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Stability analysis of linear switched systems with mixed delays

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Abstract—This paper addresses the stability of the switched systems with discrete and distributed time delays. By applying Lyapunov functional and function method, we show that, if the norm of system matrices B_i is small enough, the asymptotic stability is always achieved. Finally, a example is provided to verify technically feasibility and operability of the developed results.

Keywords—Switched system, stability, Lyapunov function, Lyapunov functional, delays.

I. Introduction

SWITCHED systems are a special class of hybrid systems which contain both continuous dynamics and discrete dynamics. In recent years, switched systems have been studied with growing interest and activity in many fields of science ranging from economics to electrical and mechanical engineering [1], [2].

Stability analysis is the most important topic for the study of switched systems (see [3]-[30] and references therein). In switched systems, there are three cases: a switching system consisted of stable subsystems, of unstable subsystems, and of both stable and unstable subsystems. Here we can observe that consisting of stable subsystems does not necessarily guarantee stability of switching system. On the other hand, as mentioned earlier a switching system composed with unstable subsystems can be stabilized by appropriate switching rule [20]. In the first case, a switched system consisting stable subsystems, one natural approach is to construct multiple Lyapunov functions. Branicky [21], [22] provided conditions for globally asymptotically stability using multiple Lyapunov functions in this case. In the second case, a switched system consisting of unstable subsystems, it will be not possible to construct a Lyapunov function for each subsystem, hence, the other option is to use single Lyapunov functions. The question is how to construct it among unstable subsystems. Wicks and Peleties [20], [23] considered a $m-$switched system consisting of m linear autonomous unstable systems. They developed a control law using a Lyapunov function having a piecewise continuous derivative. Also Schaft and Schumacher [24] provided a new switching rule-The Minimum Rule by constructed a stable linear convex combination of the unstable subsystems. By this method the asymptotic stability is achieved without chattering or sliding motion. In the third case, a switched system including both stable and unstable subsystems, the key idea is the dwell time. Hespanha and Morse [25], [26] provided a switching rule called the Average Dwell time approach. Also Zhai *et al.* [27] extended this approach, they showed that if the average dwell time is chosen large enough and the total activation time of unstable subsystems is relatively small compared to that of stable subsystems, the exponential stability of a desired degree is guaranteed.

In some switched systems, the phenomenon of time delays is rather widespread. Hence, establishing stability conditions for those systems would be beneficial. For example, stability analysis of time delay systems are reported in [28], [29], [30] and the references cited therein. There results are obtained based on the Razumikhin method and the Lyapunov-Krasovskii method. More specifically, the Razumikhin method is used in [28] to study robust stability and robust stabilization for linear systems involving a time delay and a norm bounded parametric uncertainty. In [29], some results on the robust performance of linear delayed systems are obtained based on the Lyapunov-Krasovskii functional method.

In this paper, we investigate delay-dependent stability for a class of linear switched systems with discrete and distributed time delays by means of multiple and single Lyapunov functional and multiple and single Lyapunov function methods. The rest of the paper is organized as follows: Section II provides mathematical background necessary to state the main results of the note. Section III considers the case where the subsystem matrices A_i are stable and presents some standard delay differential equations results in terms of multiple Lyapunov functional and function. In section IV, for case where the subsystem matrices A_i are unstable, we establish stability conditions by means of single Lyapunov functional and function. A numerical example is presented in section V. Finally, concluding remarks are presented in section VI.

II. System description and preliminaries

Consider the following linear switched system with discrete and distributed time delays

$$\dot{x}\left(t\right) = A_{\sigma(t)}x\left(t\right) + B_{\sigma(t)}x\left(t-\tau\right) + C_{\sigma(t)}\int_{t-h}^{t} x\left(s\right) ds, \quad (1)$$

where $x\left(t\right) \in R^n$ is the state, $h, \tau > 0$ are the discrete and distribute delays, respectively. $A_{\sigma(t)}, B_{\sigma(t)}, C_{\sigma(t)} \in R^{n\times n}$, and $\sigma\left(t\right) : \left[0,\infty\right) \rightarrow M = \left\{1, 2, \cdots, m\right\}$ is the switching function. Define $x_t\left(\theta\right) = x\left(t+\theta\right) \in \mathbb{C} = C\left(\left[-\tau^*, 0\right], R^n\right)$ and $\left\|x_t\right\|_{\tau^*} = \sup_{-\tau^*\leq\theta\leq0}\left\|x_t(\theta)\right\|$. $\left\|\cdot\right\|$ is Euclidean norm, $-\tau^* \leq \theta \leq 0$, $\tau^* = \max\left(h, \tau\right)$.

