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Stability of Property (gm) under Perturbation and Spectral Properties Type Weyl Theorems

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***Abstract*—A Banach space operator T obeys property (gm) if the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues are exactly those points λ of the spectrum for which $T-\lambda I$ is a left Drazin invertible. In this article, we study the stability of property (gm), for a bounded operator acting on a Banach space, under perturbation by finite rank operators, by nilpotent operators, by quasi-nilpotent operators, or more generally by algebraic operators commuting with T.

***Keywords*—Weyl's theorem, Weyl spectrum, polaroid operators, property (gm), property (m).

I. Introduction

THROUGHOUT this paper let $\mathcal{B}(\mathcal{X})$ denote the *algebra of bounded* operators acting on an infinite complex Banach space $\mathcal{X}$. We use I to denote the *identity* operator on $\mathcal{X}$, and $\mathcal{K}(\mathcal{X})$ to denote the ideal of all *compact* operators on $\mathcal{X}$ and $\mathcal{F}(\mathcal{X})$ to denote the ideal of all *finite rank* operators on $\mathcal{X}$. We shall denote the spectrum, the point spectrum and the approximate point spectrum of $T\in\mathcal{B}(\mathcal{X})$ by $\sigma(T),\sigma_p(T)$ and $\sigma_a(T)$, respectively. Throughout this paper, the set of all complex numbers and the complex conjugate of a complex number λ will be denoted by $\mathbb{C}$ and $\overline{\lambda}$, respectively. The closure of a set S will be denoted by $\overline{S}$ and we shall henceforth shorten $T-\lambda I$ to $T-\lambda$. If K is a subset of $\mathbb{C}$, then $\operatorname{iso} K$ denotes the set of all isolated points of K and $\operatorname{acc}K$ denotes the set of all points of accumulation of K. We use T^* to denote the adjoint of $T\in\mathcal{B}(\mathcal{X})$. For an arbitrary operator $T\in\mathcal{B}(\mathcal{X})$, $\ker(T)$ denotes its *kernel* and $\Re(T)$ denotes its *range*. We set $\alpha(T)=\dim\ker(T)$ and $\beta(T)=\dim\mathcal{X}/\Re(T)$. Let $a:=a(T)$ be the *ascent* of an operator T; i.e., the smallest nonnegative integer p such that $\ker(T^p)=\ker(T^{p+1})$. If such integer does not exist we put $a(T)=\infty$. Analogously, let $d:=d(T)$ be the *descent* of an operator T ; i.e., the smallest nonnegative integer q such that $\Re(T^q)=\Re(T^{q+1})$, and if such integer does not exist we put $d(T)=\infty$. It is well known that if $a(T)$ and $d(T)$ are both finite then $a(T)=d(T)$ [21, Proposition 38.3]. Moreover, $0<a(T-\lambda I)=d(T-\lambda I)<\infty$ precisely when λ is a pole of the resolvent of T , see Heuser [21, Proposition 50.2].

Following [20] we say that $T\in\mathcal{B}(\mathcal{X})$ has the single-valued extension property (SVEP) at point $\lambda\in\mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f:U_\lambda\longrightarrow\mathcal{H}$ which satisfies the equation $(T-\mu)f(\mu)=0$ is the constant function $f\equiv 0$. It is well-known that $T\in\mathcal{B}(\mathcal{X})$ has SVEP at every point of the resolvent $\rho(T):=\mathbb{C}\backslash\sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T\in\mathcal{B}(\mathcal{X})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [22, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Denote by

$$SF_+(\mathcal{X}):=\{T\in\mathcal{B}(\mathcal{X}):\alpha(T)<\infty\,\text{and}\,\Re(T)\,\text{is closed}\}$$

