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Stability and bifurcation analysis of a discrete Gompertz model with time delay

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Abstract—In this paper, we consider a discrete Gompertz model with time delay. Firstly, the stability of the equilibrium of the system is investigated by analyzing the characteristic equation. By choosing the time delay as a bifurcation parameter, we prove that Neimark-Sacker bifurcations occur when the delay passes a sequence of critical values. The direction and stability of the Neimark-Sacker are determined by using normal forms and centre manifold theory. Finally, some numerical simulations are given to verify the theoretical analysis.

Keywords—Gompertz system; Neimark-Sacker bifurcation; Stability; Time delay.

I. INTRODUCTION

IT is well known that Gompertz equation [1] is one of the most important models in the description of the growth law for a single species. The model reads as

$$\dot{x}(t) = -rx(t)\ln\frac{x(t)}{K}, \tag{1}$$

where $x(t)$ denotes the population density, r is a positive constant called the intrinsic growth rate, the positive constant K is usually referred to as the environment carrying capacity or saturation level, and $-r\ln\frac{x(t)}{K}$ denotes relative growth rate. Assuming that a growing population requires more food (growth and maintenance) than a saturated one (maintenance only), a further modification is to assume that the growth rate is a function of some specified delayed argument $t-\tau$ (see, e.g. [2]). The model (1) becomes

$$\dot{x}(t) = -rx(t)\ln\frac{x(t-\tau)}{K}, \tag{2}$$

where $\tau > 0$ is the time delay. System (2) is called as delayed Gompertz model. The continuous-time system (1), (2) and their similar systems have been extensively studied in the literature (see e.g., [2-4]).

But considering the need of scientific computation and real-time simulation, our interest is focused on the behaviors of discrete dynamics system corresponding to (2). Many authors considered the numerical approximation of a scalar delay differential equation by using different numerical methods, such as nonstandard finite-difference method, Euler method, Runge-Kutta method (see [5-11]). In this paper, we use the forward Euler scheme to make the discretization for system (2).

Moreover, it is also of interest to find what will happen when the system loses stability. The purpose of this paper is to discuss this version as a discrete dynamical system by using Neimark-Sacker bifurcation theory of discrete systems. We not only investigate the stability of the fixed point and the existence of the Neimark-Sacker bifurcations, but also the stability and direction of the Neimark-Sacker bifurcation of the discrete system.

The paper is organized as follows: in Section 2, we analyze the distribution of the characteristic equation associated with the discrete model, and obtain the existence of the local Neimark-Sacker bifurcation. In Section 3, the direction and stability of closed invariant curve from the Neimark-Sacker bifurcation of the discrete delay model are determined by using the theories of discrete systems in [12]. In the final section, some computer simulations are performed to illustrate the analytical results found.

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II. STABILITY ANALYSIS

Let $y(t) = \frac{x(t\tau)}{K} - 1$. Then (2) can be rewritten as

$$\dot{y}(t) = -r\tau(y(t)+1)\ln(y(t-1)+1). \tag{3}$$

We consider step size of the form $h = \frac{1}{m}$, where $m \in \mathbb{Z}_+$. The Euler method applied to this equation, yields the delay difference equation

$$y_{n+1} = y_n - rh\tau(y_n+1)\ln(y_{n-m}+1). \tag{4}$$

where u_n is an approximate value to $y(nh)$.

It is clear that Eq. (4) has a unique zero equilibrium. By introducing a new variable $Y_n = (y_n, y_{n-1}, \cdots, y_{n-m})^T$, we can rewrite (4) in the form

$$Y_{n+1} = F(Y_n, \tau). \tag{5}$$

where $F = (F_0, F_1, \cdots, F_m)^T$, and

$$F_j = \begin{cases} y_n - rh\tau(y_n+1)\ln(y_{n-m}+1), & j = 0 \\ y_{n-j+1}, & 1 \le j \le m \end{cases} \tag{6}$$