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Moreover, the ith subsystem of (1) can be written as

$$\dot{x}(t) = A_i x(t) + B_i x(t-\tau) + C_i \int_{t-h}^{t} x(s)\,ds. \quad (2)$$

For our results, we collect the following technical results.

In [31], for the linear switched system with discrete delays

$$\dot{x}(t) = A_i x(t) + B_i x(t-\tau) \quad (3)$$

with $A_i, B_i \in R^{n\times n}, i \in M$, and $\tau > 0$ is constant discrete delays, the author established the following results.

Theorem 1: Supposed that each of the subsystems of (3) is stable. If there exists a Lyapunov functional $V_i(x_t), i = 1, 2, \cdots, m$ satisfying

(i) There exist continuous and increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$, such that $\alpha(0) = \beta(0) = 0$ and

$$\alpha(\|x(t)\|) \leq V_i(x_t) \leq \beta(\|x_t\|_{\tau^*}).$$

(ii) For the ith , there is a continuous and increasing function$\psi_i(\cdot)$, such that $\psi_i(0) = 0, \forall s > 0$ with $\psi_i(s) > 0$ and

$$\dot{V}_i(x_t) \leq -\psi_i(\|x(t)\|).$$

(iii) There exists a constant $\mu > 0$, such that for all $x_t \in C([-\tau^*, 0], R^n)$,

$$V_i(x_t) \leq \mu V_j(x_t), \quad i \neq j$$

(iv) For any pair of consecutive switching times t_p, t_q of the ith subsystem, let $t_p < t_q$ and the ith mode is active at t_p and t_q, respectively. If there is a constant $0 < \xi_i < 1$ such that

$$V_i(x_{t_q}) \leq (1-\xi_i) V_i(x_{t_p}).$$

Then the trivial solution of (3) is uniformly asymptotically stable for any switching rule.

Theorem 2: Supposed that each of the subsystems of (3) is stable. If there exists a Lyapunov function $V_i(x(t)), i = 1, 2, \cdots, m$ satisfying

(i) There exist continuous and increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$, such that $\alpha(0) = \beta(0) = 0$ and

$$\alpha(\|x(t)\|) \leq V_i(x(t)) \leq \beta(\|x(t)\|).$$

(ii) For each i , there are continuous and increasing functions $\psi_i(\cdot)$ and $p_i(\cdot)$, $\psi_i(0) = 0$, $\psi_i(s) > 0, s > 0$, $p_i(s) > s, s > 0$ and

$$\dot{V}_i(x(t)) \leq -\psi_i(\|x(t)\|)$$

with $V_i(x(t+\theta)) < p_i V_i(x(t)), \theta \in [-\tau, 0]$

(iii) There is $\mu > 0$ such that, for $x \in R^n$

$$V_i(x) \leq \mu V_j(x), i \neq j.$$

(iv) For any pair of consecutive switching times t_p, t_q of the ith subsystem, let $t_p < t_q$ and the ith mode is active at t_p and t_q, respectively. If there is a constant $0 < \xi_i < 1$ such that

$$\bar{V}_i(t_q) \leq (1-\xi_i)\bar{V}_i(t_p)$$

with $\bar{V}_i(t) = \sup\limits_{-\tau^* \leq \theta \leq 0} V_i(x(t+\theta))$.

Then the trivial solution of (3) is uniformly asymptotically stable for any switching rule.

In section III, for the case of stable A_i of system (2), we are devoted to extend the above standard delay differential equations results to the systems with mixed delays by means of single Lyapunov functional and function methods. On the other hand, when all A_i are unstable, the standard delay differential equations results will not be derived. Section IV will present some sufficient conditions guaranteeing the asymptotic stability of system (1). To this end, we need the following classical results [32].

Consider the following nonlinear system

$$\dot{x} = f(t, x_t), \quad (4)$$

where $f(t, x_t)$ is continuous.

Lemma 1: For the system (4), let $f : R \times C \mapsto R^n$ take $R\times$(bounded sets of C) into bounded sets of R^n, $\mu, \nu, \omega : R^+ \mapsto R^+$ are continuous nondecreasing functions, $\mu(s), \nu(s), \omega(s) > 0$ for $s > 0$, and $\mu(0) = \nu(0) = 0$. If there are a continuous function Lyapunov functional $V(x_t) : R \times C \mapsto R$ such that

(i) $\mu(\|x(t)\|) \leq V(t, x_t) \leq \nu(\|x_t\|_{\tau^*})$.
(ii) $\dot{V}_i(t, x_t) \leq -\omega(\|x(t)\|)$.

