# Some Collineations Preserving Cross-Ratio in some Moufang-Klingenberg Planes 

Süleyman Ciftci, Atilla Akpinar and Basri Celik


#### Abstract

In this paper we are interested in Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ defined over a local alternative ring $\mathcal{A}$ of dual numbers. We show that some collineations of $\mathbf{M}(\mathcal{A})$ preserve cross-ratio.


Keywords-Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio.

## I. Introduction

The number of collineations of any projective plane is huge. For example; the Fano plane has 168 collineations, the nonDesarguesian projective Veblen-Wedderburn plane of order 9 (which is denoted by $\pi_{N}(9)$ ) has 311,040 collineations [14, p. 366]. It is easy to see that the composite of any two collineations is a collineation, as the invers of any collineation. Function composition is always associative; thus the collineations of any projective or affine plane form a group. For more detailed information about these groups, the reader is referred to the books of [11], [14].

In the Euclidean plane, Desargues established the fundemantal fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c. 300 B.C) is invariant under projection [3, p. 133]. For this reason, cross-ratio is one of the most important concepts of projective geometry.

In this paper we deal with the class (which we will denote by $\mathbf{M}(\mathcal{A})$ ) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring

$$
\mathcal{A}:=\mathbf{A}(\varepsilon)=\mathbf{A}+\mathbf{A} \varepsilon
$$

(an alternative field $\mathbf{A}, \varepsilon \notin \mathbf{A}$ and $\varepsilon^{2}=0$ ) introduced by Blunck in [7]. We will show that some collineations of $\mathbf{M}(\mathcal{A})$ from [8] preserve cross-ratio. For more information about some well-known properties of cross-ratio in the case of Moufang planes or MK-planes $\mathbf{M}(\mathcal{A})$, respectively, it can be seen the papers of [10], [4], [9] or [7], [1].

Section 2 includes some basic definitions and results from the literature.

In Section 3 we will give some collineations of $\mathbf{M}(\mathcal{A})$ from [8] and we show that the collineations preserve cross-ratio, the main result of the paper.

## II. Preliminaries

Let $\mathbf{M}=(\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation ' $\sim$ ' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$, respectively. Then

[^0]$\mathbf{M}$ is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:
(PK1) If $P, Q$ are non-neighbour points, then there is a unique line $P Q$ through $P$ and $Q$.
(PK2) If $g, h$ are non-neighbour lines, then there is a unique point $g \cap h$ on both $g$ and $h$.
(PK3) There is a projective plane $\mathbf{M}^{*}=\left(\mathbf{P}^{*}, \mathbf{L}^{*}, \in\right)$ and an incidence structure epimorphism $\Psi: \mathbf{M} \rightarrow \mathbf{M}^{*}$, such that the conditions
$$
\Psi(P)=\Psi(Q) \Leftrightarrow P \sim Q, \Psi(g)=\Psi(h) \Longleftrightarrow g \sim h
$$
hold for all $P, Q \in \mathbf{P}, g, h \in \mathbf{L}$.
A point $P \in \mathbf{P}$ is called near a line $g \in \mathbf{L}$ iff there exists a line $h \sim g$ such that $P \in h$.

Let $h, k \in \mathbf{L}, C \in \mathbf{P}, C$ is not near to $h, k$. Then the well-defined bijection

$$
\sigma:=\sigma_{C}(k, h):\left\{\begin{array}{l}
h \rightarrow k \\
X \rightarrow X C \cap k
\end{array}\right.
$$

mapping $h$ to $k$ is called a perspectivity from $h$ to $k$ with center $C$. A product of a finite number of perspectivities is called a projectivity.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of $\mathbf{M}$.
A Moufang-Klingenberg plane (MK-plane) is a PK-plane $\mathbf{M}$ that generalizes a Moufang plane, and for which $\mathbf{M}^{*}$ is a Moufang plane (for the exact definition see [2]).
An alternative ring (field) $\mathbf{R}$ is a not necessarily associative ring (field) that satisfies the alternative laws

$$
a(a b)=a^{2} b,(b a) a=b a^{2}, \forall a, b \in \mathbf{R} .
$$

An alternative ring $\mathbf{R}$ with identity element 1 is called local if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [13, Theorem 3.1]).

Lemma 2.2: The identities

$$
\begin{aligned}
x(y(x z)) & =(x y x) z \\
((y x) z) x & =y(x z x) \\
(x y)(z x) & =x(y z) x
\end{aligned}
$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [12, p. 160]).

