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Some Properties of Cut Locus of a Flat Torus

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***Abstract*—**In this article, we would like to show that there is no cut point of any point in a plane, but there exists the cut locus of a point in a flat torus. By the results, we would like to determine the structure of cut locus of a flat torus.

***Keywords*—**Cut locus, flat torus, geodesics.

I. Introduction

A great circle is the intersection of a sphere and a plane that passes through the center of the sphere. Suppose our earth is a sphere, the equator is a great circle, as is the meridian as line of longitude connecting the North Pole and the South Pole. Lines of latitude or parallels are not great circle since their centers are not the center of the earth. A geodesic in a surface is defined as the shortest path connecting two points in that surface. It was known that the shortest path between any two points on a sphere is an arc of a great circle. Thus, great circles are geodesics on a sphere [1]-[3]. There may be more than one geodesic connecting a given pair of points. For example, there are infinitely many geodesics connecting the North Pole and the South Pole on the globe.

Let a and b are antipodal points in any great circle. Thus a and b are joined by semi great circle γ. With the symmetric property on a sphere, there exists $\tilde{\gamma}$ as is the other half of great circle. Here we get $\gamma \neq \tilde{\gamma}$, $l(\gamma) = l(\tilde{\gamma})$, where $l(\cdot)$ denotes length of the curve. Let c and d are antipodal points in the same great circle as a and b. By extending the point b to the point d along the great circle, the length $l(\gamma|_{[a,c,b,d]})$ is longer than the length $l(\tilde{\gamma}|_{[a,d]})$. In this case b is called a cut point of a. Analogously a is called a cut point of b. Therefore, it is trivial that the north pole and the south pole in the same great circle are cut point to each other on a sphere.

On the right circular cylinder, the circle or parallel that is the cross sections of the cylinder, the generating curve or meridian and the helix which joined any two points are geodesics [4],[5]. Choose any two points on the cylinder; it is possible to connect them through an infinite number of helices. Among these geodesics, there exist the minimal geodesics between the two points. By the result of the great circle on a sphere, any antipodal pairs along a parallel are cut locus to each other on the cylinder. Let τ_a be the opposite meridian to a point a on the cylinder. Consider any point b in τ_a, there exists a helix γ that is a minimal geodesic joining a to b. According to the symmetric property with respect to the plane containing a and τ_a there exists $\tilde{\gamma}$ joined a to b on the opposite face of cylinder. Here we get $\gamma \neq \tilde{\gamma}$, $l(\gamma) = l(\tilde{\gamma})$. In this case b is called a cut point of a. That is, the opposite meridian to a which contained the set of all cut points to a is the cut locus of a [6]. The rigorously definition will define in the next section.

Thus it is interesting to study cut locus on a surface. In this article, we would like to determine some properties of the cut locus on a flat torus.

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II. Basic Theory

Here, let us review the basic theory concern on properties of cut locus which can be found in [2], [7]-[13].

A. Surface of Revolution

A surface of revolution is a surface obtained by rotating a plane curve in R^3 where the rotation is about a line that does not intersect the curve and is contained in the plane containing the curve. Without loss of generality we may assume that the curve is a unit speed $xz-$plane curve and the axis of rotation is the $z-$axis.

Let $c(t) := (r(t), 0, z(t))$, where $r(t) > 0$ for all t be $xz-$plane curve without self-intersection. The surface of revolution can then be covered by coordinate patch of the form $x(t,\theta) := (r(t)\cos\theta, r(t)\sin\theta, z(t)), (t,\theta) \in R^2$.

The curves on a surface of revolution obtained by holding θ constant and varying t are called meridians or longitudes, and the curves on the surface obtained by holding t constant and varying θ are circles of latitude or parallels (Fig. 1).