Then the trivial solution of (4) is uniformly asymptotically stable.

Lemma 2: For the system (4), Suppose that $f : R \times C \mapsto R^n$ takes $R\times$(bounded sets of C) into bounded sets of R^n, and $\mu, \nu, \omega : R^+ \mapsto R^+$ are continuous nondecreasing functions, $\mu(s), \nu(s), \omega(s) > 0$ are positive for $s > 0$, and $\mu(0) = \nu(0) = 0$. If there are a continuous function $V : R \times R^n \mapsto R$ and a continuous and nondecreasing function $p(s) > s$ for $s > 0$ such that

(i) $\mu(\|x(t)\|) \leq V(t, x(t)) \leq \nu(\|x(t)\|)$.
(ii) For $\theta \in [-\tau^*, 0]$, if

$$V(t+\theta, x(t+\theta)) < p(V(t, x(t)))$$

then

$$\dot{V}_i(t, x(t)) \leq -\omega(\|x(t)\|).$$

Then the trivial solution of of (4) is uniformly asymptotically stable.

III. All A_i are stable

As well known that the stability of each subsystem does not always imply stability of the whole switched systems. Hence, for system (1), this section will give the standard delay differential equations conditions by means of multiple Lyapunov functional and function method.

Firstly, for all $i = 1, 2, \cdots, m$, we define the switched regions Ω_i as follows

$$\Omega_i = \left\{ x \in R^n \left| x^T (P_i - P_{i+1}) x \geq 0, x \neq 0 \right. \right\}, \quad (5)$$

where P_i are symmetric positive definite matrices and $P_{m+1} = P_1$.

Furthermore, define the witching law *(S)*: when $x \in \Omega_i$, the ith subsystem is active.

Based on this switching rule, we now give the following results of this section.

A. Multiple Lyapunov functional method

For all $i = 1, 2, \cdots, m$, we define a Lyapunov functional for each mode of the form

$$V_i(x_t) = V_{i1} + V_{i2} + V_{i3} \tag{6}$$

with

$$\begin{aligned} V_{i1}(x_t) &= x^T(t) P_i x(t), \\ V_{i2}(x_t) &= \omega_i \int_{t-\tau}^{t} x^T(s) x(s) ds, \\ V_{i3}(x_t) &= \int_0^h ds \int_{t-s}^{t} x^T(u) x(u) du, \end{aligned}$$

where $\omega_i > 0$, P_i are defined by (5) and satisfy

$$A_i^T P_i + P_i A + hI + hP_i C_i C_i^T P_i = -Q_i \tag{7}$$

with symmetric positive definite matrices Q_i.

In this subsection, we wish to derive the standard delay differential equations conditions by the multiple Lyapunov functional (6). For this purpose, we first show that V_i satisfy the four conditions of Theorem 1.

Prosition 1: There exist continuous and increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that $\alpha(0) = \beta(0) = 0$ and

$$\alpha(\|x(t)\|) \le V_i(x_t) \le \beta(\|x_t\|_{\tau^*}). \tag{8}$$

Proof: Note that (6), let

$$\alpha(\|x(t)\|) = \min_i \lambda_{\min}(P_i) \|x(t)\|^2,$$

$$\beta(\|x_t\|_{\tau^*}) = \max_i \left(\lambda_{\max} P_i + \omega_i \tau + \tfrac{1}{2}h^2\right) \|x_t\|_{\tau^*}^2.$$

Thus, (8) follows immediately. This completes the proof. ■

Prosition 2: Suppose that the ith mode is active on $[t_k, t_{k+1})$. Let P_i, Q_i be defined by (7). If

$$\|P_i B_i\| < \frac{\lambda_{\min}(Q_i)}{2}, \tag{9}$$

then there exist $\omega_i > 0$ and a continuous, increasing function, $\psi_i : R^+ \mapsto R^+$, satisfying $\psi_i(0) = 0$ and $\psi_i(s) > 0$ for $s > 0$ such that

$$\dot{V}_i(x_t) \le -\psi_i(\|x(t)\|) \quad \forall t \in [t_k, t_{k+1}). \tag{10}$$

