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Some Results on the Generalized Higher Rank Numerical Ranges

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Abstract—In this paper, the notion of rank$-k$ numerical range of rectangular complex matrix polynomials are introduced. Some algebraic and geometrical properties are investigated. Moreover, for $\epsilon > 0$, the notion of Birkhoff-James approximate orthogonality sets for $\epsilon-$higher rank numerical ranges of rectangular matrix polynomials is also introduced and studied. The proposed definitions yield a natural generalization of the standard higher rank numerical ranges.

Keywords—Rank$-k$ numerical range, isometry, numerical range, rectangular matrix polynomials.

I. Introduction and Related Work

LET $M_{n\times m}$ be the vector space of all $n \times m$ complex matrices. For the case $n = m$, $M_{n\times n}$ is denoted by M_n; namely, the algebra of all $n\times n$ complex matrices. Throughout the paper, k, m and n are considered as positive integers and $k \leq \min\{m, n\}$. Moreover, I_k denotes the $k \times k$ identity matrix. The set of all $n \times k$ isometry matrices is denoted by $\mathcal{X}_{n,k}$, i.e., $\mathcal{X}_{n,k} = \{X \in M_{n\times k} \ : \ X^*X = I_k\}$. For the case $n = k$, $\mathcal{X}_{n,n}$ is denoted by $\mathcal{U}_n$, namely, the group of all $n \times n$ unitary matrices.

Motivation of our study comes from quantum information science. A quantum channel is a trace preserving completely positive map such as $L : M_n \to M_n$. By the structure of completely positive linear maps, e.g., see [3], there are matrices $E_1, \ldots, E_r \in M_n$ with $\sum_{j=1}^{r} E_jE_j^* = I_n$ such that $L(A) = \sum_{j=1}^{r} E_j^*AE_j$. The matrices $E_1, \ldots, E_r$ are interpreted as the error operators of the quantum channel L. Let V be a $k-$dimensional subspace of $\mathbb{C}^n$ and P be the orthogonal projection of $\mathbb{C}^n$ onto V. Then the $k-$dimensional subspace V is a quantum error correction code for the channel L if and only if there are scalars $\gamma_{ij} \in \mathbb{C}$ with $i, j \in \{1, \ldots, r\}$ such that $PE_i^*E_jP = \gamma_{ij}P$; for more information, see [7] and its references, and also see [11]. In this connection, *the rank$-k$ numerical range of* $A \in M_n$ is defined and denoted by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P, \text{ for some rank} - k \text{ orthogonal projection } P \text{ on } \mathbb{C}^n\}.$$

It is known, see [4], that

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : X^*AX = \lambda I_k, \text{ for some } \ X \in \mathcal{X}_{n,k}\}.$$

The sets $\Lambda_k(A)$, where $k \in \{1, \ldots, n\}$, are generally called higher rank numerical ranges of A. Apparently, for k=1, $\Lambda_k(A)$ reduces to the *classical numerical range of* A; namely,

$$\Lambda_1(A) = W(A) := \{x^*Ax : x \in \mathbb{C}^n, \ x^*x = 1\},$$

which has been studied extensively for many decades; e.g., see [9] and [10]. Stampfli and Williams in [14], and later Bonsall and Duncan in [2], observed that the numerical range of $A \in M_n$ can be rewritten as:

$$W(A) = \{\mu \in \mathbb{C} : \|A - \lambda I_n\|_2 \geq |\mu - \lambda|, \quad \forall\lambda \in \mathbb{C}\},$$

where $\|.\|_2$ denotes the spectral matrix norm (i.e., the matrix norm subordinate to the Euclidean vector norm). By this idea, Chorianopoulos, Karanasios and Psarrakos [5] recently introduced a definition of the numerical range for rectangular complex matrices. For any $A, B \in M_{n\times m}$ with $B \neq 0$, and any vector norm $\|.\|$ on $M_{n\times m}$, they defined the *numerical range of A with respect to B* as the compact and convex set:

$$W_{\|.\|}(A; B) = \{\mu \in \mathbb{C} : \|A - \lambda B\| \geq |\mu - \lambda|, \quad \forall\lambda \in \mathbb{C}\}. \tag{1}$$

It is clear that $W_{\|.\|_2}(A; I_n) = W(A) = \Lambda_1(A)$, where $A \in M_n$. Hence, $W_{\|.\|}(\, . \, ; .)$ is a direct generalization of the classical numerical range. It is known that $W_{\|.\|}(A; B) \neq \emptyset$ if and only if $\|B\| \geq 1$. So, to avoid trivial consideration, we assume that $\|B\| \geq 1$.