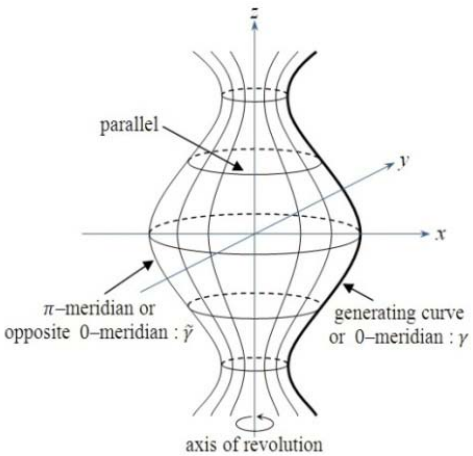

Fig. 1 Surface of revolution

A sphere of radius r is obtained by rotating a semicircle of radius r centered at the origin. A typical coordinate patch is given by

$$x(\varphi,\theta)=(r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, r\sin\varphi),(\varphi,\theta)\in R^2.$$

A torus of large radius R and small radius r is obtained by rotating a circle in the $xz-$plane with radius r and centered at the point $(R,0,0)$, as in Fig. 2, has a coordinate patch in the form

$$x(\varphi,\theta)=((R+r\cos\theta)\cos\varphi,(R+r\cos\theta)\sin\varphi, r\sin\theta),(\theta,\varphi)\in R^2.$$

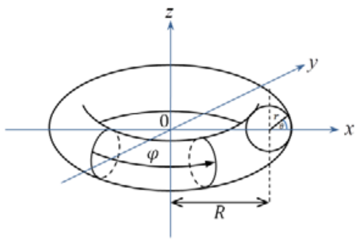

Fig. 2 Torus

A surface M is called a complete surface if every Cauchy sequence of point of M converges on M. Thus the Euclidean E^3 is complete. Moreover, any closed subset M of E^3 is complete. That is any Cauchy sequence $\{P_i\}$ of points in M is also Cauchy sequence of point in E^3, which has a limit point p. Since M is closed, $p\in M$, hence M is complete.

One of the types of geometry object are manifolds, and a surface is a two dimensional manifold. In manifold, the plane is represented by R^2, while R^1 is a real line. A unit circle in R^2 is denoted S^1, defined by $S^1=\{x\in R^2 \mid \|x\|=1\}$. If we let S^2 denotes a unit sphere in R^3, thus $S^2=\{x\in R^3 \mid \|x\|=1\}$. Then $S^1\times R^1\subset R^2\times R^1$ is a right circular cylinder in R^3 and $S^1\times S^1\subset R^2\times R^2$ is a torus in R^4.

Let $M\subset R^3$ be a complete connected surface and let $p\in M$ be a point. A vector v in R^3 is a tangent vector to M at p if there exists a curve $c:(-\varepsilon,\varepsilon)\to M$ for some number $\varepsilon>0$ such that $c(0)=p$ and $c'(0)=v$. The collection of all tangent vector to M at p is denoted T_pM, and is called the tangent plane to M at p.

B. Geodesics

Let $\gamma:[a,b]\to M$ be a C^∞ curve on a complete connected surface M. The curve γ is called a geodesic on M if $\gamma''(s)$ is orthogonal to the tangent space $T_{\gamma(s)}M$ for each $s\in[a,b]$. If $\gamma:[a,b]\to M$ is a geodesic of M then $\frac{d}{dt}|\gamma'(t)|^2=0$ or $<\gamma''(t),\gamma'(t)>=0$.

Geodesics of a plane: Let $P=\{x\in R^3 \mid <x,a>=b\}$ be a plane orthogonal to the unit vector $a\in R^3$. If $\gamma:[a,b]\to P$ is an arbitrary differentiable curve on P, thus $<\gamma(t),a>=b$ for each $t\in[a,b]$.

We have $<\gamma''(t),a>=0$, that is $\gamma''(t)\in T_{\gamma(t)}P, \forall t\in[a,b]$. Hence γ is a geodesic if and only if $\gamma''=0$, that is $\gamma(t)=ct+d$ where $c,d\in R^3$. Therefore, the geodesics of a plane are the straight lines parameterized by the arc length in the plane.