the class of all *upper semi-Fredholm* operators, and by

$$SF_-(\mathcal{X}):=\{T\in\mathcal{B}(\mathcal{X}):\beta(T)<\infty\}$$

the class of all *lower semi-Fredholm* operators. The class of all *semi-Fredholm* operators is defined by $SF_\pm(\mathcal{X}):=SF_+(\mathcal{X})\cup SF_-(\mathcal{X})$, while the class of all *Fredholm* operator is defined by $\mathfrak{F}(\mathcal{X}):=SF_+(\mathcal{X})\cap SF_-(\mathcal{X})$. For a semi-Fredholm operator T we define the *index*, $\operatorname{ind}(T)$, by $\operatorname{ind}(T)=\alpha(T)-\beta(T)$. The *upper semi-Weyl* operators are defined as the class of Fredholm operators with index less than or equal to 0, while the class of *Weyl* operators are defined as the class of Fredholm operators of index 0. These classes of operators generate the following spectra: the *Weyl spectrum* defined by

$$\sigma_w(T):=\{\lambda\in\mathbb{C}:T-\lambda\,\text{is not a Weyl operator}\},$$

the *upper semi-Weyl spectrum* defined by

$$\sigma_{SF_+^-}(T):=\{\lambda\in\mathbb{C}:T-\lambda\,\text{is not an upper semi-Weyl operator}\}.$$

Recall that an operator $T\in\mathcal{B}(\mathcal{X})$ is said to be *Browder*(resp. *upper semi-Browder, lower semi-Browder*) if T is Fredholm and $a(T)=d(T)<\infty$ (resp. T is upper semi-Fredholm and $a(T)<\infty$, T is lower semi-Fredholm and $d(T)<\infty$). The *Browder spectrum* of $T\in\mathcal{B}(\mathcal{X})$ is defined by

$$\sigma_b(T):=\{\lambda\in\mathbb{C}:T-\lambda\,\text{is not a Browder operator}\},$$

the *upper semi-Browder spectrum* is defined by

$$\sigma_{ub}(T):=\{\lambda\in\mathbb{C}:T-\lambda\,\text{is not an upper semi-Browder operator}\}.$$

Recall that an operator $T\in\mathcal{B}(\mathcal{X})$ is a *Drazin invertible* if and only if it has a finite ascent and descent. The *Drazin spectrum* is given by

$$\sigma_D(T):=\{\lambda\in\mathbb{C}:T-\lambda\quad\text{is not Drazin invertible}\}.$$

Let $\pi(T):=\{\lambda\in\mathbb{C}:a(T-\lambda)=d(T-\lambda)<\infty\}$ be the set of poles. Then $\pi^0(T):=\{\lambda\in\pi(T):\alpha(T-\lambda)<\infty\}$ is the set of poles of finite rank. We observe that $\pi(T)=\sigma(T)\setminus\sigma_D(T)$. An operator $T\in\mathcal{B}(\mathcal{X})$ is called *left Drazin*

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invertible, $T \in LD(\mathcal{X})$, if $a(T) < \infty$ and $\Re(T^{a(T)+1})$ is closed. The *left Drazin spectrum* is given by

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda \quad \text{is not left Drazin invertible}\}.$$

Let $\pi_a(T) := \{\lambda \in \sigma_a(T) : T - \lambda \,\text{is not a left Drazin invertible}\}$ be the set of left poles of T. Then $\pi_a^0(T) := \{\lambda \in \pi_a(T) : \alpha(T-\lambda) < \infty\}$ is the set of left poles of T of finite rank. We observe that $\pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T)$. According also to [21], the space $\Re((T-\lambda)^{a(T-\lambda)+1})$ is closed for each $\lambda \in \pi(T)$. Hence we have always $\pi(T) \subset \pi_a(T)$ and $\pi^0(T) \subset \pi_a^0(T)$. We say that *a-Browders theorem* holds for $T \in \mathcal{B}(\mathcal{X})$, $T \in a\mathfrak{B}$, if $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_a^0(T)$.

Recall that $T \in \mathcal{B}(\mathcal{X})$ is said to be a *Riesz operator* if $T - \lambda \in \mathfrak{F}(\mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators.