Suppose

$$P(\lambda) = A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_1\lambda + A_0 \tag{2}$$

is a rectangular matrix polynomial, where $A_i \in M_{n\times m}$ $(i \in \{0, 1, 2, \ldots, s\})$, $A_s \neq 0$, and λ is a complex variable. The study of matrix polynomials has a long history, especially with regard to their applications on higher order linear systems of differential equations; e.g., see [8], [12] and the references therein. Let $B \in M_{n\times m}$ and $\|\cdot\|$ be a vector norm on $M_{n\times m}$ such that $\|B\| \geq 1$. Moreover, let $P(\lambda)$ be an $n \times m$ matrix polynomial as in (2). Using (9), Chorianopoulos and Psarrakos [6] recently introduced and studied the *numerical range of $P(\lambda)$ with respect to B* as:

$$W_{\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : 0 \in W_{\|\cdot\|}(P(\mu); B)\}. \tag{3}$$

For the case $n = m$, $B = I_n$ and $\|\cdot\| = \|\cdot\|_2$, we have the classical numerical range of the square matrix polynomial $P(\lambda)$; namely,

$$W_{\|\cdot\|_2}[P(\lambda); I_n] = W[P(\lambda)] := \{\mu \in \mathbb{C} : x^*P(\mu)x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\}.$$

Hence, $W_{\|\cdot\|}[.;.]$ is a direct generalization of the classical numerical range of square matrix polynomials, which plays an important role in the study of overdamped vibration systems with a finite number of degrees of freedom, and it also is related to the stability theory; e.g., see [13] and its references. Recently, Aretaki and Maroulas [1] introduced the notion of higher rank numerical ranges of square complex

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matrix polynomials. Let $P(\lambda)$, as in (2), be a square matrix polynomial (i.e., $n = m$). For a positive integer $k \le n$, they defined the *rank−k numerical range of* $P(\lambda)$ as:

$$\Lambda_k[P(\lambda)] = \{\mu \in \mathbb{C} \,:\, X^*P(\mu)X = 0_k, \text{ for some } X \in \mathcal{X}_{n,k}\},$$

Where $0_k \in M_k$ is the zero matrix. It is readily verified, see [1], that

$$W_{\|\cdot\|_2}[P(\lambda); I_n] = W[P(\lambda)] = \Lambda_1[P(\lambda)] \supseteq \Lambda_2[P(\lambda)] \supseteq \cdots \supseteq \Lambda_n[P(\lambda)].$$

So, the notion of the numerical range of rectangular matrix polynomials is a generalization of the notion of higher rank numerical ranges of square matrix polynomials.
In this paper, we are going to generalize the notion of higher rank numerical ranges of square matrix polynomial to rectangular matrix polynomials. For this, we introduce the notion of rank−k numerical range of a rectangular matrix polynomial, and we investigate some algebraic and geometrical properties of this notion.

II. Main Results

In [15], the authors introduced a formula analogous to (9) to propose a definition of the higher rank numerical range of rectangular matrices. For any A, $B \in M_{n\times m}$ and any vector norm $\|\cdot\|$ on $M_{(n-k+1)\times(m-k+1)}$, where $1 \le k \le min\{n, m\}$ is a positive integer, they defined the *rank−k numerical range of A with respect to B* as

$$\Lambda_{k,\|\cdot\|}(A; B) = \{\mu \in \mathbb{C} : \|X^*(A - \lambda B)Y\| \ge |\mu - \lambda|, \quad \forall\, \lambda \in \mathbb{C},\ \forall\, (X, Y) \in \mathcal{X}\}, \tag{4}$$

where

$$\begin{cases} \mathcal{X} = \{(X, Y := \left[\begin{array}{c|c} X & 0 \\ \hline 0 & U \end{array}\right]) \,:\, X \in \mathcal{X}_{n,n-k+1},\ U \in \mathcal{U}_{m-n}\} & \text{if } m \ge n \\ \mathcal{X} = \{(X := \left[\begin{array}{c|c} Y & 0 \\ \hline 0 & U \end{array}\right], Y) \,:\, Y \in \mathcal{X}_{m,m-k+1},\ U \in \mathcal{U}_{n-m}\} & \text{if } n \ge m \end{cases} \tag{5}$$