Geodesics of a sphere: Let γ be a differentiable curve parameterized by the arc length on a unit sphere S^2 centered at a point $a\in R^3$. We have $|\gamma(t)-a|^2=r^2$ with $r>0$ for all t .By differentiating this expression two times, we obtain $<\gamma(t)-a,\gamma''(t)>=-1$. Since the tangent plane $T_{\gamma(t)}S^2(r)$ is the orthogonal component of the radius vector $\gamma(t)-a$, we may have $\gamma''(t)=-\frac{1}{r^2}(\gamma(t)-a)$. Thus, γ is a geodesic if and only if γ satisfies the differential equation

$$r^2\gamma''(t)+\gamma(t)-a=0.$$

With the condition $|\gamma(t)-a|^2=r^2$ and $|\gamma'(t)|^2=1$, we have $\gamma(t)=a+p\cos\frac{t}{r}+rv\sin\frac{t}{r}$, where $|p|^2=r^2$, $|v|^2=1$ $<p,v>=0$. That is the geodesics of a sphere are the great circles determined by the plane spanned by p and v, which passes through the center of the plane.

Geodesics of a cylinder: Let C be the right circular cylinder of unit radius whose axis is the z-axis of R^3 and let $\gamma:[a,b]\to C$ be a differentiable curve on C given by $\gamma(t)=(x(t),y(t),z(t))$. Let γ is a geodesic if and only if $(x''(t),y''(t),z''(t))\perp T_{\gamma(t)}C$ for each $t\in[a,b]$. Here we get $(x''(t),y''(t),z''(t))\parallel(x(t),y(t),0)$.

Suppose that γ is parameterized by arc length, thus $|\gamma'(t)|^2=1=x'(t)^2+y'(t)^2+z'(t)^2$.

By differentiating $x(t)^2+y(t)^2=1$ two times, we may have $x(t)x''(t)+y(t)y''(t)=-1+z(t)^2$. Hence the curve γ is a geodesic of a cylinder if and only if $x''(t)+(1-z'(t)^2)x(t)=0$, $y''(t)+(1-z'(t)^2)y(t)=0$ and $z''(t)=0$.

Suppose $r(0)=(1,0,0)$ and $r'(0)=(0,a,b)$ where $a^2+b^2=1$. Then $z(t)=bt$ and $x''(t)+a^2x(t)=y''(t)+a^2y(t)=0$, and so x and y are in the form $\lambda \sin at+\mu\cos at, \lambda,\mu\in R$. Thus $\gamma(t)=(\cos at,\sin at,bt)$ with $a^2+b^2=1$. Here we get the geodesics of a cylinder are circular helix including its straight lines and circles as a limit case.

Hopf-Rinow Theorem for a complete connected surface M is stated that every geodesic can be extended indefinitely in either direction, or else it form a closed curve, and, for a distinct points p and q on M, there exists a minimal geodesic joining p and q.

In a plane, the geodesics are straight line, and any two points p and q can be joined by a unique line segment with length $d(p,q)$. Here $d(p,q):=\inf\{l(c);c$ is piecewise C^∞ curve on M joining p and q }.

On a sphere, the geodesics are the great circles and any two points p and q which are not antipodal points can be joined by two great circular arcs (major arc and minor arc), of which only one has length $d(p,q)$. Moreover, between two antipodal points p and q there exist infinitely many great circular arcs with the same length $d(p,q)$.

On a right circular cylinder, any two points p and q on the same generating curve can be joined not only by the generating curve with the length $d(p,q)$ but also infinitely many circular helices of varying pitch, which wind around the cylinder and all are geodesics.

C. Cut Point and Cut Locus

Let $\gamma|_{[0,t_1]}$ be a unit speed minimal geodesics emanating from a point $p=\gamma(0)$ of a complete connected surface M .If for all number $t_2>t_1$, for all geodesic extension $\gamma|_{[0,t_2]}$ is not minimal anymore, then $\gamma(t_1)$ is called a cut point of p along γ.

The cut locus of a point p is the set of all cut points along a minimal geodesic emanating from p and denotes the set by C_p.