Suppose that $T \in \mathcal{B}(\mathcal{X})$ and R is a Riesz operator commuting with T. Then it follows from [26, Theorem 1] and [29, Proposition 5] that

$$\begin{aligned} \sigma_b(T) &= \sigma_b(T+R); &(1)\\ \sigma_w(T) &= \sigma_w(T+R); &(2)\\ \sigma_{ub}(T) &= \sigma_{ub}(T+R); &(3)\\ \sigma_{SF_+^-}(T) &= \sigma_{SF_+^-}(T+R). &(4) \end{aligned}$$

Let $E(T) := \{\lambda \in \operatorname{iso}\sigma(T) : \alpha(T-\lambda) > 0\}$ be the set of all isolated eigenvalues of T and $E_a(T) := \{\lambda \in \operatorname{iso}\sigma_a(T) : \alpha(T-\lambda) > 0\}$ be the set of all eigenvalues of T that are isolated in $\sigma_a(T)$. Then $E^0(T) := \{\lambda \in E(T) : \alpha(T-\lambda) < \infty\}$ is the set of all isolated eigenvalues of T of finite multiplicity and $E_a^0(T) := \{\lambda \in E_a(T) : \alpha(T-\lambda) < \infty\}$ is the set of all eigenvalues of T that are isolated in $\sigma_a(T)$ of finite multiplicity. According to Coburn [19], *Weyl's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_w(T) = E^0(T)$, and that *Browder's theorem* holds for T if $\Delta(T) = \pi^0(T)$. According to Rakočević [25], an operator $T \in B(X)$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$. It is known [25] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in \mathcal{B}(\mathcal{X})$ and a non negative integer n define $T_{[n]}$ to be the restriction T to $\Re(T^n)$ viewed as a map from $\Re(T^n)$ to $\Re(T^n)$(in particular $T_{[0]} = T$). If for some integer n the range space $\Re(T^n)$ is closed and $T_{[n]}$ is an upper (resp., lower) semi-Fredholm operator, then T is called *upper* (resp., *lower*) *semi-B-Fredholm* operator. In this case index of T is defined as the index of semi-B-Fredholm operator $T_{[n]}$. A *semi-B-Fredholm operator* is an upper or lower semi-Fredholm operator [12]. Moreover, if $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm* operator [11]. An operator T is called a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not } B\text{-Weyl operator}\}$ [13].

Following [14], we say that *generalized Weyl's theorem* holds for $T \in \mathcal{B}(\mathcal{X})$, $T \in g\mathcal{W}$ if $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T) = E(T)$ and that *generalized Browder's theorem* holds for $T \in \mathcal{B}(\mathcal{X})$, $T \in g\mathfrak{B}$, if $\Delta^g(T) = \pi(T)$. It is proved in [9, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem. In [15, Theorem 3.9], it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(T) = \pi(T)$, it is proved in [16, Theorem 2.9] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let $SBF_+(\mathcal{X})$ be the class of all *upper semi-B-Fredholm operators*,

$$SBF_+^-(\mathcal{X}) := \{T \in SBF_+(\mathcal{X}) : \operatorname{ind}(T) \leq 0\}.$$

The *upper B-Weyl spectrum* of T is defined by

$$\sigma_{SBF_+^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathcal{X})\}.$$

We say that *generalized a-Weyl's theorem* holds for $T \in \mathcal{B}(\mathcal{X})$, $T \in ga\mathcal{W}$, if $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$ and that $T \in \mathcal{B}(\mathcal{X})$ obeys *generalized a-Browders theorem*, $T \in ga\mathfrak{B}$, if $\Delta_a^g(T) = \pi_a(T)$. It is proved in [9, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [15, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \pi_a(T)$ it is proved in [16, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

II. Property (gm) for Bounded Linear Operators

Definition 1. Let $T \in \mathcal{B}(\mathcal{X})$. We say that T obeys

(i) property (gw) if $\Delta_a^g(T) = E(T)$ [10].
(ii) property (gR) if $\sigma_a(T) \setminus \sigma_{LD}(T) = E(T)$ [6].
(iii) property (gt) if $\Delta_+^g(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$ [27].

In [27, Theorem 2.6] the author proved that T obeys property (gt) if and only if T obeys property (gw) and $\sigma(T) = \sigma_a(T)$.