At first, we state some results from [15] which are useful in our discussion. Recall that, in a complex normed space $(X, \|.\|)$, for any $\epsilon \in [0, 1)$, two vectors ϕ and ψ are said to be Birkhoff-James ϵ−orthogonal, denoted by $\phi \perp^{\epsilon}_{BJ} \psi$, if $\|\phi + \lambda\psi\| \ge \sqrt{1 - \epsilon^2}\|\phi\|$ for all $\lambda \in \mathbb{C}$. For the case $\epsilon = 0$, we write $\phi \perp_{BJ} \psi$ instead $\phi \perp^{0}_{BJ} \psi$. Also, Let $1 \le k_2 \le k_1 \le \min\{n, m\}$ be two positive integers. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k_2+1)\times(m-k_2+1)}$. Define $|||\cdot|||$ on $M_{(n-k_1+1)\times(m-k_1+1)}$ by

$$|||Z||| = \| \left(\begin{array}{c|c} Z & 0 \\ \hline 0 & 0_{k_1-k_2} \end{array}\right) \|, \tag{6}$$

where $Z \in M_{(n-k_1+1)\times(m-k_1+1)}$, and $0_{k_1-k_2} \in M_{k_1-k_2}$ is the zero matrix.

Theorem 1. *Let A, $B \in M_{n\times m}$ and $1 \le k \le \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $\mathcal{X}$ be the set as in (5). Then the following assertions are true:*
(i) $\Lambda_{k,\|\cdot\|}(A; B) = \bigcap_{(X,Y)\in\mathcal{X}} W_{\|\cdot\|}(X^*AY; X^*BY)$. Consequently, $\Lambda_{k,\|\cdot\|}(A; B)$ is a compact and convex set in $\mathbb{C}$. For the case $k = 1$, if the vector norm $\|\cdot\|$ is unitarily invariant, then

$$\Lambda_1(A; B) = W_{\|\cdot\|}(A; B);$$

(ii) For the case $n = m$, $\Lambda_{k,\|\cdot\|}(A; B) = \bigcap_{X\in\mathcal{X}_{n,n-k+1}} W_{\|\cdot\|}(X^*AX; X^*BX)$. Consequently, if $\|\cdot\| = \|\cdot\|_2$ and $B = I_n$, then

$$\Lambda_{k,\|\cdot\|}(A; I_n) = \Lambda_k(A);$$

(iii) $\Lambda_{k,\|\cdot\|}(UAV; UBV) = \Lambda_{k,\|\cdot\|}(A; B)$, where for the case $m \ge n$, $U \in \mathcal{U}_n$ and $V = \left(\begin{array}{c|c} U^* & 0 \\ \hline 0 & * \end{array}\right) \in \mathcal{U}_m$, and for the other case, i.e., $n \ge m$, $V \in \mathcal{U}_m$ and $U = \left(\begin{array}{c|c} V^* & 0 \\ \hline 0 & * \end{array}\right) \in \mathcal{U}_n$;
(iv) Let $1 \le k_2 \le k_1 \le \min\{n, m\}$ be two positive integers, $\|\cdot\|$ be a unitarily invariant norm on $M_{(n-k_2+1)\times(m-k_2+1)}$ and $|||\cdot|||$ be the vector norm on $M_{(n-k_1+1)\times(m-k_1+1)}$ as in (6). Then

$$\Lambda_{k_1,|||\cdot|||}(A; B) \subseteq \Lambda_{k_2,\|\cdot\|}(A; B);$$

(v) If $\|X^*BY\| > 1$ for all $(X, Y) \in \mathcal{X}$, then $\Lambda_{k,\|\cdot\|}(A; B) \supseteq \bigcap_{(X,Y)\in\mathcal{X}}\{\mu \in \mathbb{C} : X^*BY \perp_{BJ} X^*(A - \mu B)Y\}$. The equality holds if $\|X^*BY\| = 1$ for all $(X, Y) \in \mathcal{X}$;
(vi) For any nonzero $b \in \mathbb{C}$,