Klingenberg Lemma: If $\gamma(t_1)$ is the cut point of $p=\gamma(0)$ along γ then there exist two distinct minimal geodesics α and β emanating from p to $\gamma(t_1)$ such that $l(\alpha)=l(\beta)$.

D. Covering Space

Let $\tilde{M}$ and M be subsets of R^3. We will call $\pi:\tilde{M}\to M$, a covering map if

1. π is continuous and $\pi(\tilde{M})=M$,
2. each point $p\in M$ has an open neighborhood U_p in M such that for each p, $\pi^{-1}(U_p)$ is a disjoint union of open sets in $\tilde{M}$.

Then $\tilde{M}$ is called a covering space of M.

Here we have, $\pi:\tilde{M}\to M$ is a universal covering space if and only if $\tilde{M}$ is simply connected and $\pi:\tilde{M}\to M$ is a covering space.

For example, let $P\subset R^3$ be a plane in R^3. By fixing a point $q\in P$ and two orthogonal unit vectors $e_1,e_2\in P$ with origin at q_0, the coordinates (u,v) for any point $p\in P$ are given by $q-q_0=ue_1+ve_2$.

Let $S=\{(x,y,z)\in R^3 \mid x^2+y^2=1\}$ be the right circular cylinder whose axis is the z-axis, and let $\pi:P\to S$ be the map defined by $\pi(u,v)=(\cos u,\sin u,v)$.

The geometrical meaning of this map is to wrap the plane P around the cylinder S into an infinite number of times.

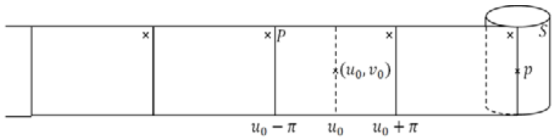

Fig. 3 The map $\pi:P\to S$

We would like to show that π is a covering map. Let consider $(u_0,v_0)\in P$, the mapping is limiting to the band $R=\{(u,v)\in P, u_0-\pi\le u\le u_0+\pi\}$ which covers S minus the generating curve. Thus π is continuous and $\pi(P)=S$.

Next, let p is any point on S and $U_p=S-\tau_p$ where τ_p is the opposite meridian to p. We would like to show that $\pi^{-1}(U_p)$ is a disjoint union of open set of $\tilde{M}$.

Let $(u_0,v_0)\in P$ be a point such that $\pi(u_0,v_0)=p$ and choose the band V_n given by

$$V_n=\{(u,v)\in P \mid u_0+(2n-1)\pi<u<u_0+(2n+1)\pi\},$$

$n=0,\pm1,\pm2,\ldots$. It is clearly that if $m\ne n$ then $V_n\cap V_m=\phi$ and $\bigcup_n V_n=\pi^{-1}(U_p)$.

Thus, the plane P is a covering space of the cylinder S. Since P is simply connected, here we get $\pi:P\to S$ is a universal covering space which is a flat cylinder in this article.

We now intend for the torus case. Let R^2 be a plane with coordinates (x,y) and $T_{m,n}:R^2\to R^2$ be the map translation $T_{m,n}(x,y)=(x+m,y+n)$ where m and n are any integers. Consider the equivalence relation on R^2 given by $(x,y)\sim(x+m,y+n)$ where m and n are any integers. Let $\pi:R^2\to T$ be the normal projection map $\pi(x,y)=\{T_{m,n}(x,y)$ for all integer m,n }. Thus, in each open unit square whose vertex have integer coordinates, there is only one representative of T and T be a torus obtained by identifying opposite side of a square and $\pi:R^2\to T$ is a universal covering space which is a flat torus in this article. (Fig. 4) The point $p_i, i = 1,...,4$ on each corner of a flat torus in R^2 is the only one point p on a torus T, that is $p_i = p, i = 1,...,4$.