Proof: Since $\|P_i B_i\| < \frac{\lambda_{\min}(Q_i)}{2}$, then there exist $\omega_i > 0$ such that $\|P_i B_i\| \le \omega_i < \frac{\lambda_{\min}(Q_i)}{2}$, assume that ω_i satisfy (6), then

$$\begin{aligned} \dot{V}_{i1}(x_t) &= 2x^T(t) P_i \dot{x}(t), \\ \dot{V}_{i2}(x_t) &= \omega_i \left(x^T(t) x(t) - x^T(t-\tau) x(t-\tau)\right), \\ \dot{V}_{i3}(x_t) &= hx^T(t) x(t) - \int_{t-h}^{t} x^T(s) x(s) ds. \end{aligned}$$

Along the system (2), taking (6), (7) and (9) into account yields that

$$\begin{aligned} \dot{V}_i(x_t) &= x^T(t) \left(A_i^T P_i + P_i A + hI + hP_i C_i C_i^T P_i\right) x(t) \\ &\quad + 2x^T(t) P_i B_i x(t-\tau) + \omega_i \left(\|x(t)\|^2 - \|x(t-\tau)\|^2\right) \\ &\quad - \int_{t-h}^{t} \left[x(s) - C_i^T P_i x(t)\right]^T \left[x(s) - C_i^T P_i x(t)\right] ds \\ &\le -\lambda_{\min}(Q_i) \|x(t)\|^2 + \|P_i B_i\| \left(\|x(t)\|^2 + \|x(t-\tau)\|^2\right) \\ &\quad + \omega_i \left(\|x(t)\|^2 - \|x(t-\tau)\|^2\right) \\ &\le -\lambda_{\min}(Q_i) \|x(t)\|^2 + \omega_i \left(\|x(t)\|^2 + \|x(t-\tau)\|^2\right) \\ &\quad + \omega_i \left(\|x(t)\|^2 - \|x(t-\tau)\|^2\right) \\ &= -\left(\lambda_{\min}(Q_i) - 2\omega_i\right) \|x(t)\|^2. \end{aligned}$$

Hence, let $\psi_i(\|x(t)\|) = (\lambda_{\min}(Q_i) - 2\omega_i) \|x(t)\|^2$, (10) thus holds. This completes the proof. ■

Prosition 3: There exists a constant $\mu > 0$ such that, $\forall x_t \in C([-\tau^*, 0], R^n)$,

$$V_i(x_t) \le \mu V_j(x_t) \quad i \ne j. \tag{11}$$

Proof: Since

$$\begin{aligned} V_{i1}(x_t) &= x^T(t) P_i x(t) \le \lambda_{\max}(P_i) x^T(t) x(t) \\ &= \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)} \lambda_{\min}(P_j) x^T(t) x(t) \\ &\le \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)} x^T(t) P_j x(t), \\ V_{i2}(x_t) &= \omega_i \int_{t-\tau}^{t} x^T(s) x(s) ds = \frac{\omega_i}{\omega_j} \omega_j \int_{t-\tau}^{t} x^T(s) x(s) ds \end{aligned}$$

Now, pick $\mu = \max\left\{\sup_{i,j} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)}, \sup_{i,j} \frac{\omega_i}{\omega_j}, 1\right\}$, it follows from (6) that

$$V_i(x_t) = V_{i1} + V_{i2} + V_{i3} \le \mu V_j(x_t).$$

This completes the proof. ■

Prosition 4: [31] Under the switching law (S), there is a constant $\xi_i \in (0, 1)$ such that

$$V_i\left(x_{t_q}\right) \le (1 - \xi_i) V_i\left(x_{t_p}\right), \tag{12}$$

where t_p, t_q is defined by Theorem 1.

Based on the above argument, we now present the main result of this subsection.

Theorem 3: Let P_i, Q_i defined by (7), if

$$\|P_i B_i\| < \frac{\lambda_{\min}(Q_i)}{2}, \tag{13}$$

then the trivial solution of switched system (1) with the switching rule (S) is uniformly asymptotically stable.

Proof: Let $V_i(x_t)$ be given by (6), take (13) and Proposition 1-4 into account, it follows from that $V_i(x_t)$ satisfy the statement (i)-(iv) of Theorem 1, the rest proof is essentially same as Theorem 1. This completes the proof. ■

B. Multiple Lyapunov function method

For all $i = 1, 2, \cdots, m$, define a Lyapunov function $V_i(x(t))$ for each subsystem as follows

$$V_i(x(t)) = x^T(t) P_i x(t), \tag{14}$$

where P_i is a symmetric positive definite matrix satisfying

$$A_i^T P_i + P_i A_i + hP_i C_i C_i^T P_i = -Q_i \tag{15}$$

with symmetric positive definite matrix Q_i.