$$\begin{cases} \text{if } |b| = 1, \text{ then } \Lambda_{k,\|\cdot\|}(A; bB) = b^{-1}\Lambda_{k,\|\cdot\|}(A; B); \\ \text{if } |b| < 1, \text{ then } \Lambda_{k,\|\cdot\|}(A; bB) \subseteq b^{-1}\Lambda_{k,\|\cdot\|}(A; B); \\ \text{if } |b| > 1, \text{ then } \Lambda_{k,\|\cdot\|}(A; bB) \supseteq b^{-1}\Lambda_{k,\|\cdot\|}(A; B); \end{cases}$$

(vii) $\Lambda_{k,\|\cdot\|}(aA + bB; B) = a\Lambda_{k,\|\cdot\|}(A; B) + b$, where $a, b \in \mathbb{C}$.

It is natural to use a formula analogous to (3) to propose a definition of the higher rank numerical range of rectangular matrix polynomials.

Definition 1. Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \le k \le \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. The rank−k numerical range of $P(\lambda)$ with respect to B is defined and denoted by

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : 0 \in \Lambda_{k,\|\cdot\|}(P(\mu); B)\}.$$

The sets $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$, where $k \in \{1, 2, \ldots, \min\{n, m\}\}$ is a positive integer, are generally called the *higher rank numerical range of $P(\lambda)$ with respect to B*.

Theorem 2. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \le k \le \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then*

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \bigcap_{(X,Y)\in\mathcal{X}} W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY],$$

where $\mathcal{X}$ is the set as in (5) and $X^*P(\lambda)Y = (X^*A_lY)\lambda^l + \cdots + (X^*A_1Y)\lambda + (X^*A_0Y)$. If $k=1$, and the vector norm $\|\cdot\|$ is unitarily invariant, then

$$\Lambda_{1,\|\cdot\|}[P(\lambda); B] = W_{\|\cdot\|}[P(\lambda); B].$$

Proof: Using Definition 1 and Theorem 1(i), the first equality is easy to verify. If $k=1$, and the vector norm $\|\cdot\|$ is unitarily invariant on $M_{n\times m}$, then by Theorem 1(i), the second equality can be also easily verify by the first result. ■

Theorem 3. *Let $B \in M_n$, $P(\lambda)$, as in (2), be a square matrix polynomial (i.e., n=m) and $1 \le k \le n$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on M_{n-k+1}. Then*

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \bigcap_{X\in\mathcal{X}_{n,n-k+1}} W_{\|\cdot\|}[X^*P(\lambda)X; X^*BX].$$

Consequently, for the case $\|\cdot\| = \|\cdot\|_2$ and $B = I_n$,

$$\Lambda_{k,\|\cdot\|_2}[P(\lambda); I_n] = \Lambda_k[P(\lambda)].$$

Proof: The results follows directly from Theorem 1(ii), or Theorem 2. ■

Remark 1. Theorems 2 and 3 show that the notion of rank$-k$ numerical range of rectangular matrix polynomials can be considered as generalizations of the numerical range of rectangular matrix polynomials and the rank$-k$ numerical range of square matrix polynomials.

In the following proposition, we state some basic properties of higher rank numerical ranges of rectangular matrix polynomials. For this, we need the following lemma.

Lemma 1. *Let $B \in M_{n\times m}$ and $P(\lambda)$ be a rectangular matrix polynomial as in (2). Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $0 \ne \alpha \in \mathbb{C}$. Then the following assertions are true:*
(i) $W_{\|\cdot\|}[\alpha P(\lambda); B] = W_{\|\cdot\|}[P(\lambda); B]$, $W_{\|\cdot\|}[P(\alpha\lambda); B] = \alpha^{-1}W_{\|\cdot\|}[P(\lambda); B]$ *and* $W_{\|\cdot\|}[P(\lambda+\alpha); B] = W_{\|\cdot\|}[P(\lambda); B] - \alpha$;
(ii) If $R(\lambda) = \lambda^l P(\frac{1}{\lambda}) := A_0\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l$ *is the reverse matrix polynomial of* $P(\lambda)$, *then*

$$W_{\|\cdot\|}[R(\lambda); B]\backslash\{0\} = \{\mu \in \mathbb{C} : \frac{1}{\mu} \in W_{\|\cdot\|}[P(\lambda); B],\ \mu \ne 0\}.$$