Clearly the origin is a fixed point of (5), and the linear part of (5) is

$$Y_{n+1} = AY_n \tag{7}$$

where

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & -rh\tau \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 & 0 \end{pmatrix}$$

The characteristic equation of A is given by

$$a(\lambda) := \lambda^{m+1} - \lambda^m + rh\tau = 0. \quad (8)$$

It is well known that the stability of the zero equilibrium solution of (5) depends on the distribution of the zeros of the roots of (8). In this paper, we will employ the results from Zhang et al. [6] and He et al. [10] to analyze the distribution of the zeros of the characteristic Eq. (8). In order to proof the existence of the local Neimark-Sacker bifurcation at fixed point, we need some lemmas as follows.

Lemma 2.1.(see[6,10]) Suppose that $\hat{B} \subset R$ is a bounded, closed, and connected set, $f(\lambda,\tau) = \lambda^m + p_1(\tau)\lambda^{m-1} + p_2(\tau)\lambda^{m-2} + \cdots + p_m(\tau)$ is continuous in $(\lambda,\tau) \in C \times \hat{B}$. Then as τ varies, the sum of the order of the zeros of $f(\lambda,\tau)$ out of the unit circle

$$\{\lambda \in C : |\lambda| > 1\}$$

can change only if a zero appears on or crosses the unit circle.

Lemma 2.2. All roots of Eq. (8) have modulus less than one for sufficiently small positive $\tau > 0$.

Proof. When $\tau = 0$, (8) becomes

$$\lambda^{m+1} - \lambda^m = 0.$$

The equation has, at $\tau = 0$, an m-fold root $\lambda = 0$, and a simple root $\lambda = 1$.

Consider the root $\lambda(\tau)$ such that $\lambda(0) = 1$. This root depends continuously on τ and is a differential function of τ. From (8), we have

$$\frac{d\lambda}{d\tau} = \frac{rh}{m\lambda^{m-1} - (m+1)\lambda^m} \quad (9)$$

and

$$\frac{d\bar{\lambda}}{d\tau} = \frac{rh}{m\bar{\lambda}^{m-1} - (m+1)\bar{\lambda}^m}. \quad (10)$$

We have

$$\frac{d|\lambda|^2}{d\tau}|_{\tau=0,\lambda=1} = [\lambda\frac{d\bar{\lambda}}{d\tau} + \bar{\lambda}\frac{d\lambda}{d\tau}]|_{\tau=0,\lambda=1} = -2rh < 0.$$

So with the increasing of $\tau > 0$, λ cannot cross $\lambda = 1$. Consequently, all roots of Eq. (8) lie in the unit circle for sufficiently small positive $\tau > 0$. □

A Neimark-Sacker bifurcation occurs when a complex conjugate pair of eigenvalues of A cross the unit circle as τ varies. We have to find values of τ such that there are roots on the unit circle. Denote the roots on the unit circle by $e^{i\omega^*}$. Then

$$e^{i\omega^*} - 1 + rh\tau e^{-im\omega^*} = 0. \quad (11)$$

Separating the real part and imaginary part from Eq. (11), there are

$$\cos\omega^* + rh\tau^*\cos m\omega^* = 1 \quad (12)$$

and

$$\sin\omega^* - rh\tau^*\sin m\omega^* = 0. \quad (13)$$

So

$$\cos\omega^* = 1 - \frac{1}{2}(rh\tau^*)^2. \quad (14)$$

Summarizing the discussion above, we obtain that the roots $e^{\pm i\omega^*}$ of Eq. (8) with modulus one satisfy

$$\begin{cases} \cos\omega^* = 1 - \frac{1}{2}(rh\tau^*)^2, \\ \tau^* = \frac{\sin\omega^*}{rh\sin m\omega^*}, \\ h = \frac{1}{m}, \end{cases} \quad (15)$$

It is clear that there exists an infinite sequence of values of the time delay parameter $0 < \tau_0 < \tau_1 < \cdots < \tau_j < \cdots$ satisfying Eq. (15).