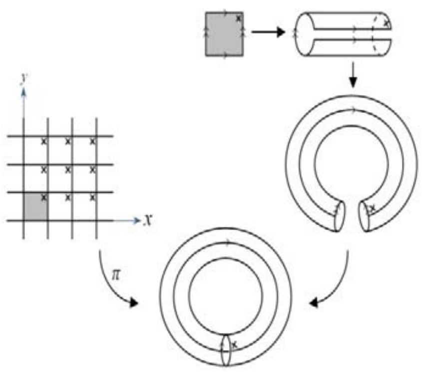

Fig. 4 The map $\pi : R^2 \to T$

Let $M := S^1 \times R^1$ is a right circular cylinder with the metric $ds^2 = dr^2 + r^2 d\theta^2$ where (r,θ) denote a polar coordinates. By the universal covering space $\pi : \tilde{M} \to M$, we get $\tilde{M} := R^1 \times R^1$ is a flat cylinder with the metric $ds^2 = dr^2 + d\theta^2$. For a torus $M := S^1 \times S^1$, the metric is $ds^2 = r^2 d\phi^2 + (R + r\cos\varphi)^2 d\theta^2$ where (θ,φ) rectangular coordinates are, R is a large radius and r is a small radius. Here we get a flat torus $\tilde{M} := R^1 \times R^1$ with the metric $ds^2 = d\theta^2 + d\varphi^2$.

III. The Main Results

In this section we would like to verify some statements that we have mentioned in the first section then we will present the main results of this article.

Lemma 1: Any line segment in a plane is a minimal geodesic.

Lemma 2: There is no cut point of any point in a plane.

Proof: Without loss of generality we may assume $p := (0,0)$ be a point in a plane P.

Suppose there exists a cut point q of p along a minimal geodesic γ joining $p = \gamma(0)$ to $q = \gamma(t_1)$. Thus for all number $t_2 > t_1$, for all geodesics extension $\gamma|_{[0,t_2]}$ is not minimal anymore.

Here we get a contradiction, by Lemma 1.

Lemma 3: Let M be a complete simply connected surface. Let
p be a point on M. The cut locus C_p of $p \in M$ has a local tree structure which does not contain a cycle.

Proof: Suppose the cut locus C_p of $p \in M$ contains a cycle in M.

From Jordan theorem, there exists an interior bounded by C_p. Let x be a point in an interior. Since M is complete and simply connected, by Hopf-Rinow theorem, there exists a minimal geodesic γ joining p to x and γ intersects C_p at q. Here we get $\gamma|_{[p,x]}$ is the extension of $\gamma|_{[p,q]}$.

Since q is a point in C_p, thus q is a cut point.

Since q is a cut point of p, the extension $\gamma|_{[p,q]}$ is not minimal anymore. This is a contradiction.

That is C_p of $p \in M$ does not contain a cycle in a complete simply connected surface M.

Lemma 4: The cut point of any point p on a sphere is the antipodal point to p.

Proof: We may suppose that $C_N = \{S\}$ where N is the north pole and S is south pole.

There exists the meridian joining N to S. Since a meridian is a half circle of the great circle and is a minimal geodesic, here we get by Klingenberg Lemma, S is a cut point of N.

Hence $\{S\} \subset C_N$.

Conversely, let $\gamma : [0,\infty) \to S^2(1)$ be a curve on a unit sphere $S^2(1)$ such that $N := \gamma(0)$ and $S := \gamma(\pi)$. Here we get $\gamma|_{[0,\pi]}$ is a half circle of the great circle.

Consider the geodesic extension $\gamma|_{[0,t]}$ where $2\pi > t > \pi$. here we get $l(\gamma|_{[0,t]}) = t$, and

$$l(\tilde{\gamma}) = 2\pi - l(\gamma|_{[0,t]}) = 2\pi - t < 2\pi - \pi = \pi.$$

Thus the geodesic extension is not minimal anymore.

Then $C_N \subset \{S\}$.

Therefore, the cut locus of any point in a sphere is the antipodal point to that point.

A. On the Cut Locus in a Flat Cylinder

Lemma 5: Let $\tilde{M} := (R^1 \times R^1, ds^2 = dr^2 + d\theta^2)$ denote a flat cylinder of revolution. Let p be a point on $\tilde{M}$ with $\theta(p) = 0$, then the cut locus C_p is the opposite meridian to p.