ISSN: 2517-9934
Vol:3, No:11, 2009

We summarize some basic concepts about the coordinatization of MK-planes from [2].

Let $\mathbf{R}$ be a local alternative ring. Then $\mathbf{M R}=(\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$
\begin{aligned}
\mathbf{P}= & \{(x, y, 1) \mid x, y \in \mathbf{R}\} \cup\{(1, y, z) \mid y \in \mathbf{R} z \in \mathbf{I}\} \\
& \cup\{(w, 1, z) \mid w, z \in \mathbf{I}\} \\
\mathbf{L}= & \{[m, 1, p] \mid m, p \in \mathbf{R}\} \cup\{[1, n, p] \mid p \in \mathbf{R} n \in \mathbf{I}\} \\
& \cup\{[q, n, 1] \mid q, n \in \mathbf{I}\} \\
{[m, 1, p]=} & \{(x, x m+p, 1) \mid x \in \mathbf{R}\} \\
& \cup\{(1, z p+m, z) \mid z \in \mathbf{I}\}, \\
{[1, n, p]=} & \{(y n+p, y, 1) \mid y \in \mathbf{R}\} \\
& \cup\{(z p+n, 1, z) \mid z \in \mathbf{I}\} \\
{[q, n, 1]=} & \{(1, y, y n+q) \mid y \in \mathbf{R}\} \\
& \cup\{(w, 1, w q+n) \mid w \in \mathbf{I}\}, \\
P= & \left(x_{1}, x_{2}, x_{3}\right) \sim\left(y_{1}, y_{2}, y_{3}\right)=Q \\
& \left.\Leftrightarrow x_{i}-y_{i} \in \mathbf{I}(i=1,2,3)\right), \forall P, Q \in \mathbf{P}, \\
g= & {\left[x_{1}, x_{2}, x_{3}\right] \sim\left[y_{1}, y_{2}, y_{3}\right]=h } \\
& \left.\Leftrightarrow x_{i}-y_{i} \in \mathbf{I}(i=1,2,3)\right), \forall g, h \in \mathbf{L} .
\end{aligned}
$$

Now it is time to give the following theorem from [2].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let $\mathbf{A}$ be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider

$$
\mathcal{A}:=\mathbf{A}(\varepsilon)=\mathbf{A}+\mathbf{A} \varepsilon
$$

with componentwise addition and multiplication as follows:

$$
\left(a_{1}+a_{2} \varepsilon\right)\left(b_{1}+b_{2} \varepsilon\right)=a_{1} b_{1}+\left(a_{1} b_{2}+a_{2} b_{1}\right) \varepsilon
$$

where $a_{i}, b_{i} \in \mathbf{A}$ for $i=1,2$. Then $\mathcal{A}$ is a local alternative ring with ideal $\mathbf{I}=\mathbf{A} \varepsilon$ of non-units. The set of formal inverses of the non-units of $\mathcal{A}$ is denoted as $\mathbf{I}^{-1}$. Calculations with the elements of $\mathbf{I}^{-1}$ are defined as follows [6]:

$$
\begin{aligned}
(a \varepsilon)^{-1}+t & :=(a \varepsilon)^{-1}:=t+(a \varepsilon)^{-1} \\
q(a \varepsilon)^{-1} & :=\left(a q^{-1} \varepsilon\right)^{-1} \\
(a \varepsilon)^{-1} q & :=\left(q^{-1} a \varepsilon\right)^{-1} \\
\left((a \varepsilon)^{-1}\right)^{-1} & :=a \varepsilon
\end{aligned}
$$

where $(a \varepsilon)^{-1} \in \mathbf{I}^{-1}, t \in \mathcal{A}, q \in \mathcal{A} \backslash \mathbf{I}$. (Other terms are not defined.). For more information about $\mathcal{A}$ and its relation to MK-planes, the reader is referred to the papers of Blunck [6], [7]. In [7], the centre $\mathbf{Z}(\mathcal{A})$ is defined to be the (commutative, associative) subring of $\mathcal{A}$ which is commuting and associating with all elements of $\mathcal{A}$. It is $\mathbf{Z}(\mathcal{A}):=\mathbf{Z}(\varepsilon)=\mathbf{Z}+\mathbf{Z} \varepsilon$, where $\mathbf{Z}=\{z \in \mathbf{A} \mid z a=a z, \forall a \in \mathbf{A}\}$ is the centre of $\mathbf{A}$. If $\mathbf{A}$ is not associative, then $\mathbf{A}$ is a Cayley division algebra over its centre $\mathbf{Z}$.