Lemma 2.3. Let $\lambda(\tau) = r(\tau)e^{i\omega(\tau)}$ be a root of (8) near $\tau = \tau^*$ satisfying $r(\tau^*) = 1$ and $\omega(\tau^*) = \omega^*$. Then

$$\frac{dr^2(\tau)}{d\tau}|_{\tau=\tau^*,\omega=\omega^*} > 0.$$

Proof. From (12) and (13), we obtain that

$$\cos m\omega^* = \frac{1-\cos\omega^*}{rh\tau^*}, \quad (16)$$

$$\sin m\omega^* = \frac{\sin\omega^*}{rh\tau^*}, \quad (17)$$

It is easy to see that

$$\begin{aligned} \cos(m+1)\omega^* &= \cos m\omega^*\cos\omega^* - \sin m\omega^*\sin\omega^* \\ &= \frac{\cos\omega^* - 1}{rh\tau^*}. \end{aligned} \quad (18)$$

From (9), (10) and using (16) – (18), we have

$$\begin{aligned} \frac{dr^2}{d\tau}|_{\tau=\tau^*,\omega=\omega^*} &= \frac{d|\lambda|^2}{d\tau}|_{\tau=\tau^*,\omega=\omega^*} \\ &= [\lambda\frac{d\bar{\lambda}}{d\tau} + \bar{\lambda}\frac{d\lambda}{d\tau}]|_{\tau=\tau^*,\omega=\omega^*} \\ &= \frac{2(2m+1)(1-\cos\omega^*)}{\tau^*\left|me^{i(m-1)\omega^*} - (m+1)e^{im\omega^*}\right|^2} > 0 \end{aligned}$$

This completes the proof. □

Applying Lemmas 2.1 – 2.3, we have the following Lemma.

Lemma 2.4. Eq. (8) has a pair of simple roots $e^{\pm i\omega^*}$ on the unit circle when $\tau = \tau_j,\ j = 0,1,2,\cdots$. Furthermore, if $\tau \in [0,\tau_0)$, then all the roots of Eq. (8) have modulus less than one; If $\tau > \tau_0$, then Eq. (8) has at least a couple of roots with modulus more than one.

Lemma 2.4 immediately lead to stability of the zero equilibrium of Eq. (4). So we have the following results on stability and bifurcation in system (4).

Theorem 2.1. there exists a sequence of values of the time delay parameter $0 < \tau_0 < \tau_1 < \cdots < \tau_j < \cdots$ such that the zero equilibrium of Eq. (4) is asymptotically stable for $\tau \in [0,\tau_0)$ and unstable for $\tau > \tau_0$. Eq. (4) undergoes a Neimark-Sacker bifurcation at the zero equilibrium when $\tau = \tau_j,\ j = 0,1,2,\cdots$, where τ_j satisfies (15).

III. Direction and Stability of the Neimark-Sacker Bifurcation in Discrete Model

In the previous section, we obtain the conditions under which a family of periodic solutions bifurcate from the steady state at the critical value $\tau = \tau_j,\ j = 0,1,2,\cdots$. Without loss of generality, denote the critical value $\tau = \tau_j$ by τ^*. In this section, following the idea of Hassard et al. [13], we shall study the direction, stability and the period of the bifurcating periodic solution when $\tau = \tau^*$ in the discrete Gompertz model. The method, we used is based on the theories of discrete system by Kuznetsov [12].