Definition 2. ([28]) Let $T \in \mathcal{B}(\mathcal{X})$. Then we say that T obeys property (gm) if

$$\sigma(T) \setminus \sigma_{LD}(T) = E(T).$$

Generalized Weyl's theorem corresponds to the half of property (gm), in the following sense:

Theorem 1. *([28]) If $T \in \mathcal{B}(\mathcal{X})$ then the following assertions are equivalent:*

1) *Property (gm) holds for T;*
2) *T satisfies generalized Weyl's theorem and $\sigma_{LD}(T) = \sigma_{BW}(T)$.*

Theorem 2. *Let $T \in \mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent:*

(i) *Property (gt) holds for T;*
(ii) *T satisfies property (gm) and $\sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$.*

Proof: (i)$\Longrightarrow$(ii) As T has property (gt), we have T satisfies generalized Browder's theorem and so $\sigma_{LD}(T) =$

$\sigma_{SBF_+^-}(T)$. Therefore,

$$E(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{LD}(T).$$

That is, T satisfies property (gm).
(ii)$\Longrightarrow$(i) Suppose that T obeys property (m) and $\sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$. Then

$$E(T) = \sigma(T) \setminus \sigma_{LD}(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T).$$

That is, T obeys property (gt). ■

Remark 1. Let $T \in \mathcal{B}(\mathcal{X})$. If T^* has the SVEP, then it is known from [23, Page 35] that $\sigma(T) = \sigma_a(T)$ and from [3, Theorem 2.9] we have $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_D(T)$. Hence $E_a(T) = E(T)$, $\Delta^g(T) = \Delta_a^g(T)$, $\Delta_+^g(T) = \Delta^g(T)$ and $\sigma(T) \setminus \sigma_{LD}(T) = \Delta^g(T)$.

Theorem 3. *Let $T \in \mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent:*

(i) *Property (gm) holds for T;*
(ii) *T satisfies property (gR) and $\sigma(T) = \sigma_a(T)$.*

Proof: (i)$\Longrightarrow$(ii) Assume that T obeys property (gm). It then follows from Theorem 1 that T satisfies generalized Weyl's theorem and $\sigma_{LD}(T) = \sigma_{BW}(T)$ and hence T satisfies generalized Browder's theorem and $\pi(T) = E(T)$. Therefore

$$\pi_a(T) = \sigma_a(T)\setminus\sigma_{LD}(T) \subseteq \sigma(T)\setminus\sigma_{LD}(T) = E(T) = \pi(T) \subseteq \pi_a(T).$$

So, $E(T) = \pi_a(T)$, i.e, T obeys property (gR) and $\sigma(T) = \sigma_a(T)$.
(ii)$\Longrightarrow$(i) Suppose that T satisfies property (gR) and $\sigma(T) = \sigma_a(T)$. Then

$$\pi_a(T) = E(T) = \sigma_a(T) \setminus \sigma_{LD}(T) = \sigma(T) \setminus \sigma_{LD}(T).$$

That is, T obeys property (gm). ■

A bounded operator $T \in \mathcal{B}(\mathcal{X})$ is said to be finitely isoloid (respectively, finitely a-isoloid) if every isolated point of $\sigma(T)$ (respectively, every isolated point of $\sigma_a(T)$) is an eigenvalue of T having finite multiplicity.

Theorem 4. *Suppose that $T \in \mathcal{B}(\mathcal{X})$ is a finitely a-isoloid operator and suppose that there is an injective quasi-nilpotent operator Q which commutes with T. Then T obeys property (gm).*

Proof: First we note that $E(T) = \emptyset$. Indeed, suppose that $\lambda \in E(T)$. Then λ is an isolated point of $\sigma(T)$ and hence belong to $\sigma_a(T)$. Thus $\lambda \in \text{iso}\,\sigma_a(T)$, so that $0 < \alpha(T-\lambda) < \infty$, since T is finitely a-isoloid. But from [6, Lemma 3.6] we also have $\alpha(T-\lambda) = 0$, and this is impossible. Therefore, $E(T) = \emptyset$.