Throughout this paper we assume $\operatorname{char} \mathbf{A} \neq \mathbf{2}$ and we restrict ourselves to the MK-planes $\mathbf{M}(\mathcal{A})$.

Blunck [7] gives the following algebraic definition of the cross-ratio for the points on the line $g:=[1,0,0]$ in $\mathbf{M}(\mathcal{A})$.

$$
\begin{aligned}
& (A, B ; C, D):=(a, b ; c, d) \\
& =<\left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right)> \\
& (Z, B ; C, D):=\left(z^{-1}, b ; c, d\right) \\
& =<\left((1-d z)^{-1}(b-d)\right)\left((b-c)^{-1}(1-c z)\right)> \\
& (A, Z ; C, D):=\left(a, z^{-1} ; c, d\right) \\
& =<\left((a-d)^{-1}(1-d z)\right)\left((1-c z)^{-1}(a-c)\right)> \\
& (A, B ; Z, D):=\left(a, b ; z^{-1}, d\right) \\
& =<\left((a-d)^{-1}(b-d)\right)\left((1-z b)^{-1}(1-z a)\right)> \\
& (A, B ; C, Z):=\left(a, b ; c, z^{-1}\right) \\
& =<\left((1-z a)^{-1}(1-z b)\right)\left((b-c)^{-1}(a-c)\right)>
\end{aligned}
$$

where $A=(0, a, 1), B=(0, b, 1), C=(0, c, 1), D=$ $(0, d, 1), Z=(0,1, z)$ are pairwise non-neighbour points of $g$ and $\langle x\rangle=\left\{y^{-1} x y \mid \quad y \in \mathcal{A}\right\}$.

In [6, Theorem 2], it is shown that the transformations

$$
\begin{aligned}
t_{u}(x) & =x+u ; u \in \mathcal{A} \\
r_{u}(x) & =x u ; u \in \mathcal{A} \backslash \mathbf{I} \\
i(x) & =x^{-1} \\
l_{u}(x) & =u x=\left(i r_{u}^{-1} i\right)(x) ; u \in \mathcal{A} \backslash \mathbf{I}
\end{aligned}
$$

which are defined on the line $g$ preserve cross-ratios. In [5, Corollary (iii)], it is also shown that the group generated by these transformations, which is denoted by $\Lambda$, equals to the group of projectivities of a line in $\mathbf{M}(\mathcal{A})$. The elements preserving cross-ratio of the group $\Lambda$ defined on $g$ will act a very important role in the proof of Theorem 3.1.

We give the following result from [1, Theorem 8]. This result states a simple way for calculation of the cross-ratio of the points on any line in $\mathbf{M}(\mathcal{A})$.

Theorem 2.2: Let $\{O, U, V, E\}$ be the basis of $\mathbf{M}(\mathcal{A})$ where $O=(0,0,1), U=(1,0,0), V=(0,1,0), E=$ $(1,1,1)$ (see $[2$, Section 4]). Then, according to types of lines, the cross-ratio of the points on the line $l$ can be calculated as follows:

If $A, B, C, D$ and $Z$ are the pairwise non-neighbour points
(a) of the line $l=[m, 1, k]$, where $A=(a, a m+k, 1), B=$ $(b, b m+k, 1), C=(c, c m+k, 1), D=(d, d m+k, 1)$ are not near to the line $U V=[0,0,1]$ and $Z=$ $(1, m+z p, z)$ is near to $U V$,
(b) of the line $l=[1, n, p]$, where $A=(a n+p, a, 1), B=$ $(b n+p, b, 1), C=(c n+p, c, 1), D=(d n+p, d, 1)$ are not neighbour to $V$ and $Z=(n+z p, 1, z) \sim V$,
(c) of the line $l=[q, n, 1]$, where $A=(1, a, q+a n), B=$ $(1, b, q+b n), C=(1, c, q+c n), D=(1, d, q+d n)$ are not neighbour to $V$ and $Z=(z, 1, z q+n) \sim V$,
then

| $(A, B ; C, D)$ | $=(a, b ; c, d)$ |
| ---: | :--- |
| $(Z, B ; C, D)$ | $=\left(z^{-1}, b ; c, d\right)$ |
| $(A, Z ; C, D)$ | $=\left(a, z^{-1} ; c, d\right)$ |
| $(A, B ; Z, D)$ | $=\left(a, b ; z^{-1}, d\right)$ |
| $(A, B ; C, Z)$ | $=\left(a, b ; c, z^{-1}\right)$. |

We can give an important theorem, from [1, Theorem 9], about cross-ratio.