Similarly, in this subsection, we wish to derive the standard differential equations conditions by using Lyapunov function (14). To this end, we shall show that V_i satisfy the four statements of Theorem 2. Obviously, V_i satisfy the statements (i) and (iii). Now it remains to prove that, for the Lyapunov function (14), the statements (ii) and (iv) are true.

Prosition 5: Suppose that the ith mode of system (1) is active on $[t_k, t_{k+1})$. For $i \in \{1, 2, \ldots, m\}$, let $\kappa = \lambda_{\max}(P_i)/\lambda_{\min}(P_i)$. If V_i defined by (14) satisfy

$$\|P_i B_i\| < \frac{\lambda_{\min}(Q_i) - h\kappa}{1+\kappa}. \tag{16}$$

Then there exist $q_i > 1$ and $\psi_i : R^+ \mapsto R^+$ with $\psi_i(0) = 0$ and $\psi_i(s) > 0$ for $s > 0$, such that

$$\dot{V}_i(x_t) \leq -\psi_i(\|x(t)\|) \quad \forall t \in [t_k, t_{k+1}), \tag{17}$$

when

$$q_i V_i(x(t)) > V_i(x(t+\theta)) \quad \theta \in [-\tau^*, 0], \tag{18}$$

where P_i ,Q_i are defined by (15).

Proof: Let V_i be given by (14), when $t \in [t_k, t_{k+1})$, along the trajectories of systems (2), we have

$$\begin{aligned}
&\dot{V}_i(x(t)) = \dot{x}^T(t) P_i x(t) + x^T(t) P_i \dot{x}(t) \\
&= x^T(t)\left(A_i^T P_i + P_i A_i\right) x(t) + 2x^T(t) P_i B_i x(t-\tau) \\
&\quad + \int_{t-h}^{t} \left[x^T(s) C_i^T P_i x(t) + x^T(t) C_i^T P_i x(s)\right] ds \\
&= x^T(t)\left(A_i^T P_i + P_i A + h P_i C_i C_i^T P_i\right) x(t) \\
&\quad + 2x^T(t) P_i B_i x(t-\tau) + \int_{t-h}^{t} x^T(s) x(s) ds \\
&\quad - \int_{t-h}^{t} \left[x(s) - C_i^T P_i x(t)\right]^T \left[x(s) - C_i^T P_i x(t)\right] ds \\
&\leq x^T(t)\left(A_i^T P_i + P_i A + h P_i C_i C_i^T P_i\right) x(t) \\
&\quad + 2x^T(t) P_i B_i x(t-\tau) + \int_{t-h}^{t} x^T(s) x(s) ds \\
&\leq -\lambda_{min}(Q_i)\|x(t)\|^2 + \|P_i B_i\|\left(\|x(t)\|^2 + \|x(t-\tau)\|^2\right) \\
&\quad + \int_{t-h}^{t} x^T(s) x(s) ds.
\end{aligned}$$

By (16), there exist $q_i > 1$ such that

$$\|P_i B_i\|_i \leq \frac{\lambda_{\min}(Q_i) - h q_i \kappa}{1 + q_i \kappa} < \frac{\lambda_{\min}(Q_i) - h\kappa}{1+\kappa}.$$

Since

$$q_i V_i(x(t)) > V_i(x(t+\theta)) \quad \theta \in [-\tau^*, 0].$$

This leads to that

$$q_i \kappa \|x(t)\|^2 > \|x(t+\theta)\|^2 \quad \theta \in [-\tau^*, 0].$$

We proceed to get that

$$\begin{aligned}
\dot{V}_i(x(t)) &\leq -\lambda_{min}(Q_i)\|x(t)\|^2 + \|P_i B_i\|\Big(\|x(t)\|^2 \\
&\quad + q_i\kappa\|x(t)\|^2\Big) + h q_i \kappa \|x(t)\|^2 \\
&= -\left(\lambda_{\min}(Q_i) - (1+q_i\kappa)\|P_i B_i\| - h q_i \kappa\right)\|x(t)\|^2.
\end{aligned}$$