Proof: Let $\tau_p = \{q \in \tilde{M} \mid \theta(q) = \pi\}$ be the opposite meridian to $p \in \tilde{M}$.

For any q of τ_p, there exists a minimal geodesic γ joining p to q. According to the symmetric property, there exists $\tilde{\gamma}$ which is the reflection of γ with respect to τ_p.

Here we get $\tilde{\gamma} \neq \gamma$ and $l(\tilde{\gamma}) = l(\gamma)$. Thus by Klingenberg lemma, q is a cut point of p.

Hence $\tau_p \subset C_p$.

Conversely, suppose q is a cut point of p and $q \notin \tau_p$.

Let $0 < \theta(q) < \pi$.

Since q is a cut point of p, there exists a minimal geodesic γ joining p to q where $p = \gamma(0)$ and $q = \gamma(t_1)$. For all number $t_2 > t_1$, for all geodesic extension $\gamma|_{[0,t_2]}$ is not minimal anymore. Here we get a contradiction by Lemma 1.

Hence $C_p \subset \tau_p$.

Therefore, cut locus of any point in a flat cylinder is the opposite meridian to that point.

B. On the Cut Locus in a Flat Torus

Lemma 6: Let $\tilde{M} := (R^1 \times R^1, ds^2 = dr^2 + d\theta^2)$ denote a flat torus. Let p be a point on $\tilde{M}$ such that $(\theta(p), \varphi(p)) = (0,0)$ and $p_i = p, i = 1,...,4$, then the cut locus C_p of p is the meridian opposite to p union the parallel opposite to p.

Proof: Let q be a point on $\tilde{M}$. If $\theta(q) = \pi = \varphi(q)$ it is clearly that q is a cut point of p.

Thus we will consider the case $\theta(q) \neq \pi$ or $\varphi(q) \neq \pi$.

Let q be a point on $\tilde{M}$ such that $\varphi(q) = \pi$, $0 < \theta(q) < 2\pi$.

Since $\tilde{M}$ is complete, there exists a minimal geodesic γ joining p_1 to q. With the symmetric property to the parallel $\varphi = \pi$, there exists a minimal geodesic $\tilde{\gamma}$ joining p_4 to q. Thus q is a cut point of p along γ. If we consider a minimal geodesic joining p_2 to q and p_3 to q, the result is the same as above.

The proof is similar if we choose $\theta(q) = \pi$, $0 < \varphi(q) < 2\pi$.

Here we get $\{(\theta,\varphi) \in \tilde{M} \,|\, \theta = \pi \cup \varphi = \pi\} \subset C_p$.

Conversely, suppose q is a cut point of p and $q \notin \{(\theta,\varphi) \in \tilde{M} \,|\, \theta = \pi \cup \varphi = \pi\}$. Let $0 < \theta(q) < 2\pi$ and $0 < \varphi(q) < 2\pi$.

Since q is a cut point of p, there exists a minimal geodesic γ joining p to q where $p = \gamma(0)$ and $q = \gamma(\theta_1)$. For all number $\theta_2 > \theta_1$, for all geodesic extension $\gamma|_{[0,\theta_2]}$ is not minimal anymore. Here we get a contradiction by Lemma 1. Therefore the proof is completed.

The flat torus that we had discussed in Lemma 6 is generated by a rectangle as a covering space. The rectangle is a special case of a parallelogram. In general, a parallelogram with the angle equals π is called a rectangle. Thus we would like to determine the structure of the cut locus of a general flat torus which a covering space is a parallelogram by using the property of the orthogonal bisectors.

Let p and q be distinct points in a plane P. A point x is equidistant to p and to q if $d(p,x) = d(q,x)$. The set of all points equidistant to p and to q is the line orthogonal and passes through the midpoint of the line segment joining p to q. This line is called the orthogonal bisector.

The orthogonal bisector to each side of a triangle meets at one point interior to the triangle. Thus the triangle is divided into three sectors, each sector bounded by two orthogonal bisectors and two sides of triangle adjacent to the orthogonal bisectors.