Proposition 1. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2), and $1 \le k \le \min\{n,m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then the following assertions are true:*
(i) $\Lambda_{k,\|\cdot\|}[P(\alpha\lambda); B] = \alpha^{-1}\Lambda_{k,\|\cdot\|}[P(\lambda); B]$, *and* $\Lambda_{k,\|\cdot\|}[\alpha P(\lambda); B] = \Lambda_{k,\|\cdot\|}[P(\lambda); B]$, *where $\alpha \in \mathbb{C}$ is nonzero;*
(ii) $\Lambda_{k,\|\cdot\|}[P(\lambda+\alpha); B] = \Lambda_{k,\|\cdot\|}[P(\lambda); B] - \alpha$, *where* $\alpha \in \mathbb{C}$.
(iii) If $R(\lambda) = \lambda^l P(\frac{1}{\lambda}) := A_0\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l$, *then*

$$\Lambda_{k,\|\cdot\|}[R(\lambda); B] \setminus \{0\} = \{\frac{1}{\mu} : \mu \in \Lambda_{k,\|\cdot\|}[P(\lambda); B],\ \mu \ne 0\}.$$

Proof: Let $\mathcal{X}$ be the set as in (5) and $(X,Y) \in \mathcal{X}$ be given. By setting

$$\begin{aligned} Q(\lambda) :&= X^*P(\lambda)Y = (X^*A_lY)\lambda^l + \cdots + (X^*A_1Y)\lambda + \\ &(X^*A_0Y), \end{aligned}$$

and using Lemma 1(i), we have

$$\begin{aligned} W_{\|\cdot\|}[X^*P(\alpha\lambda)Y; X^*BY] &= W_{\|\cdot\|}[Q(\alpha\lambda); X^*BY] \\ &= \alpha^{-1}W_{\|\cdot\|}[Q(\lambda); X^*BY] \\ &= \alpha^{-1}W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY], \end{aligned}$$

$$\begin{aligned} W_{\|\cdot\|}[X^*P(\lambda+\alpha)Y; X^*BY] &= W_{\|\cdot\|}[Q(\lambda+\alpha); X^*BY] \\ &= W_{\|\cdot\|}[Q(\lambda); X^*BY] - \alpha \\ &= W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY] \\ &\quad - \alpha. \end{aligned}$$

So, the results in (i) and (ii) follow from Theorem 2. By Theorem 2 and Lemma 1(ii), we have:

$$\begin{aligned} \mu \ne 0,\ \mu \in \Lambda_{k,\|\cdot\|}[R(\lambda); B] &\Longleftrightarrow \forall (X,Y) \in \mathcal{X},\ \mu \in \\ &\quad W_{\|\cdot\|}[X^*R(\lambda)Y; X^*BY], \mu \ne 0 \\ &\Longleftrightarrow \forall (X,Y) \in \mathcal{X},\ \frac{1}{\mu} \in \\ &\quad W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY], \mu \ne 0 \\ &\Longleftrightarrow \frac{1}{\mu} \in \Lambda_{k,\|\cdot\|}[P(\lambda); B], \mu \ne 0. \end{aligned}$$

So, the set equality in (ii) holds. ■

In the following proposition, we investigate the closeness of the rank$-k$ numerical range of rectangular matrix polynomials.

Proposition 2. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \le k \le \min\{n,m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ is a closed set in $\mathbb{C}$.*

Proof: In view of Theorem 2, it is enough to show that for every $(X,Y) \in \mathcal{X}$, where $\mathcal{X}$ is the set as in (5), $W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$ is closed. Let $\{\mu_t\}_{t=1}^{\infty} \subseteq W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$, and $\lim_{t\to\infty}\mu_t = \mu$. We will show that $\mu \in W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$. For this, let $\lambda \in \mathbb{C}$ be arbitrary. By (3) and (9), we have

$$\|X^*P(\mu_t)Y - \lambda X^*BY\| \ge |\lambda|$$

for all $t \in \mathbb{N}$. Since $\|\cdot\|$ and $P(\cdot)$ are continuous functions, the above inequality shows that

$$\|X^*P(\mu)Y - \lambda X^*BY\| \ge |\lambda|.$$

So, by (3) and (9), $\mu \in W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$, and hence, the result holds. ■

The following example shows that the higher rank numerical range of rectangular matrix polynomials need not be a bounded set, and hence a compact set in $\mathbb{C}$.