Finally, we can conclude that

$$\dot{V}_i(x_t) \leq -\psi_i(\|x(t)\|)$$

with

$$\psi_i(\|x(t)\|) = \left(\lambda_{\min}(Q_i) - (1+q_i\kappa)\|P_i B_i\| - h q_i \kappa\right)\|x(t)\|^2.$$

This completes the proof. ■

Prosition 6: [31] Under the switching law (S), let $V_i, i = 1, \cdots, m$ are defined by (14), then there exist $\xi_i \in (0, 1)$, such that

$$\bar{V}_i(t_q) \leq (1-\xi_i)\bar{V}_i(t_p) \tag{19}$$

with t_p, t_q given by Theorem 2 and

$$\bar{V}_i(t) = \sup_{-\tau^* \leq \theta \leq 0} V_i(x(t+\theta)).$$

Theorem 4: Let P_i, Q_i are defined by (15), if

$$\|P_i B_i\| < \frac{\lambda_{\min}(Q_i) - h\kappa}{1+\kappa},$$

then the trivial solution of (1) is uniformly asymptotically stable for some appropriate switching rule.

Proof: Define the Lyapunov function as (14), based on the above argument, then we know that such function satisfies the statements $(i)-(iv)$ of Theorem 2. The rest of the proof follows the same lines as the proof of Theorem 2. ■

IV. All A_i are unstable

In this section, for the case when A_i are unstable, we are dedicated to derive the sufficient condition guaranteeing the stability of system (1) by means of single Lyapunov functional and function methods. For the unstable subsystems, the stability of the whole switched system may be achieved by designing a appropriate switching law. To begin with, we first make the following two assumptions:

(i) C_i are constant matrix, i.e., $C_i = C$ for $i = 1, 2, \cdots, m$.
(ii) For the system matrices $A_i, i = 1, \cdots, m$, there exist a Hurwitz convex combination

$$\gamma_{\alpha_1,\cdots,\alpha_m}(A_1, \cdots, A_m) = \left\{\sum_{i=1}^{m} \alpha_i A_i \,\middle|\, 0 < \alpha_i < 1, \sum_{i=1}^{m} \alpha_i = 1\right\}.$$

Under the two hypotheses, we shall present the following results.

A. Single Lyapunov functional method

For the switched systems with unstable subsystem, as a classical approach, single Lyapunov can be apply to deal with the stability of such systems. Here, we first consider single Lyapunov functional method.

Theorem 5: Let $A \in \gamma_{\alpha_1,\cdots,\alpha_m}(A_1, \cdots, A_m)$, if there exists a constant $\xi > 1$ satisfying

$$\|P B_i\| < \frac{\lambda_{\min}(Q)}{2\xi}, \tag{20}$$

where P, Q are symmetric positive definite matrices satisfying

$$A^T P + PA + hI + hPCC^T P = -Q. \tag{21}$$

Then the trivial solution of system (1) is uniformly asymptotically stable for some appropriate switching rule.

Proof: First of all, we shall construct the switching region. Since $A \in \gamma_{\alpha_1,\cdots,\alpha_m}(A_1,\cdots,A_m)$, then, for all $i=1,\cdots,m$, there exist $\alpha_i \in (0,1)$ with $\sum_{i=1}^m \alpha_i = 1$ such that $A = \sum_{i=1}^m \alpha_i A_i$. Taking (21) into account, we obtain

$$\begin{aligned}
&A^T P + PA + hI + hPCC^T P \\
&= \left(\sum_{i=1}^m \alpha_i A_i^T\right) P + P\left(\sum_{i=1}^m \alpha_i A_i\right) + hI + hPCC^T P \\
&= \left(\sum_{i=1}^m \alpha_i\right) A_i^T P + \left(\sum_{i=1}^m \alpha_i\right) PA_i + \left(\sum_{i=1}^m \alpha_i\right) hI \\
&\qquad + \left(\sum_{i=1}^m \alpha_i\right) hPCC^T P \\
&= \left(\sum_{i=1}^m \alpha_i\right)\left(A_i^T P + PA_i + hI + hPCC^T P\right) \\
&= -Q.
\end{aligned}$$

Note that $\alpha_i > 0$ and Q symmetric positive definite matrix, then it follows that, for at least one i,

$$A_i^T P + PA_i + hI + hPCC^T P < 0.$$

Now, we define domains $\Omega_i, i = 1,\cdots,m$ as follows

$$\Omega_i = \left\{x \in R^n | x^T\left(A_i^T P + PA_i + hI + hPCC^T P\right)x \le -x^T Qx\right\}. \quad (22)$$