Theorem 2.3: In $\mathbf{M}(\mathcal{A})$, perspectivities preserve crossratios.

In the next section, we deal with some collineations preserving cross-ratio in $\mathbf{M}(\mathcal{A})$.

## III. Some Collineations Preserving Cross-Ratio.

In this section we would like to show that the following collineations we will introduce from [8] preserve cross-ratios. Now we start with giving the collineations, where $w, z, q, n \in$ A:

For any $u \notin \mathbf{I}$, the map $\mathrm{L}_{u}$ transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow(u x, u y u, 1) \\
(1, y, z \varepsilon) & \rightarrow\left(1, y u,\left(z u^{-1}\right) \varepsilon\right) \\
(w \varepsilon, 1, z \varepsilon) & \rightarrow\left(\left(u^{-1} w\right) \varepsilon, 1,\left(u^{-1} z u^{-1}\right) \varepsilon\right) \\
{[m, 1, k] } & \rightarrow[m u, 1, u k u] \\
{[1, n \varepsilon, p] } & \rightarrow\left[1,\left(u^{-1} n\right) \varepsilon, u p\right] \\
{[q \varepsilon, n \varepsilon, 1] } & \rightarrow\left[\left(q u^{-1}\right) \varepsilon,\left(u^{-1} n u^{-1}\right) \varepsilon, 1\right] .
\end{aligned}
$$

For any $u \notin \mathbf{I}$, the map $F_{u}$ transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow(u x u, u y, 1) \\
(1, y, z \varepsilon) & \rightarrow\left(1, u^{-1} y,\left(u^{-1} z u^{-1}\right) \varepsilon\right) \\
(w \varepsilon, 1, z \varepsilon) & \rightarrow\left((w u) \varepsilon, 1,\left(z u^{-1}\right) \varepsilon\right) \\
{[m, 1, k] } & \rightarrow\left[u^{-1} m, 1, u k\right] \\
{[1, n \varepsilon, p] } & \rightarrow[1,(n u) \varepsilon, u p u] \\
{[q \varepsilon, n \varepsilon, 1] } & \rightarrow\left[\left(u^{-1} q u^{-1}\right) \varepsilon,\left(n u^{-1}\right) \varepsilon, 1\right] .
\end{aligned}
$$

For any $\alpha, \beta \in \mathbf{Z}(\mathcal{A}), \alpha, \beta \notin \mathbf{I}$, the map $S_{\alpha, \beta}$ transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow(x \beta, y \alpha, 1) \\
(1, y, z \varepsilon) & \rightarrow\left(1, \beta^{-1} y \alpha,\left(\beta^{-1} z\right) \varepsilon\right) \\
(w \varepsilon, 1, z \varepsilon) & \rightarrow\left(\left(\alpha^{-1} w \beta\right) \varepsilon, 1,\left(\alpha^{-1} z\right) \varepsilon\right) \\
{[m, 1, k] } & \rightarrow\left[\beta^{-1} m \alpha, 1, k \alpha\right] \\
{[1, n \varepsilon, p] } & \rightarrow\left[1,\left(\alpha^{-1} n \beta\right) \varepsilon, p \beta\right] \\
{[q \varepsilon, n \varepsilon, 1] } & \rightarrow\left[\left(\beta^{-1} q\right) \varepsilon,\left(\alpha^{-1} n\right) \varepsilon, 1\right] .
\end{aligned}
$$

The map $\mathrm{I}_{2}$ transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow\left(y^{-1} x, y^{-1}, 1\right) \quad \text { if } \quad y \notin \mathbf{I} \\
(x, y, 1) & \rightarrow\left(1, x^{-1}, x^{-1} y\right) \quad \text { if } y \in \mathbf{I} \wedge x \notin \mathbf{I} \\
(x, y, 1) & \rightarrow(x, 1, y) \quad \text { if } y \in \mathbf{I} \wedge x \in \mathbf{I} \\
(1, y, z \varepsilon) & \rightarrow\left(y^{-1},\left(y^{-1} z\right) \varepsilon, 1\right) \quad \text { if } \quad y \notin \mathbf{I} \\
(1, y, z \varepsilon) & \rightarrow(1, z \varepsilon, y) \quad \text { if } \quad y \in \mathbf{I} \\
(w \varepsilon, 1, z \varepsilon) & \rightarrow(w \varepsilon, z \varepsilon, 1)
\end{aligned}
$$