In order to show property (gm) holds for T, it suffices to prove that $\sigma(T)\setminus\sigma_{LD}(T) = \emptyset$. Let $\lambda \in \sigma(T)\setminus\sigma_{LD}(T)$. Then $\lambda \in \sigma(T)$ and $T-\lambda$ is left Drazin invertible. We distinguish two cases. Firstly, if $\lambda \in \sigma_a(T)$. By [2, Theorem 2.7] λ is an isolated point of $\sigma_a(T)$, and since T is finitely a-isoloid we then have $\alpha(T-\lambda) < \infty$. Again by [6, Lemma 3.6] we then conclude that $T-\lambda$ is injective. On the other hand, by [4, Lemma 2.4], we have $T-\lambda \in SF_+(\mathcal{X})$, so $T-\lambda$ has closed range and hence $T-\lambda$ is bounded below, i.e., $\lambda \notin \sigma_a(T)$, a contradiction. If $\lambda \notin \sigma_a(T)$, then $\lambda \notin \pi_a(T)$. Hence $\lambda \in \sigma_{LD}(T)$, a contradiction. Therefore, $\sigma(T)\setminus\sigma_{LD}(T) = \emptyset$, and consequently T satisfies property (gm). ■

III. Property (gm) under Perturbations by Finite Rank Operators

We begin with the following lemmas in order to give the proof of the main result in this section.

Lemma 1. *([24, Lemma 2.1]) Let $T \in \mathcal{B}(\mathcal{X})$. If F is an arbitrary finite rank operator on $\mathcal{X}$, such that $FT = TF$, then for all $\mu \in \mathbb{C}$:*

$$\mu \in acc\,\sigma(T) \Longleftrightarrow \mu \in acc\,\sigma(T+F).$$

Remark 2. If $T \in \mathcal{B}(\mathcal{X})$ is an isoloid and F is an arbitrary finite rank operator on $\mathcal{X}$, such that $FT = TF$, then it follows from Lemma 1 that

$$E(T+F) \cap \sigma(T) \subset E(T).$$

Remark 3. We conclude from [17, Theorem 2.1] that if $T \in \mathcal{B}(\mathcal{X})$ and $F \in \mathcal{F}(\mathcal{X})$ such that $TF = FT$, then

$$\sigma_{LD}(T) = \sigma_{LD}(T+F). \tag{5}$$

Recall that $T \in \mathcal{B}(\mathcal{X})$ is isolated, provided that all isolated points of $\sigma(T)$ are eigenvalues of T. $T \in \mathcal{B}(\mathcal{X})$ is a-isolated provided that all isolated points of $\sigma_a(T)$ are eigenvalues of T. It is well-known that $\partial\sigma(T) \subseteq \sigma_a(T)$, so all isolated points of $\sigma(T)$ are also isolated points of $\sigma_a(T)$. Now it is obvious that if T is a-isolated, then it is also isolated.

Theorem 5. *Let $T \in \mathcal{B}(\mathcal{X})$. Suppose that F is an arbitrary finite rank operator and $TF = FT$. If T is isoloid and property (gm) holds for T, then property (gm) holds for $T+F$.*

Proof: It is enough to prove that $0 \in \sigma(T+F)\setminus\sigma_{LD}(T+F)$ if and only if $0 \in E(T+F)$.

Firstly we prove that if $0 \in \sigma(T+F) \setminus \sigma_{LD}(T+F)$, then $T+F$ is left Drazin invertible and $0 < \alpha(T+F)$. We need to prove that $0 \in \text{iso}\sigma(T+F)$. It follows that $T \in LD(\mathcal{X})$, so $0 \notin \sigma_{LD}(T)$. It is possible that $0 \notin \sigma(T)$. In this case we get from Lemma 1 that $0 \notin \text{acc}\sigma(T)$ and hence $0 \notin \text{acc}\sigma(T+F)$, so $0 \in E(T+F)$. The second possibility is that $0 \in \sigma(T)$. Since property (gm) holds for T, we get that $0 \notin \text{acc}\sigma(T)$ and again $0 \in E(T+F)$.