It is easy to show that $\bigcup_{i=1}^m \Omega_i = R^n$. To prevent a sliding motion (a motion of a trajectory along a boundary between two switching regions or chattering phenomenon, we thus construct the switching regions

$$\tilde{\Omega}_i = \left\{x \in R^n : x^T\left(A_i^T P + P_i A_i + hI + hPCC^T P\right)x \le -\frac{1}{\xi}x^T Qx\right\}, \quad (23)$$

where $\xi > 1$ is given by (20). Obviously, $\Omega_i \subset \tilde{\Omega}_i$ and $\bigcup_{i=1}^m \tilde{\Omega}_i = R^n$.

Next, we are going to design the switching law. To this end, define

$$i(x) = \arg\min x^T\left(A_i^T P + PA_i + hI + hPCC^T P\right)x.$$

This function is known as the *minimum rule*. Now, the switching law $(\tilde{S})$ can be given by:

(S_0) Choose the initial mode, i_0, by the minimum rule applied to $x(t_0)$.
(S_1) Stay in the ith mode as long as the state satisfies $x \in \tilde{\Omega}_i$.
(S_3) If the state hits the boundary of $\tilde{\Omega}_i$, determine the jth mode according to the minimum rule and switch to the jth mode.

Based on the above argument, for stability, define the Lyapunov functional as follows

$$V(x_t) = x^T(t)Px(t) + \omega\int_{t-\tau}^{t} x^T(s)x(s)\,ds, \quad (24)$$

where P is given by (21), $\omega > 0$. It is easy to show that

$$\alpha(\|x(t)\|) \le V(x_t) \le \beta(\|x_t\|_{\tau^*}) \quad (25)$$

with

$$\alpha(\|x(t)\|) = \lambda_{\min}(P)\|x(t)\|^2,$$
$$\beta(\|x_t\|_{\tau^*}) = \max(\lambda_{\max}P + \omega\tau)\|x_t\|_{\tau^*}^2.$$

Taking (20) into account, then it follows that, for some $\omega > 0$,

$$\|PB_i\| \le \omega < \frac{\lambda_{\min}(Q)}{2\xi}.$$

On the other hand, similar to Proposition 2, one can show that

$$\dot{V}(x_t) \le -\gamma(\|x(t)\|)\,\forall t \in [t_k, t_{k+1}) \quad (26)$$

with $\gamma(\|x(t)\|) = \left(\lambda_{\min}(Q_i)/\xi - 2\omega\right)\|x(t)\|^2$.

Finally, taking (25), (26) and Lemma 1 into account, we thus conclude that the trivial solution of system (1) is uniformly asymptotically stable under the switching rule $(\tilde{S})$. ■

B. Single Lyapunov function method

In this subsection, the single Lyapunov function method shall be used to derive the stability result for the switched system (1).

Theorem 6: Let $A \in \gamma_{\alpha_1,\cdots,\alpha_m}(A_1,\cdots,A_m)$. If there exists a constant $\xi > 1$ satisfying

$$\|PB_i\| < \frac{\lambda_{\min}(Q) - h\bar{\kappa}}{\xi(1+\bar{\kappa})}, \quad (27)$$

where $\bar{\kappa} = \lambda_{\max}(P)/\lambda_{\min}(P)$, and P, Q are symmetric positive definite matrices satisfying

$$A^T P + PA + hI + hPCC^T P = -Q. \quad (28)$$

Then the trivial solution of (1) is uniformly asymptotically stable for some appropriate switching rule.

Proof: Let the switching law $(\tilde{S})$ be defined as Theorem 5. Consider the following Lyapunov function

$$V(x(t)) = x^T(t)Px(t) \quad (29)$$

Clearly, $V(x(t))$ satisfies the statement (i) of Lemma 2. Now, it remains to show that $V(x(t))$ satisfies the statement (ii) of Lemma 2. From (27), it follows that there exists a constant $q > 1$ such that

$$\|PB_i\| < \frac{\lambda_{\min}(Q) - hq\bar{\kappa}}{\xi(1+q\bar{\kappa})} < \frac{\lambda_{\min}(Q) - h\bar{\kappa}}{\xi(1+\bar{\kappa})}. \quad (30)$$