$$
\begin{aligned}
{[m, 1, k] } & \rightarrow\left[-m k^{-1}, 1, k^{-1}\right] \quad \text { if } \quad k \notin \mathbf{I} \\
{[m, 1, k] } & \rightarrow\left[1,-k m^{-1}, m^{-1}\right] \quad \text { if } \quad k \in \mathbf{I} \wedge m \notin \mathbf{I} \\
{[m, 1, k] } & \rightarrow[m, k, 1] \quad \text { if } \quad k \in \mathbf{I} \wedge m \in \mathbf{I} \\
{[1, n \varepsilon, p] } & \rightarrow\left[p^{-1}, 1,-\left(n p^{-1}\right) \varepsilon\right] \quad \text { if } \quad p \notin \mathbf{I} \\
{[1, n \varepsilon, p] } & \rightarrow[1, p, n \varepsilon] \quad \text { if } \quad p \in \mathbf{I} \\
{[q \varepsilon, n \varepsilon, 1] } & \rightarrow[q \varepsilon, 1, n \varepsilon] .
\end{aligned}
$$

Now we are ready to give the main result of the paper.
Theorem 3.1: The collineations $\mathrm{L}_{u}, \mathrm{~F}_{u}, \mathrm{~S}_{\alpha, \beta}$ and $\mathrm{I}_{2}$ preserve cross-ratio.

Proof: Let $A, B, C, D$ and $Z$ be the points with the property given in the statement of Theorem 2.2. Then, it is obvious that

$$
\begin{align*}
(A, B ; C, D) & =(a, b ; c, d)  \tag{1}\\
(Z, B ; C, D) & =\left(z^{-1}, b ; c, d\right) \\
(A, Z ; C, D) & =\left(a, z^{-1} ; c, d\right) \\
(A, B ; Z, D) & =\left(a, b ; z^{-1}, d\right) \\
(A, B ; C, Z) & =\left(a, b ; c, z^{-1}\right),
\end{align*}
$$

where $z \in \mathbf{I}$. In this case we must find the effect of $\varphi$ to the points of any line where $\varphi$ is any one of collineations $\mathrm{L}_{u}, \mathrm{~F}_{u}$, $\mathrm{S}_{\alpha, \beta}$, and $\mathrm{I}_{2}$.
i) Let $\varphi=\mathrm{L}_{u}$. If $l=[m, 1, k]$, then

$$
\begin{aligned}
\varphi(X) & =\varphi(x, x m+k, 1)=(u x, u(x m+k) u, 1) \\
\varphi(Z) & =\varphi(1, m+z k, z)=\left(1,(m+z k) u, z u^{-1}\right)
\end{aligned}
$$

and $\varphi(l)=[m u, 1, u k u]$. From (a) of Theorem 2.2, we obtain

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & =(u a, u b ; u c, u d) \\
& =\sigma \quad(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(u z^{-1}, u b ; u c, u d\right) \\
& ={ }^{\sigma}\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=l_{u^{-1}} \in \Lambda$.
If $l=[1, n, p]$, then

$$
\begin{aligned}
\varphi(X) & =\varphi(x n+p, x, 1)=(u(x n+p), u x u, 1) \\
\varphi(Z) & =\varphi(n+z p, 1, z)=\left(u^{-1}(n+z p), 1, u^{-1} z u^{-1}\right)
\end{aligned}
$$

and $\varphi(l)=\left[1, u^{-1} n, u p\right]$. From (b) of Theorem 2.2, we have

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & = \\
& =(u a u, u b u ; u c u, u d u) \\
& =\sigma, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(u z^{-1} u, u b u ; u c u, u d u\right) \\
& ={ }^{\sigma}\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

# International Journal of Engineering, Mathematical and Physical Sciences <br> ISSN: 2517-9934 <br> Vol:3, No:11, 2009 

where $\sigma=l_{u^{-1}} \circ r_{u^{-1}} \in \Lambda$.
If $l=[q, n, 1]$, then

$$
\begin{aligned}
\varphi(X) & =\varphi(1, x, q+x n)=\left(1, x u,(q+x n) u^{-1}\right) \\
\varphi(Z) & =\varphi(z, 1, z q+n)=\left(u^{-1} z, 1, u^{-1}(z q+n) u^{-1}\right)
\end{aligned}
$$

and $\varphi(l)=\left[q u^{-1}, u^{-1} n u^{-1}, 1\right]$. From (c) of Theorem 2.2, we obtain