To prove the opposite implication, suppose that $0 \in E(T+F)$. Then $0 \in \text{iso}\,\sigma(T+F)$ and $0 < \alpha(T+F)$. Hence $0 \notin \text{acc}\sigma(T)$ and so it follows that $0 \leq \alpha(T)$. Again we distinguish two cases. Firstly, if $0 \notin \sigma(T)$, then $T \in LD(\mathcal{X})(\mathcal{X})$ and by Remark 3 $T+F \in LD(\mathcal{X})$, $0 \in \sigma(T+F) \setminus \sigma_{LD}(T+F)$. On the other hand, if $0 \in \sigma(T)$ then $0 \in \text{iso}\sigma(T)$. Since T is isoloid, we get that $0 < \alpha(T)$ and $0 \notin \sigma_{LD}(T)$. Now, we have $T \in LD(\mathcal{X})$, $T+F \in LD(\mathcal{X})$ and $0 \in \sigma(T+F) \setminus \sigma_{LD}(T+F)$. ■

Example 1. Let $S : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ be an injective quasinilpotent operator which is not nilpotent. We define T on the Banach space $\mathcal{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = I \oplus S$. Then $\sigma(T) = \sigma_a(T) = \{0, 1\}$ and $E(T) = \{1\}$. It follows

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that $\sigma_{BW}(T)=\{0\}$ and hence $\sigma_{SBF_+^-}(T)=\sigma_{LD}(T)=\{0\}$. Hence $\sigma(T)\setminus\sigma_{LD}(T)=E(T)$ and T obeys property (gm).

We define the operator U on $\ell^2(\mathbb{N})$ by $U(x_1,x_2,\cdots):=(-x_1,0,0,\cdots)$ and $F=U\oplus 0$ on the Banach space $\mathcal{X}=\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N})$. Then F is a finite rank operator commuting with T. On the other hand, $\sigma(T+F)=\sigma_a(T+F)=\{0,1\}$ and $E(T+F)=\{0,1\}$. As $\sigma_{LD}(T+F)=\sigma_{LD}(T)=\{0\}$, then $\sigma(T+F)\setminus\sigma_{LD}(T+F)=\{1\}\neq E(T+F)$ and $T+F$ does not satisfy property (gm).

Theorem 6. *Let $T\in\mathcal{B}(\mathcal{X})$ and let F be a finite rank operator commuting with T. If T satisfies property (gm), then the following properties are equivalent.*

(i) *$T+F$ satisfies property (gm);*
(ii) *$E(T)=E(T+F)$.*

Proof: Assume that $T+F$ satisfies property (gm), then

$$\sigma(T+F)\setminus\sigma_{LD}(T+F)=E(T+F).$$

As $\sigma(T+F)=\sigma(T)$ and $\sigma_{LD}(T)=\sigma_{LD}(T+F)$ then $\sigma(T)\setminus\sigma_{LD}(T)=E(T+F)$. Since T obeys property (gm), then $E(T)=\sigma(T)\setminus\sigma_{LD}(T)$. So, $E(T)=E(T+F)$. Conversely, assume that $E(T+F)=E(T)$, then as T obeys property (gm) we have

$$E(T+F)=E(T)=\sigma(T)\setminus\sigma_{LD}(T)=\sigma(T+F)\setminus\sigma_{LD}(T+F)$$

and hence $T+F$ obeys property (gm). ■

Lemma 2. *([30]) Let $T\in\mathcal{B}(\mathcal{X})$ and let $F\in\mathcal{B}(\mathcal{X})$ with $F^n\in\mathcal{F}(\mathcal{X})$ for some $n\in\mathbb{N}$. If T commutes with F, then*

$$\sigma_{BW}(T)=\sigma_{BW}(T+F). \tag{6}$$
$$\sigma_{D}(T)=\sigma_{D}(T+F). \tag{7}$$
$$\sigma_{LD}(T)=\sigma_{LD}(T+F). \tag{8}$$

Theorem 7. *Let $T\in\mathcal{B}(\mathcal{X})$ be an isoloid and let $F\in\mathcal{B}(\mathcal{X})$ with $F^n\in\mathcal{F}(\mathcal{X})$ for some $n\in\mathbb{N}$. If T commutes with F, then*

$$E(T)=E(T+F).$$

Proof: Let $\lambda\in E(T+F)$. Then λ is an isolated point of $\sigma(T+F)$, and since $\alpha(T+F-\lambda)>0$ we then have $\lambda\in\sigma(T+F)=\sigma(T)$. Therefore, it follows from Remark 2 that $\lambda\in E(T)$. By symmetry, we have the other inclusion. ■