Similar to Proposition 5, we can show that

$$\dot{V}(x(t)) \le -\gamma(\|x(t)\|), \quad (31)$$

when

$$V(t+\theta, x(t+\theta)) < p(V(t, x(t))), \quad \forall\theta \in [-\tau^*, 0],$$

where

$$p\left(V\left(t,x\left(t\right)\right)\right)=qV\left(x\left(t\right)\right),$$

$$\gamma\left(\|x(t)\|\right)=\left(\frac{\lambda_{\min}Q}{\xi}-\|PB_i\|\left(1+q\bar{\kappa}\right)-hq\bar{\kappa}\right)\|x(t)\|^2.$$

Hence, from (31), we know that $V\left(x\left(t\right)\right)$ satisfies the statement (ii) of Lemma 2. Finally, we can conclude that the trivial solution of (1) is uniformly asymptotically stable for the switching rule $(\tilde{S})$. ■

V. Numerical examples

In this section, two examples will be presented to show the validity of the main results derived above.

Example 1: Consider the switched system (1) with $h=0.2$ and

$$A_1=\begin{bmatrix}-5 & 2 & 4\\ -3 & -5 & 1\\ 2 & 4 & -6\end{bmatrix};C_1=\begin{bmatrix}-5 & 2 & 2\\ 0 & -1 & 4\\ 2 & 1.4 & -5.4\end{bmatrix};$$

$$B_1=\begin{bmatrix}-0.02 & 0.01 & 0.03\\ 0.03 & -0.02 & 0.05\\ 0.04 & 0.02 & -0.03\end{bmatrix};$$

$$A_2=\begin{bmatrix}-5.4 & 2.1 & 3.4\\ -3 & -2.5 & 1\\ 2 & 4 & -6.1\end{bmatrix};C_2=\begin{bmatrix}-5 & 2.2 & 1.2\\ 0 & -2.1 & -4\\ 2.1 & 1.5 & -5.3\end{bmatrix};$$

$$B_2=\begin{bmatrix}0.013 & -0.02 & -0.034\\ 0.05 & 0.062 & 0.15\\ 0.042 & 0.022 & 0.033\end{bmatrix}.$$

It is easy to see that each subsystem is stable. Therefore, from Theorem 3, by solving the equation (7), we have

$$P_1=\begin{bmatrix}1.3488 & -0.3412 & -0.2010\\ -0.3412 & 1.7609 & 0.0132\\ -0.2010 & 0.0132 & 1.2461\end{bmatrix};$$

$$P_2=\begin{bmatrix}1.1441 & 0.4623 & -0.4040\\ 0.4623 & 0.9304 & -0.8812\\ -0.4040 & -0.8812 & 1.9912\end{bmatrix};$$

$$Q_1=\begin{bmatrix}3.0735 & -1.6897 & -0.7643\\ -1.6897 & 5.0403 & 0.2198\\ -0.7643 & 0.2198 & 2.4540\end{bmatrix};$$

$$Q_2=\begin{bmatrix}3.4112 & 2.6756 & -3.4638\\ 2.6756 & 3.7999 & -5.7093\\ -3.4638 & -5.7093 & 10.0265\end{bmatrix}.$$

Moreover, a straightforward calculation follows that

$$\|P_1B_1\|=0.1062<\frac{\lambda_{\min}\left(Q_i\right)}{2}=0.8351;$$

$$\|P_2B_2\|=0.1128<\frac{\lambda_{\min}\left(Q_i\right)}{2}=0.1233$$

and

$$\Omega_1=\{(x_1,x_2,x_3)|-0.2047x_1^2-0.8305x_2^2+0.7451x_3^2 +1.6070x_1x_2-0.4060x_1x_3+1.7888x_2x_3<0\};$$

$$\Omega_2=\{(x_1,x_2,x_3)|0.2047x_1^2+0.8305x_2^2-0.7451x_3^2 -1.6070x_1x_2+0.4060x_1x_3-1.7888x_2x_3<0\}$$

Meanwhile, define the switching law (S):

when $x\left(t\right)\in\tilde{\Omega}_i, i=1,2$, the ith subsystem is activated.

Then according to Theorem 3, the trivial solution of (1) is uniformly asymptotically stable.

VI. Conclusion

In this paper, we studied the stability of a class of linear switched system with mixed delays. First, in the stable subsystems case, some standard delay differential equations results have been presented by means of multiple Lyapunov functional and function methods. Second, in the unstable subsystems case, we derived some sufficient conditions guaranteeing the uniform asymptotic stability by using single Lyapunov functional and function methods.

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