theta_s^{*t}, s = 1,\ldots,K_t, t = 1,\ldots,H$ be the optimal solution to the master problem.
- **Step 2. Solve Recourse Problem**
  Solve the recourse problem for stage t scenario $s, s = 1,\ldots,K_t, t = 1,\ldots,H$.
- **Step 3. Add Feasibility Cuts**
  If the recourse problem for stage t scenario s is infeasible, the feasibility cut (1) is added to the formulation of the master problem. Go to Step.1.
- **Step 4. Add Optimality Cuts**
  Calculate $Q_{s'}^t(w^{*s,t-1}), \forall s' \in D^t(s), s = 1,\ldots,K_t, t = 0,\ldots,H-1$. If $\theta_{s'}^{*t} < (1-\varepsilon)Q_{s'}^t(w^{*s,t-1}), s' \in D^t(s)$, the optimality cut (2) is added to the formulation of the master problem ($\varepsilon > 0$: tolerance). Go to Step 1.
- **Step 5. Convergence Check**
  If no optimality cuts are added, then stop.

Since all of the integer variables are aggregate level decision variables, the master problem becomes a mixed integer programming problem. The upper bound for the optimal objective value is calcurated as follows.

$$\begin{aligned} \text{Upper Bound} = & \sum_{i=1}^{n}(r_i^{1,0} w_i^{*1,0} + f_i^{1,0} v_i^{*1,0}) \\ & + \sum_{s=1}^{K_1} p_s^1 \sum_{i=1}^{n}(r_i^{s1} w_i^{*s1} + f_i^{s1} v_i^{*s1}) \\ & \vdots \\ & + \sum_{s=1}^{K_H} p_s^H \sum_{i=1}^{n}(r_i^{sH} w_i^{*sH} + f_i^{sH} v_i^{*sH}) \\ & + \sum_{s=1}^{K_1} p_s^1 Q_s^1(w^{*1,0}) \\ & \vdots \\ & + \sum_{s=1}^{K_H} p_s^H Q_s^H(w^{*\alpha(s,H),H-1}) \end{aligned} \tag{3}$$

The value of upper bound for the optimal objective value can be adopted as the approximate optimal objective value.

IV. Numerical Experiments

The L-shaped method for the multistage stochastic electric power generation problem was implemented using AMPL [5] (CPU: Intel core 2 duo E8500, 3.16GHz). The whole framework of the algorithm was coded in AMPL. The mathematical programming problems were solved by linear programming/branch-and-bound solver CPLEX 10.0. We consider the following two problems.

- **Problem 1.** $n = 20, m = 4, H = 4, K_4 = 8$.
- **Problem 2.** $n = 60, m = 4, H = 6, K_6 = 16$.

In both problems, the load patterns are modeled by yearly load duration curves. The scenarios are generated from the power demand of stage 1 scenario 1 by adding an increase in demand as shown as follows.

TABLE I
DEMAND INCREASES FOR DIFFERENT SCENARIOS

Problem 1		stage					
Scenario	Probability	1	2	3	4		
1	0.125	0	0	0	+10%		
2	0.125	0	0	+10%	+20%		
3	0.125	0	+10%	0	+10%		
4	0.125	0	+10%	+10%	+20%		
5	0.125	+10%	0	0	+10%		
6	0.125	+10%	0	+10%	+20%		
7	0.125	+10%	+10%	0	+10%		
8	0.125	+10%	+10%	+10%	+20%		
Problem 2		stage					
Scenario	Probability	1	2	3	4	5	6
1	0.0625	0	0	0	+10%	+10%	+20%
2	0.0625	0	0	0	+10%	+30%	+30%
3	0.0625	0	0	+10%	+20%	+10%	+20%
4	0.0625	0	0	+10%	+20%	+30%	+30%
5	0.0625	0	+10%	0	+10%	+10%	+20%
6	0.0625	0	+10%	0	+10%	+30%	+30%
7	0.0625	0	+10%	+10%	+20%	+10%	+20%
8	0.0625	0	+10%	+10%	+20%	+30%	+30%
9	0.0625	+10%	0	0	+10%	+10%	+20%
10	0.0625	+10%	0	0	+10%	+30%	+30%
11	0.0625	+10%	0	+10%	+20%	+10%	+20%
12	0.0625	+10%	0	+10%	+20%	+30%	+30%
13	0.0625	+10%	+10%	0	+10%	+10%	+20%
14	0.0625	+10%	+10%	0	+10%	+30%	+30%
15	0.0625	+10%	+10%	+10%	+20%	+10%	+20%
16	0.0625	+10%	+10%	+10%	+20%	+30%	+30%

The structure of the scenario trees of Problem 1 and 2 are shown as follows.

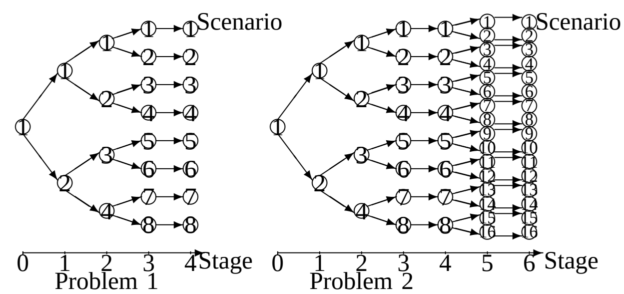

Fig. 4. Scenario tree of problem 1 and 2

The L-shaped method is applied to the master problem. We set $\varepsilon = 0.01(\%)$ in step 4 of the L-shaped method. In solving the master problem or the deterministic equivalent problem using branch-and-bound, the search is terminated when the best value of lower bound times $(1+10^{-4})$ is greater than or equal to the best integer objective value. From the optimal solution of the master problem, the upper bound for the optimal objective value can be calculated. The results of the numerical experiments are shown as follows.

TABLE II
RESULTS FOR PROBLEM 1

Iteration Number	Master Problem Optimal Objective Value	Number of Added Cuts
1	66375.0	4 feasibility cuts
2	69610	22 optimality cuts
3	98043	11 optimality cuts
4	98099	0 cuts
Optimal Cost	98103	

TABLE III
RESULTS FOR PROBLEM 2

Iteration Number	Master Problem Optimal Objective Value	Number of Added Cuts
1	118799	28 feasibility cuts
2	142859	54 optimality cuts
3	170384	2 optimality cuts
4	170391	0 cuts
Optimal Cost	170393	

V. CONCLUSION

We developed a multistage stochastic programming model for the electric power generation problem. By utilizing the property of block separable recourse, the L-shaped method solves the problem effectively.

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