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Some Results on Parallel Alternating Two-stage Methods

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***Abstract*—**In this paper, we present parallel alternating two-stage methods for solving linear system *Ax=b*, where *A* is a symmetric positive definite matrix. And we give some convergence results of these methods for nonsingular linear system.

***Keywords*—**alternating two-stage, convergence, linear system, parallel.

I. Introduction

FOR the solution of the large linear system

$$Ax = b, \tag{1}$$

where A is a $n \times n$ square matrix, and x and b are n-dimensional vectors, the basic iterative method is

$$Mx_{k+1} = Nx_k + b, \quad k = 0, 1, 2, \ldots \tag{2}$$

where $A = M - N$ and M is nonsingular.

Given a splitting $A = M - N$ (M is nonsingular), a classical iterative method produces the following iteration

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b, \quad k = 0, 1, 2 \ldots \tag{3}$$

where $M^{-1}N$ is called the iteration matrix of the method. On the other hand, a two-stage method consists of approximating the linear system (3) by using another iterative procedure (inner iterations). That is, consider the splitting $M = F - G$ and perform, at each outer step l, *s(l)* inner iterations of the iterative procedure induced by this splitting. Thus, the resulting method is two-stage iterative method:

$$x_{k+1} = (F^{-1}G)^{s(k)}x_k + \sum_{j=0}^{s(k)-1}(F^{-1}G)^j F^{-1}(Nx_k + b)$$

, k = 0, 1, 2 …

Alternating two-stage iterative method [1] has been studied to approximate the linear system (2) by using a inner iteration. Let $M = P - Q = R - S$ be two splittings of matrix *M*. In order to approximate (2), for each *k, k = 1, 2 , …*, we perform *s(k)* inner iterations of the general class of iterative methods of the form

$$\begin{aligned} y_{j-\frac{1}{2}} &= P^{-1}Qy_{j-1} + P^{-1}(Nx_{k-1} + b) \\ y_j &= R^{-1}Sy_{j-\frac{1}{2}} + R^{-1}(Nx_{k-1} + b) \end{aligned} \quad . \ j = 1, 2 \ldots s(k)$$

Thus, the resulting method is alternating two-stage iterative method:

$$x_k = (R^{-1}SP^{-1}Q)^{s(k)}x_{k-1} + \sum_{j=0}^{s(k)-1}(R^{-1}SP^{-1}Q)^j R^{-1}(SP^{-1} + I)(Nx_{k-1} + b) \quad . \ k = 1, 2 \ldots$$

On the other hand, with the development of parallel computation in recent years, the utilization of the parallel algorithms for the solution of large nonsingular linear system has become effective. Now we introduce the parallel alternating two-stage methods.

Given a parallel multisplitting of *A*, s.t.

(i) $A = M_l - N_l$,

(ii) $M_l = P_l - Q_l = R_l - S_l$,

(iii) $E_l \geq 0$ *and* $\sum_{l=1}^{a} E_l = I$,

where l = 1, 2, … a and *I* is the identity matrix.

Suppose that we have a multiprocessor with α processors connected to a host processor, that is, the same number of processors as splittings, and that all processors have the last update vector x_{k-1}, then the l-th processor only computes those entries of the vector

$$\begin{aligned} y_{l,j-\frac{1}{2}} &= P_l^{-1}Q_l y_{l,j-1} + P_l^{-1}(N_l x_{k-1} + b) \\ y_{l,j} &= R_l^{-1}S_l y_{l,j-\frac{1}{2}} + R_l^{-1}(N_l x_{k-1} + b) \end{aligned} \quad , \ j = 1, 2 \ldots s(k)$$

,

With $y_{l,0} = x_{k-1}$, or equivalently

$$y_{l,j} = R_l^{-1}S_l P_l^{-1}Q_l y_{l,j-1} + R_l^{-1}(S_l P_l^{-1} + I)(N_l x_{k-1} + b),$$

j = 1, 2, … s(k),

which correspond to the nonzero diagonal entries of E_l. The processor then scales these entries so as to be able to deliver the results to the host processor, performing the parallel multisplitting scheme