Rewrite Eq. (4) as

$$\begin{aligned} y_{n+1} &= y_n - rh\tau y_{n-m} + \tfrac{1}{2}rh\tau(-2y_n y_{n-m} + y_{n-m}^2) \\ &\quad + \tfrac{1}{6}rh\tau(3y_n y_{n-m}^2 - 2y_{n-m}^3) + O(|y_n^2 + y_{n-m}^2|^2). \end{aligned}$$

So system (5) is turned into

$$Y_{n+1} = AY_n + \tfrac{1}{2}B(Y_n, Y_n) + \tfrac{1}{6}C(Y_n, Y_n, Y_n) + O(\| Y_n \|^4), \tag{19}$$

where

$$\begin{aligned} B(Y_n, Y_n) &= (b_0(Y_n, Y_n), 0, \cdots, 0), \\ C(Y_n, Y_n, Y_n) &= (c_0(Y_n, Y_n, Y_n), 0, \cdots, 0), \end{aligned}$$

and

$$\begin{aligned} b_0(\phi, \psi) &= rh\tau(-\phi_0\psi_m - \phi_m\psi_0 + \phi_m\psi_m), \\ c_0(\phi, \psi, \eta) &= rh\tau(\phi_0\psi_m\eta_m + \phi_m\psi_0\eta_m \\ &\quad + \phi_m\psi_m\eta_0 - 2\phi_m\psi_m\eta_m). \end{aligned} \tag{20}$$

Let $q \in \mathbb{C}^{m+1}$ be an eigenvector of A corresponding to $e^{i\omega^*}$, then

$$Aq = e^{i\omega^*}q, \qquad A\bar{q} = e^{-i\omega^*}\bar{q}.$$

We also introduce an adjoint eigenvector $q^* \in \mathbb{C}^{m+1}$ having the properties

$$A^T q^* = e^{-i\omega^*}q^*, \qquad A^T\bar{q}^* = e^{i\omega^*}\bar{q}^*,$$

and satisfying the normalization $< q^*, q >= 1$, where $< q^*, q >= \sum_{j=0}^{m} \bar{q}_j^* q_j$.

Lemma 3.1. Let $q = (q_0, q_1, \cdots, q_m)^T$ be the eigenvector of A corresponding to the eigenvalue $e^{i\omega^*}$ and $q^* = (q_0^*, q_1^*, \cdots, q_m^*)^T$ be the eigenvector of A^T corresponding to the eigenvalue $e^{-i\omega^*}$, then

$$\begin{aligned} q &= (1, e^{-i\omega^*}, \cdots, e^{-im\omega^*})^T, \\ q^* &= \bar{D}(1, \alpha e^{im\omega^*}, \cdots, \alpha e^{i\omega^*})^T. \end{aligned} \tag{21}$$

where $\alpha = -rh\tau$ and $D = (1 + mrh\tau e^{i(m+1)\omega^*})^{-1}$

Proof. Let $q = (q_0, q_1, \cdots, q_m)^T$ be the eigenvector of A corresponding to the eigenvalue $e^{i\omega^*}$, then

$$q_j = e^{i\omega^*}q_m, \quad j = 1, \cdots, m, \tag{22}$$

Setting $q_0 = 1$, we obtain that $q = (q_0, q_1, \cdots, q_m)^T$ is the eigenvector of A corresponding to the eigenvalue $e^{i\omega^*}$.

Similarly, assign q^* satisfies $A^T q^* = \bar{z}q^*$ with $\bar{z} = e^{-i\omega_0}$, then the following identities hold

$$\begin{cases} q_j^* = e^{-i\omega^*}q_{j-1}^*, \quad j = 2, \cdots, m, \\ -rh\tau q_0^* = e^{-i\omega^*}q_m^*. \end{cases} \tag{23}$$

Let $q_m^* = \alpha e^{i\omega^*}\bar{D}$, then

$$q^* = \bar{D}(1, \alpha e^{im\omega^*}, \alpha e^{i(m-1)\omega^*}, \cdots, \alpha e^{i2\omega^*}, \alpha e^{i\omega^*})^T.$$

From normalization $< q^*, q >= 1$ and computation, we get

$$D = (1 + mrh\tau e^{i(m+1)\omega^*})^{-1}. \ \square$$

Let T^c denote a real eigenspace corresponding to $e^{\pm i\omega^*}$, which is two dimensional and is spanned by $\{Re(q), Im(q)\}$ and T^s a real eigenspace corresponding to all eigenvalues of A^T other than $e^{\pm i\omega^*}$ is $(m-1)$ dimensional.