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Example 1. Let $P(\lambda) = \lambda A - I_2$, where $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2$. By Theorem 3, we have:

$$\begin{aligned} \Lambda_{1,\|\cdot\|_2}[P(\lambda); I_2] &= W[P(\lambda)] \\ &= \{\mu \in \mathbb{C} : \ (x^*Ax)\mu = 1, \ \textit{for some} \\ &\quad x \in \mathbb{C}^2 \textit{and } x^*x = 1\} \\ &= \{\mu \in \mathbb{C} : \ t\mu = 1, \ \text{for some } t \in [-1, 1]\} \\ &= \{\frac{1}{t} : \ t \in [-1, 1], \ t \neq 0\} \\ &= (-\infty, -1] \bigcup [1, +\infty). \end{aligned}$$

So, $\Lambda_{1,\|\cdot\|_2}[P(\lambda); I_2]$ is unbounded.

In the following Theorem, we investigate the boundedness of $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$. For this, we need the following Lemma.

Lemma 2. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leq k \leq \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{n\times m}$. Then the following assertions are true:*
(i) If $0 \notin W_{\|\cdot\|}(A_l; B)$, then $W_{\|\cdot\|}[P(\lambda); B]$ is bounded.
(ii) Suppose $0 \in W_{\|\cdot\|}(A_l; B)$ and 0 is not an isolated point of $W_{\|\cdot\|}[R(\lambda); B]$, where

$$R(\lambda) = \lambda^l P(\frac{1}{\lambda}) := A_0\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l.$$

Then $W_{\|\cdot\|}[P(\lambda); B]$ is unbounded.

Theorem 4. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leq k \leq \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then the following assertions are true:*
(i) If $0 \notin \Lambda_{k,\|\cdot\|}(A_l; B)$, then $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ is bounded.
(ii) Suppose $0 \in \Lambda_{k,\|\cdot\|}(A_l; B)$ and 0 is not an isolated point of $\Lambda_{k,\|\cdot\|}[R(\lambda); B]$, where

$$R(\lambda) = \lambda^l P(\frac{1}{\lambda}) := A_0\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l.$$

Then $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ is unbounded.

Proof: (i); Since $0 \notin \Lambda_{k,\|\cdot\|}(A_l; B)$, by Theorem $1(i)$, there exists $(X, Y) \in \mathcal{X}$, such that

$$0 \notin W_{\|\cdot\|}(X^*A_lY; X^*BY),$$

where $\mathcal{X}$ is the set as in (5). Using Lemma $2(i)$, $W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$ is a bounded set in $\mathbb{C}$, and hence, by Theorem 2, $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ is bounded.
Since $0 \in \Lambda_{k,\|\cdot\|}(A_l; B)$, by Definition 1, it follows that $0 \in \Lambda_{k,\|\cdot\|}[R(\lambda); B]$. Moreover, since 0 is not an isolated point of the set $\Lambda_{k,\|\cdot\|}[R(\lambda); B]$, there is a sequence $\{\mu_k\}_{k\in\mathbb{N}} \subseteq \Lambda_{k,\|\cdot\|}[R(\lambda); B] \setminus \{0\}$ that converges to the origin. So, by Proposition $1(iii)$, we have

$$\{\mu_k^{-1}\}_{k\in\mathbb{N}} \subseteq \Lambda_{k,\|\cdot\|}[P(\lambda); B],$$

and hence, the result in (ii) follows from this fact that the sequence $\{\mu_k^{-1}\}_{k\in\mathbb{N}}$ is unbounded. ■

III. Additional Results

In this section, we investigate some algebraic properties of the higher rank numerical range of rectangular matrix polynomials.