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$$x_k = H(k)x_{k-1} + W(k)b \quad , \text{ k = 1, 2…} \tag{4}$$

where

$$H(k) = \sum_{l=1}^{a} E_l[(R_l^{-1}S_lP_l^{-1}Q_l)^{s(l,k)} + \sum_{j=0}^{s(l,k)-1}(R_l^{-1}S_lP_l^{-1}Q_l)^j R^{-1}(S_lP_l^{-1}+I)N_l]$$

and

$$W(k) = \sum_{l=1}^{a} E_l \sum_{j=0}^{s(l,k)-1}(R_l^{-1}S_lP_l^{-1}Q_l)^j (R^{-1}S_lP_l^{-1}+R_l)$$

Then, we can obtain the next algorithm:

Algorithm 1 (PATS):
for any given initial vector x_0
for k = 1, 2, … until convergent
for l = 1, 2, … α
$y_{l,0} = x_{k-1}$
for j = 1, 2, … s(l, k)

$$P_l y_{l,j-\frac{1}{2}} = Q_l y_{l,j-1} + (N_l x_{k-1} + b)$$

$$R_l y_{l,j} = S_l y_{l,j-\frac{1}{2}} + (N_l x_{k-1} + b)$$

$$x_k = \sum_{l=1}^{a} E_l y_{l,j}$$

Usually, we say that an parallel alternating two-stage method is stationary when *s(l, k) = s*, for all *l*, *k*, while an parallel alternating two-stage method is non-stationary if the number of inner iterations changes with the outer iteration *k*. We call them *SPATS* method and *NSPATS* method, respectively.

In this paper, our study concentrates on the parallel alternating two-stage method. With this aim, in the next section, we introduce the notation and preliminaries needed in this paper. In section 3, we present convergence conditions of these methods for solving nonsingular linear systems *Ax=b* when matrix A is a symmetric positive definite matrix. In section 4, we also give two relaxation parallel alternating two-stage methods, *RPATS I* and *RPATS II*. And we analyze their convergence conditions.

II. Notation and Preliminaries

We need the following definitions and results.

We say that a vector x is nonnegative, denoted $x \geq 0$, if all of its entries are nonnegative.

Definition 1: Let A be an $n \times n$ real matrix. We say that the splitting $A = M - N$ is *P*-regular if $M^T + N$ is definite positive.

Lemma 1: ([2]). Let $T_1, T_2, \cdots T_k, \cdots$ be a sequence of nonnegative matrices in $R^{n\times n}$. If exist a real number $0 < \theta < 1$, and a vector $v > 0$ in R^n, such that $T_j v \leq \theta v$, for *j = 1, 2 , …*, then $\rho(H(k)) \leq \theta < 1$, where $H(k) = T_k T_{k-1} \cdots T_2 T_1$, and therefore $\lim_{k\to\infty} H(k) = 0$.

Lemma 2: For an *n*-vector $x > 0$, the monotonic vector norm $\| y \|_x = \min\{a > 0 : -ax \leq y \leq ax\} = \max_{1\leq i\leq n} | \frac{y_i}{x_i} |$, which induces the matrix norm $\| A \|_x$ that satisfies $\| A \|_x = \| Ax \|_x$, and $\| |A| x \|_x = \| A \|_x$. Furthermore, if $|A| x \leq \beta x$ for some $\beta \geq 0$, then $\| A \|_x \leq \beta$ [3]. Moreover, a real matrix A is positive semidefinite if

$$x^T Ax \geq 0 \text{ for all } x \neq 0 \tag{5}$$

and positive definite if strict inequality holds in (5) [4]. Furthermore, it is well-known that for a symmetric positive definite matrix A the expression $\|x\|_A = \sqrt{x^T Ax}$ defines a vector norm. We denote by $\|.\|_A$ both the vector norm defined by A and the corresponding induced matrix norm.