For any $x \in \mathbb{R}^{m+1}$, we have its decomposition

$$x = zq + \bar{z}\bar{q} + y,$$

where $z \in \mathbb{C}$, $zq + \bar{z}\bar{q} \in T^c$, $y \in T^s$. The complex variable z can be viewed as a new coordinate on T^c. Now we adopt the computation process introduced by Kuznetsov ([12], pp. 184-186), we have that the restriction of the Eq. (19) to the centre manifold, up to cubic term is given by

$$z \to e^{i\omega^*}z + \tfrac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \tfrac{g_{02}}{2}\bar{z}^2 + \tfrac{g_{21}}{2}z^2\bar{z} + \cdots,$$

where

$$\begin{aligned} g_{20} &= < q^*, B(q, q) >, \\ g_{11} &= < q^*, B(q, \bar{q}) >, \\ g_{02} &= < q^*, B(\bar{q}, \bar{q}) >, \\ g_{21} &= < q^*, C(q, q, \bar{q}) > \\ &\quad -2 < q^*, B(q, (I - A^{-1})B(q, \bar{q})) > \\ &\quad + < q^*, B(\bar{q}, (\lambda^2 I - A)^{-1}B(q, q)) > \\ &\quad - \tfrac{1-2\lambda}{\lambda^2-\lambda} < q^*, B(q, q) > \times < q^*, B(q, \bar{q}) > \\ &\quad - \tfrac{2}{1-\bar{\lambda}}| < q^*, B(\bar{q}, \bar{q}) > |^2 \\ &\quad - \tfrac{1}{\bar{\lambda}^2-\bar{\lambda}}| < q^*, B(\bar{q}, \bar{q}) > |^2. \end{aligned} \tag{24}$$

Define

$$c_1(\tau) = \frac{g_{20}g_{11}(2\lambda + \bar{\lambda} - 3)}{2(\lambda - 1)(\lambda^2 - \lambda)} + \frac{|g_{11}|^2}{1 - \bar{\lambda}} + \frac{|g_{02}|^2}{2(\lambda^2 - \bar{\lambda})} + \frac{g_{21}}{2}, \tag{25}$$

Substituting $\lambda = e^{-i\omega^*}$ into (24) and (25), we can obtain $c_1(\tau^*)$.

Lemma 3.2. (See [14].) Given the map (5) and assume
(1) $\lambda(\tau) = r(\tau)e^{i\omega(\tau)}$, where $r(\tau^*) = 1, r'(\tau^*) \neq 0$ and $\omega(\tau^*) = \omega^*$;
(2) $e^{ik\omega^*} \neq 1$ for $k = 1, 2, 3, 4$;
(3) $Re[e^{-i\omega^*}c_1(\tau^*)] \neq 0$.

Then an invariant closed curve, topologically equivalent to a circle, for map (5) exists for τ in a one side neighborhood of τ^*. The radius of the invariant curve grows like $O(\sqrt{|\tau - \tau^*|})$. One of the four cases below applies:

(1) $r'(\tau^*) > 0, Re[e^{-i\omega^*}c_1(\tau^*)] < 0$. The origin is asymptotically stable for $\tau < \tau^*$ and unstable for $\tau > \tau^*$. An attracting invariant closed curve exists for $\tau > \tau^*$.
(2) $r'(\tau^*) > 0, Re[e^{-i\omega^*}c_1(\tau^*)] > 0$. The origin is asymptotically stable for $\tau < \tau^*$ and unstable for $\tau > \tau^*$. An repelling invariant closed curve exists for $\tau < \tau^*$.
(3) $r'(\tau^*) < 0, Re[e^{-i\omega^*}c_1(\tau^*)] < 0$. The origin is asymptotically stable for $\tau > \tau^*$ and unstable for $\tau < \tau^*$. An attracting invariant closed curve exists for $\tau < \tau^*$.
(4) $r'(\tau^*) < 0, Re[e^{-i\omega^*}c_1(\tau^*)] > 0$. The origin is asymptotically stable for $\tau > \tau^*$ and unstable for $\tau < \tau^*$. An repelling invariant closed curve exists for $\tau > \tau^*$.