Let q be the meeting point for the orthogonal bisectors of a triangle $p_1 p_2 p_3$ thus $d(p_1,q) = d(p_2,q) = d(p_3,q)$.

Let consider a parallelogram $p_1 p_2 p_3 p_4$, $\measuredangle p_1 \angle \frac{\pi}{2}$. There are two meeting points x and y of the orthogonal bisector to each side of the parallelogram.

If we join x to y, this line is the orthogonal bisector to the minor diagonal of the parallelogram or the diagonal opposite to p_1.

The midpoint of the line segment between x and y is the bisector of the main diagonal of the parallelogram or the diagonal passes through p_1.

If we are identifying the opposite side of the parallelogram, we get the torus with $p_i = p, i = 1,...,4$.

Let α be the orthogonal bisector to the minor diagonal of the parallelogram, and $\beta_i, i = 1,...,4$ be the orthogonal bisector to each side of parallelogram.

If $\measuredangle p = \frac{\pi}{2}$, a parallelogram is then a rectangle. For the rectangle, the two meeting points x and y will be the same point or $x = y$. Here we get only two orthogonal bisectors which intersect at x, and x is also the bisector of both diagonal of rectangle. The union of the two orthogonal bisectors of the rectangle is the cut locus of point p where $(\varphi(p), \theta(p)) = (0,0)$, which has been proved in Lemma 6.

From the knowledge above, we may assume that the cut locus of the flat torus generated by the parallelogram as a covering space is the union of all orthogonal bisector interior the parallelogram. Thus, we will state the main theorem of this article as following.

Theorem: Let $\tilde{M} = (R^1 \times R^1, ds^2 = d\theta^2 + d\varphi)$ denote a flat torus. Suppose (θ,φ) are not rectangular coordinates, then the cut locus of a point $p_i = p, i = 1,...,4$ on $\tilde{M}$ is the union of the orthogonal bisector to each side of the flat torus and the coincident line joining the intersection between each pairs of the orthogonal bisector.

Proof: Without loss of generality, we may assume that $\measuredangle(\theta,\varphi) \angle \frac{\pi}{2}$, and we will consider one half of the flat torus which is divided by the minor diagonal.

Let p be a point on $\tilde{M}$ such that $p_i = p, i = 1,...,4$ and $(\theta(p), \varphi(p)) = (0,0)$. Let L_p be the union of the orthogonal bisector of the triangle that is a half part of a flat torus.

Suppose q is a point on the orthogonal bisector incident to the line segment joining p_1 to p_2, by the property of the orthogonal bisector of a triangle, there exist two minimal geodesics γ and $\tilde{\gamma}$ joining p_1 to q and p_2 to q. Thus q is a cut point of p along γ.

Here we get $L_p \subset C_p$.

Conversely, suppose q is a cut point of p_1 and $q \notin L_p$.

We may assume that q is in the sector that contain p_1.Since q is a cut point of p_1, there exists a minimal geodesic γ joining p_1 to q. The other geodesic emanating from p_2, p_3, p_4 to q are all longer than γ since they intersect the appropriate orthogonal bisector, while γ is not intersect any orthogonal bisectors.

Since q is a cut point of p_1, $\gamma|[p_1, q]$ is a minimal geodesic joining $p_1 = \gamma(0)$ to $q = \gamma(\theta_1)$. For all number $\theta_2 > \theta_1$, for all $\gamma|[p, \gamma(\theta_2)]$ is not minimal anymore. Here we get a contradiction by Lemma 1.

Hence $q \in L_p$.Thus $C_q \subset L_p$.

For this reason the proof is completed.

IV. Conclusion

Here we have proved that no cut point of any point p in a plane. For any point p on a flat cylinder, there exists the cut locus of p as is the opposite meridian to p. For any point p of a flat torus generated by the rectangle as a covering space, cut locus is the opposite meridian to p union the parallel that is a geodesic opposite to p. The structure of cut locus of a general flat torus which generate by the parallelogram as a covering space is the union of the orthogonal bisector to each side of the flat torus and the incident line joining the intersection between each pairs of the orthogonal bisector.

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