Proposition 3. *Let $B \in M_{n\times m}$ and $P(\lambda) = q(\lambda)B$, where $q(\lambda)$ is a scalar polynomial . Moreover, let $1 \leq k \leq \min\{n, m\}$ be a positive integer and $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then*

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : q(\mu) = 0\}.$$

Proof: Using Definition 1 and Theorem $1(vii)$, we have:

$$\begin{aligned} \mu \in \Lambda_{k,\|\cdot\|}[P(\lambda); B] \Longleftrightarrow 0 \in \Lambda_{k,\|\cdot\|}(P(\mu); B) = \\ \Lambda_{k,\|\cdot\|}(q(\mu)B; B) = \{q(\mu)\} \\ \Longleftrightarrow q(\mu) = 0. \end{aligned}$$

So, the result holds. ■

In the following theorem, we show that the rank$-k$ numerical range of rectangular matrix polynomials is invariant under some unitary matrices.

Theorem 5. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2), and $1 \leq k \leq \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then*

$$\Lambda_{k,\|\cdot\|}[UP(\lambda)V; UBV] = \Lambda_{k,\|\cdot\|}[P(\lambda); B],$$

where for the case $m \geq n$, $U \in \mathcal{U}_n$ and $V = \left(\begin{array}{c|c} U^* & 0 \\ \hline 0 & * \end{array}\right) \in \mathcal{U}_m$, and for the other case, i.e., $n \geq m$, $V \in \mathcal{U}_m$ and $U = \left(\begin{array}{c|c} V^* & 0 \\ \hline 0 & * \end{array}\right) \in \mathcal{U}_n$. Also, $UP(\lambda)V = (UA_lV)\lambda^l + \cdots + (UA_1V)\lambda + (UA_0V)$.

Proof: Using Definition 1 and Theorem $1(iii)$, the result is easy to verify. ■

In the following theorem, we state the relationship between higher rank numerical range of rectangular matrix polynomials.

Theorem 6. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leq k_2 \leq k_1 \leq \min\{n, m\}$ be two positive integers. Moreover, let $\|\cdot\|$ be a unitarily invariant norm on $M_{(n-k_2+1)\times(m-k_2+1)}$ and $|||\cdot|||$ be the vector norm on $M_{(n-k_1+1)\times(m-k_1+1)}$ as in (6). Then*

$$\Lambda_{k_1,|||\cdot|||}[P(\lambda); B] \subseteq \Lambda_{k_2,\|\cdot\|}[P(\lambda); B].$$

Proof: Let $\mu \in \Lambda_{k_1,|||\cdot|||}(P(\mu); B)$, be given. So, by Definition 1, $0 \in \Lambda_{k_1,|||\cdot|||}(P(\mu); B)$, and hence, by Theorem 1 (iv), $0 \in \Lambda_{k_2,||\cdot||}(P(\mu); B)$. Hence, the proof is complete. ■

Using Definition 1 and Theorem 1 (v), we have the following proposition

Proposition 4. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leq k \leq \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on*

$M_{(n-k+1)\times(m-k+1)}$ and $\mathcal{X}$ be the set as in (5). Then the following assertions are true:
(i) If $\|X^*BY\| = 1$ for all $(X,Y) \in \mathcal{X}$, then

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \bigcap_{(X,Y)\in\mathcal{X}} \{\mu \in \mathbb{C} : X^*BY \perp_{BJ} X^*P(\mu)Y\};$$

(ii) If $\|X^*BY\| > 1$ for all $(X,Y) \in \mathcal{X}$, then

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] \supseteq \bigcap_{(X,Y)\in\mathcal{X}} \{\mu \in \mathbb{C} : X^*BY \perp_{BJ} X^*P(\mu)Y\}.$$

The following proposition follows from Definition 1 and Theorem $1(vi)$.

Proposition 5. *Let $B \in M_{n\times m}$, $0 \neq b \in \mathbb{C}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leq k \leq \min\{n,m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then the following assertions are true:*
(i) If $|b| = 1$, then $\Lambda_{k,\|\cdot\|}[P(\lambda); bB] = \Lambda_{k,\|\cdot\|}[P(\lambda); B]$;
(ii) If $|b| < 1$, then $\Lambda_{k,\|\cdot\|}[P(\lambda); bB] \subseteq \Lambda_{k,\|\cdot\|}[P(\lambda); B]$;
(iii) If $|b| > 1$, then $\Lambda_{k,\|\cdot\|}[P(\lambda); bB] \supseteq \Lambda_{k,\|\cdot\|}[P(\lambda); B]$.