From the discussion in Section 2, we know that $r'(\tau^*) > 0$, therefore, by Lemma 3.2 we have the following result.

Theorem 3.1. For Eq. (4), the zero equilibrium is asymptotically stable for $\tau < \tau^*$, and unstable for $\tau > \tau^*$. An attracting (repelling) invariant closed curve exists for $\tau > \tau^*$ if $Re[e^{-i\omega^*}c_1(\tau^*)] < 0 (> 0)$.

IV. Computer simulation

In this section, we will confirm our theoretical analysis by numerical simulation. We give an example of system (4) with $r = 1, m = 20, h = 0.05$. Then Eq. (4) becomes

$$y_{n+1} = y_n - 0.05\tau(y_n + 1)\ln(y_{n-20} + 1). \tag{26}$$

From Eq. (15), it follows that $\tau_0 = 1.5321$ is the Neimark-Sacker bifurcation value.

In Fig. 1, we show the waveform plot and phase plot for (26) with initial values $y_j = 0.1$ $(j = 0, 1, \cdots, 20)$ for $\tau = 1.5 < \tau_0 = 1.5321$. The zero equilibrium of Eq. (26) is asymptotically stable. In Fig. 2, we show the waveform plot for (26) with initial values $y_j = 0.1$ $(j = 0, 1, \cdots, 20)$. The zero equilibrium of (26) is unstable for $\tau = 1.533 > \tau_0 = 1.5321$. When τ varies and passes through $\tau_0 = 1.5321$, the equilibrium loses its stability and a periodic solution bifurcates from the equilibrium for $\tau = 1.533 > \tau_0 = 1.5321$. That is the delay difference Eq. (26) which has a Neimark-Sacker bifurcation at τ_0.

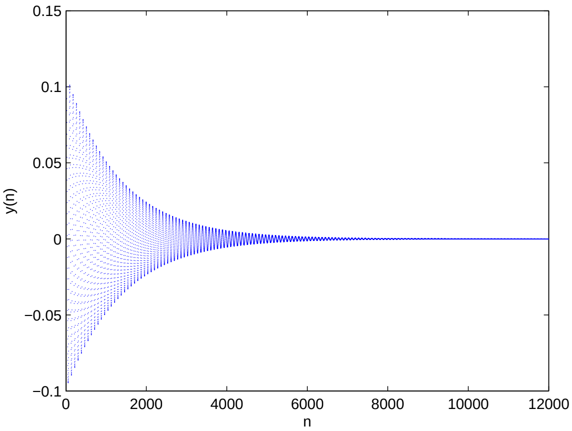

Fig.1. The equilibrium u^* of (26) is asymptotically stable for $\tau = 1.5 < \tau_0 = 1.5321$.

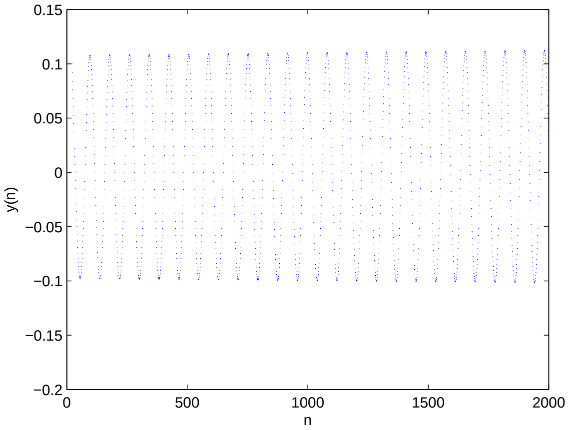

Fig.2. A bifurcating periodic solution appears for $\tau = 1.533 > \tau_0 = 1.5321$.

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