Theorem 8. *Let $T\in\mathcal{B}(\mathcal{X})$ be an isoloid obeys property (gm) and let $F\in\mathcal{B}(\mathcal{X})$ with $F^n\in\mathcal{F}(\mathcal{X})$ for some $n\in\mathbb{N}$. If T commutes with F, then $T+F$ obeys property (gm).*

Proof: As T obeys property (gm). Then

$$\begin{aligned}E(T)&=\sigma(T)\setminus\sigma_{LD}(T)\\&=\sigma(T+F)\setminus\sigma_{LD}(T+F)\qquad\text{(by Lemma 2)}\\&=E(T+F)\qquad\text{(by Theorem 7)}.\end{aligned}$$

Hence, $T+F$ obeys property (gm). ■

IV. Property (gm) under Perturbation by Quasi-Nilpotent Operators

First, observe that if Q is quasi-nilpotent and commutes with $T\in\mathcal{B}(\mathcal{X})$ then

$$\sigma(T)=\sigma(T+Q)\quad\text{and}\quad\sigma_a(T)=\sigma_a(T+Q). \tag{9}$$

In particular both equalities holds for commuting nilpotent operators.

Suppose that $T\in\mathcal{B}(\mathcal{X})$ and that $N\in\mathcal{B}(\mathcal{X})$ is a nilpotent operator commuting with T. Then from the proof of [18, Theorem 3.5], we have

$$\alpha(T+N)>0\Longleftrightarrow\alpha(T)>0. \tag{10}$$

Hence by Equation (9), we have the following equation:

$$E(T+N)=E(T). \tag{11}$$

Lemma 3. *Suppose that $T\in\mathcal{B}(\mathcal{X})$ and that $N\in\mathcal{B}(\mathcal{X})$ is a nilpotent operator commuting with T. Then*

$$\sigma_{LD}(T+N)=\sigma_{LD}(T). \tag{12}$$

Proof: It follows from [30, Corllary 3.8] that $\pi_a(T+N)=\pi_a(T)$. Then

$$\begin{aligned}\sigma_{LD}(T+N)&=\sigma_a(T+N)\setminus\pi_a(T+N)\\&=\sigma_a(T)\setminus\pi_a(T+N)\quad\text{(by Equation 9)}\\&=\sigma_a(T)\setminus\pi_a(T)\\&=\sigma_{LD}(T).\end{aligned}$$

So, the proof of the lemma is achieved. ■

Theorem 9. *Suppose that $T\in\mathcal{B}(\mathcal{X})$ has property (gm) and that $N\in\mathcal{B}(\mathcal{X})$ is a nilpotent operator commuting with T. Then $T+N$ has property (gm).*

Proof: As T obeys property (gm), we have

$$\begin{aligned}&E(T+N)=E(T)\quad\text{(by Equation 11)}\\&=\sigma(T)\setminus\sigma_{LD}(T)\\&=\sigma(T+N)\setminus\sigma_{LD}(T+N)\quad\text{(by Equation (9)and Lemma 3}\end{aligned}$$

That is, $T+N$ obeys property (gm). ■

The following example shows that property (gm) is not stable under commuting quasi-nilpotent perturbations.

Example 2. Let $Q:\ell^2(\mathbb{N})\longrightarrow\ell^2(\mathbb{N})$ be a quasi-nilpotent operator defined by

$$Q(x_1,x_2,\cdots):=\left(\frac{x_2}{2},\frac{x_3}{3},\cdots\right)\text{ for all }(x_n)\in\frac{x_2}{2}.$$

Then Q is quasi-nilpotent, $\sigma(Q)=\sigma_{LD}(Q)=\{0\}$ and $E(T)=\{0\}$. Take $T=0$. Clearly, T satisfies property (gm), but $T+Q=Q$ fails to satisfy property (gm).