III. Convergence Theorems

Firstly, we deal with the convergence of the *PATS* method when A is a symmetric matrix.

Theorem 1 [5]: Let A be a symmetric positive definite matrix. Let $A = M - N = P - Q$ be *P*-regular splittings. Consider the matrix $T = P^{-1}QM^{-1}N$, then $\rho(T) < 1$. Moreover, the unique splitting $A = B - C$ induced by the iteration matrix *T*, such that $T = B^{-1}C$ is also *P*-regular .

Theorem 2 [6]: Let A be a symmetric positive definite matrix. Consider $A = M - N$ such that M is Hermitian and N is positive semidefinite. Let $M = F - G$ be a P-regular splitting. Assume further that the sequence of inner iterations $\{s(l)\}_{l=0}^{\infty}$ remains bounded. Then the two-stage iterative method converges to the solution of the linear system (1) for any initial vector x_o .

Theorem 3: Let A be a symmetric positive definite matrix. Consider $A = M_l - N_l$, *l=1, 2, …* , such that M_l is Hermitian and N_l is positive semidefinite. Let $M_l = P_l - Q_l = R_l - S_l$ be P-regular splittings. Assume further that the sequence of inner iterations $\{s(l)\}_{l=0}^{\infty}$ remains bounded. Then the *PATS* method converges to the solution of the linear system (1) for any initial vector x_o .

Proof: We suppose that $H(k) = \sum_{l=1}^{\alpha} E_l T_l(k)$, *l=1, 2, …* , *k = 0, 1, 2, …* where

$$T_l(k) = (R_l^{-1}S_lP_l^{-1}Q_l)^{s(l,k)} + \sum_{j=0}^{s(l,k)-1}(R_l^{-1}S_lP_l^{-1}Q_l)^j R^{-1}(S_lP_l^{-1}+I)N_l$$
$$= (R_l^{-1}S_lP_l^{-1}Q_l)^{s(l,k)} + [I-(R_l^{-1}S_lP_l^{-1}Q_l)^{s(l,k)}]M_l^{-1}N_l$$

From Theorem 1, there is a unique pair of matrices B_l, C_l, *l=1, 2, …*, such that $R_l^{-1}S_lP_l^{-1}Q_l = B_l^{-1}C_l$ and $M_l = B_l - C_l$, *l=1, 2, …*, is a *P*-regular splitting. Thus, $H(k)=\sum_{l=1}^{\alpha}E_lT_l(k)$ is the iteration matrix of a non-stationary two-stage method for the matrix $A_l = M_l - N_l$, with M_l Hermitian and N_l positive semidefinite and $M_l = B_l - C_l$ being *P*-regular. So $H(k)=\sum_{l=1}^{\alpha}E_lT_l(k)$, *l=1, 2, …, k = 0, 1, 2, …*, where

$$T_l(k) = (B_l^{-1}C_l)^{s(l,k)} + [I-(B_l^{-1}C_l)^{s(l,k)}]M_l^{-1}N_l$$
$$= I - P_l(k)A,$$

where $P_l(k) = [I-(B_l^{-1}C_l)^{s(l,k)}]M_l^{-1}$,

$$\|T_l(k)\|_A = \|I-P_l(k)A\|_A = \left\|A^{\frac{1}{2}}[I-P_l(k)A]A^{-\frac{1}{2}}\right\|_2,$$

$A^{\frac{1}{2}}[I-P_l(k)A]A^{-\frac{1}{2}}$ is symmetric and

$$\left\|A^{\frac{1}{2}}[I-P_l(k)A]A^{-\frac{1}{2}}\right\|_2 = \rho(A^{\frac{1}{2}}[I-P_l(k)A]A^{-\frac{1}{2}})$$
$$= \rho(I-P_l(k)A) = \rho(T_l(k)).$$