Corollary 1. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leq k \leq \min\{n,m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. If $\|B\| > 1$, Then*

$$\Lambda_{k,\|\cdot\|}[P(\lambda); \|B\|^{-1}B] \subseteq \Lambda_{k,\|\cdot\|}[P(\lambda); B]$$

Let $A, B \in M_{n\times m}$, $1 \leq k \leq \min\{n,m\}$ be a positive integer, and $\mathcal{X}$ be the set as in (5). Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $0 \leq \epsilon < 1$. The Birkhoff-James $\epsilon-$ orthogonality set of A with respect to B is defined and denoted, [6], by

$$W^{\epsilon}_{\|\cdot\|}(A; B) = \{\mu \in \mathbb{C} : \|A - \lambda B\| \geq \sqrt{1-\epsilon^2}\|B\||\mu - \lambda|, \\ \forall \lambda \in \mathbb{C}\}.$$

Also, the rank$-k, \epsilon$ numerical range of A with respect to B is defined and denoted, e.g., see [15], by

$$\Lambda^{\epsilon}_{k,\|\cdot\|}(A; B) = \{\mu \in \mathbb{C} : \|X^*(A - \lambda B)Y\| \geq \sqrt{1-\epsilon^2} \\ \|X^*BY\||\mu - \lambda|, \ \forall \lambda \in \mathbb{C}, \forall (X,Y) \in \mathcal{X}\},$$

and by [15], we have

$$\Lambda^{\epsilon}_{k,\|\cdot\|}(A; B) = \bigcap_{(X,Y)\in\mathcal{X}} W^{\epsilon}_{\|\cdot\|}(X^*AY; X^*BY), \qquad (7)$$

$$\Lambda^{\epsilon}_{k,\|\cdot\|}(A; B) = \bigcap_{(X,Y)\in\mathcal{X}} \{\mu \in \mathbb{C} : X^*BY \perp^{\epsilon}_{BJ} X^*(A - \mu B)Y\}. \qquad (8)$$

Moreover, let $P(\lambda)$ be a rectangular matrix polynomial as in (rpoly). The Birkhoff-James $\epsilon-$orthogonality set of $P(\lambda)$ with respect to B is defined and denoted, e.g., see [6], by

$$W^{\epsilon}_{\|\cdot\|}(A; B) = \{\mu \in \mathbb{C} : 0 \in W^{\epsilon}_{\|\cdot\|}(P(\mu); B)\}. \qquad (9)$$

By this idea, at the end of this section, we introduce and study the notion of rank-k, ϵ numerical range of rectangular matrix polynomials.

Definition 2. Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2), $1 \leq k \leq \min\{n,m\}$ be a positive integer and $\mathcal{X}$ be the set as in (5). Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $0 \leq \epsilon < 1$. The rank$-k, \epsilon$ numerical range of $P(\lambda)$ with respect to B is defined and denoted by

$$\Lambda^{\epsilon}_{k,\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : 0 \in \Lambda^{\epsilon}_{k,\|\cdot\|}(P(\mu); B)\}.$$

It is clear that:

$$\Lambda^{\epsilon}_{k,\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : \|X^*(P(\mu) - \lambda B)Y\| \geq \sqrt{1-\epsilon^2} \\ \|X^*BY\||\lambda|, \ \forall \lambda \in \mathbb{C}, \forall (X,Y) \in \mathcal{X}\}.$$

Using Definition 2 and Relations (7), (8), (9), and Theorem 2, we have the following theorem.

Theorem 7. *Let $B \in M_{n\times m}$, $P(\lambda)$ be a matrix polynomial as in (2), $1 \leq k \leq \min\{n,m\}$ be a positive integer and $\mathcal{X}$ be the set as in (5). Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $0 \leq \epsilon < 1$. Then*

$$\Lambda^{\epsilon}_{k,\|\cdot\|}[P(\lambda); B] = \bigcap_{(X,Y)\in\mathcal{X}} W^{\epsilon}_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY] \\ = \bigcap_{(X,Y)\in\mathcal{X}} \{\mu \in \mathbb{C} : X^*BY \perp^{\epsilon}_{BJ} X^*P(\mu)Y\}.$$

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