A bounded operator $T\in\mathcal{B}(\mathcal{X})$ is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T and that $T\in\mathcal{B}(\mathcal{X})$ is said to be a-polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T. It is known that T is polaroid if and only if T^* is polaroid and evidently,

$$T\ a\text{-polaroid}\Longrightarrow T\text{ polaroid}, \tag{13}$$

while, in general, the converse does not hold.

Theorem 10. *Let $T \in \mathcal{B}(\mathcal{X})$ obeys property (gm). If T is a-polaroid and finitely isoloid, Q is a quasi-nilpotent operator which commutes with T, then $T+Q$ obeys property (gm).*

Proof: It follows from [6, Theorem 4.8] that $T+Q$ is a-polaroid and hence by [6, Theorem 3.2], we have $T+Q$ obeys property (gR). As T obeys property (gm), we have by Theorem 3 that T satisfies property (gR) and $\sigma(T) = \sigma_a(T)$. Therefore,

$$\begin{aligned} E(T+Q) &= \sigma_a(T+Q) \setminus \sigma_{LD}(T+Q) \\ &= \sigma_a(T) \setminus \sigma_{LD}(T+Q) \\ &= \sigma(T) \setminus \sigma_{LD}(T+Q) \\ &= \sigma(T+Q) \setminus \sigma_{LD}(T+Q). \end{aligned}$$

That is, $T+Q$ obeys property (gm). ■

V. Property (gm) under Perturbations by Algebraic Operators

We shall consider algebraic perturbations of operators satisfying property (gm).

A bounded linear operator T is said to be *algebraic* if there exists a non-trivial polynomial h such that $h(T) = 0$. From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. A nilpotent operator is a trivial example of an algebraic operator. Also finite rank operators K are algebraic; more generally, if K^n is a finite rank operator for some $n \in \mathbb{N}$ then K is algebraic. Clearly, if T is algebraic then its dual T^* is algebraic, as well as T' in the case of Hilbert space operators.

Let $H_{nc}(T)$ denotes the set of all complex-valued functions f, defined and regular in some neighborhood of $\sigma(T)$, such that f is not constant on the connected components of its domain of definition.

Theorem 11. *Suppose that $T \in \mathcal{B}(\mathcal{X})$ and $K \in \mathcal{B}(\mathcal{X})$ is an algebraic operator which commutes with T.*

(i) *If T^* is hereditarily polaroid and has SVEP, then $T+K$ obeys property (gm).*
(ii) *If T is hereditarily polaroid and has SVEP, then T^*+K^* obeys property (gm).*

Proof: (i) Obviously, K^* is algebraic and commutes with T^*. Moreover, by [7, Theorem 2.15], we have T^*+K^* is polaroid, or equivalently, $T+K$ is polaroid. Since T^* has SVEP then by [5, Theorem 2.14], we have T^*+K^* has SVEP . Therefore, $T+K$ obeys property (gm) by [28, Theorem 3.4 (i)].
(ii) It follows from the proof of Theorem 2.15 of [7] that $T+K$ is polaroid and hence by duality T^*+K^* is polaroid. Since T has SVEP then it follows from [5, Theorem 2.14] that $T+K$ has SVEP. Therefore, T^*+K^* obeys property (m) by [28, Theorem 3.3 (ii)]. ■

Theorem 12. *Suppose that $T \in \mathcal{B}(\mathcal{X})$ and $K \in \mathcal{B}(\mathcal{X})$ is an algebraic operator which commutes with T.*

(i) *If T^* is hereditarily polaroid and has SVEP, then $f(T+K)$ obeys property (gm) for all $f \in H_{nc}(\sigma(T))$.*
(ii) *If T is hereditarily polaroid and has SVEP, then $f(T^*+K^*)$ obeys property (gm) for all $f \in H_{nc}(\sigma(T))$.*

Proof: (i) We conclude from [7, Theorem 2.15] that $T+K$ is polaroid and hence by [8, Lemma 3.11], we have $f(T+K)$ is polaroid and from [5, Theorem 2.14] that T^*+K^* has SVEP. The SVEP of T^*+K^* entails the SVEP for $f(T^*+K^*)$ by [1, Theorem 2.40]. So, $f(T+K)$ obeys property (m) by [28, Theorem 3.4 (i)].
(ii) The proof of part (ii) is analogous. ■

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