From Theorem 1, $\rho(B_l^{-1}C_l)<1$, $\lim_{s(l,k)\to\infty}(B_l^{-1}C_l)^{s(l,k)}=0$, so $\lim_{s(l,k)\to\infty}\rho(T_l(k)) = \rho(M_l^{-1}N_l)<1$, thus

$$\rho(T_l(k)) < \rho(M_l^{-1}N_l) + \frac{1}{2}[1-\rho(M_l^{-1}N_l)]$$
$$= \frac{1}{2}[1+\rho(M_l^{-1}N_l)] < 1$$

Let

$\theta_k = \max\{\rho(T_l(k)), s(l,k)=1,2,\cdots; \frac{1}{2}[1+\rho(M_l^{-1}N_l)]\}$,

obviously, $\theta_k < 1$.

We have $\rho(T_l(k)) \le \theta_k < 1$, *l=1, 2, …, k = 0, 1, 2, …*, let $\theta = \max\{\theta_1, \theta_2, \cdots \theta_k\}$, we can obtain

$$\|H(k)\|_A = \left\|\sum_{l=1}^{\alpha}E_lT_l(k)\right\|_A \le \sum_{l=1}^{\alpha}\|E_l\|_A\|T_l(k)\|_A$$
$$= \sum_{l=1}^{\alpha}\|E_l\|_A\rho(T_l(k)) \le \sum_{l=1}^{\alpha}\|E_l\|_A\theta_k \le \sum_{l=1}^{\alpha}\|E_l\|_A\theta = \theta$$

Since $\rho(H(k)) \le \|H(k)\|$, it gets $\rho(H(k)) \le \theta < 1$, *k = 0, 1, 2, …*. Thus the proof is completed.

IV. Relaxation Iteration Methods

Now we introduce the relaxation factor to the parallel alternating two-stage methods. Then, we can obtain the following two algorithms:

Algorithm 2 (RPATS I):

for any given initial vector x_0 and $\omega \in (0,1)$

for k = 1, 2, … until convergent

for l = 1, 2, … α

$y_{l,0} = x_{k-1}$

for j = 1, 2, … s(l, k)

$$P_ly_{l,j-\frac{1}{2}} = Q_ly_{l,j-1} + (N_lx_{k-1}+b)$$
$$R_ly_{l,j} = S_ly_{l,j-\frac{1}{2}} + (N_lx_{k-1}+b)$$
$$x_k = \omega\sum_{l=1}^{a}E_ly_{l,j} + (1-\omega)x_{k-1}$$

Algorithm 3 (RPATS II):

for any given initial vector x_0 and $\omega \in (0,1)$

for k = 1, 2, … until convergent

for l = 1, 2, … α

$y_{l,0} = x_{k-1}$

for j = 1, 2, … s(l, k)

$$P_ly_{l,j-\frac{1}{2}} = \omega(Q_ly_{l,j-1} + (N_lx_{k-1}+b)) + (1-\omega)P_ly_{l,j-1}$$
$$R_ly_{l,j} = \omega(S_ly_{l,j-\frac{1}{2}} + (N_lx_{k-1}+b)) + (1-\omega)R_ly_{l,j-\frac{1}{2}}$$
$$x_k = \sum_{l=1}^{a}E_ly_{l,j}$$

For the relaxation parallel alternating two-stage methods (*RPATS I and RPATS II*), we can similarly get the following convergence theorems.

Theorem 4: Let A be a Hermitian positive definite matrix. Consider $A = M_l - N_l$, *l = 1, 2, …*, such that M_l is Hermitian and N_l is positive semidefinite. Let $M_l = P_l - Q_l = R_l - S_l$ be P-regular splittings, and $0<\omega<1$. Assume further that the sequence of inner iterations $\{s(l)\}_{l=0}^{\infty}$ remains bounded. Then the relaxation parallel alternating two-stage methods (*RPATS I and RPATS*

II) converge to the solution of the linear system (1) for any initial vector x_0 .

The proof of the Theorem 4 is similar to the proof of Theorem 3, so we omit it.

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