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & = \\
& =(a u, b u ; c u, d u) \\
& ={ }^{\sigma}(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(z^{-1} u, b u ; c u, d u\right) \\
& =^{\sigma}\left(z^{-1}, b ; c, d\right)
\end{aligned}
$$

where $\sigma=r_{u^{-1}} \in \Lambda$.
ii) Let $\varphi=\mathrm{F}_{u}$. If $l=[m, 1, k]$, then

$$
\begin{aligned}
\varphi(X) & =\varphi(x, x m+k, 1)=(u x u, u(x m+k), 1) \\
\varphi(Z) & =\varphi(1, m+z k, z)=\left(1, u^{-1}(m+z k), u^{-1} z u^{-1}\right)
\end{aligned}
$$

and $\varphi(l)=\left[u^{-1} m, 1, u k\right]$. From (a) of Theorem 2.2, we have $(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D))=(u a u, u b u ; u c u, u d u)$

$$
={ }^{\sigma} \quad(a, b ; c, d)
$$

$(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) \quad=\quad\left(u z^{-1} u, u b u ; u c u, u d u\right)$

$$
={ }^{\sigma} \quad\left(z^{-1}, b ; c, d\right),
$$

where $\sigma=l_{u^{-1}} \circ r_{u^{-1}} \in \Lambda$.
If $l=[1, n, p]$, then

$$
\begin{aligned}
\varphi(X) & =\varphi(x n+p, x, 1)=(u(x n+p) u, u x, 1) \\
\varphi(Z) & =\varphi(n+z p, 1, z)=\left((n+z p) u, 1, z u^{-1}\right)
\end{aligned}
$$

and $\varphi(l)=[1, n u, u p u]$. From (b) of Theorem 2.2, we obtain

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & =\quad(u a, u b ; u c, u d) \\
& ={ }^{\sigma} \quad(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(u z^{-1}, u b ; u c, u d\right) \\
& ={ }^{\sigma} \quad\left(z^{-1}, b ; c, d\right)
\end{aligned}
$$

where $\sigma=l_{u^{-1}} \in \Lambda$.
If $l=[q, n, 1]$, then

$$
\begin{aligned}
\varphi(X) & =\varphi(1, x, q+x n)=\left(1, u^{-1} x, u^{-1}(q+x n) u^{-1}\right) \\
\varphi(Z) & =\varphi(z, 1, z q+n)=\left(z u, 1,(z q+n) u^{-1}\right)
\end{aligned}
$$

and $\varphi(l)=\left[u^{-1} q u^{-1}, n u^{-1}, 1\right]$. From (c) of Theorem 2.2, we have

$$
\begin{aligned}
& (\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) \\
& =\left(u^{-1} a, u^{-1} b ; u^{-1} c, u^{-1} d\right)=^{\sigma}(a, b ; c, d) \\
& (\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) \\
& =\left(u^{-1} z^{-1}, u^{-1} b ; u^{-1} c, u^{-1} d\right)=^{\sigma}\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=l_{u} \in \Lambda$.
iii) Let $\varphi=\mathbf{S}_{\alpha, \beta}$. If $l=[m, 1, k]$, , then

$$
\begin{aligned}
\varphi(X) & =\varphi(x, x m+k, 1)=(x \beta,(x m+k) \alpha, 1) \\
\varphi(Z) & =\varphi(1, m+z k, z)=\left(1, \beta^{-1}(m+z k) \alpha, \beta^{-1} z\right)
\end{aligned}
$$

and $\varphi(l)=\left[\beta^{-1} m \alpha, 1, k \alpha\right]$. From (a) of Theorem 2.2, we obtain

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & = \\
& =(a \beta, b \beta ; c \beta, d \beta) \\
& =\quad(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & = \\
& =\left(z^{-1} \beta, b \beta ; c \beta, d \beta\right) \\
& =^{\sigma} \quad\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=r_{\beta^{-1}} \in \Lambda$.
If $l=[1, n, p]$, then

$$
\begin{aligned}
\varphi(X) & =\varphi(x n+p, x, 1)=((x n+p) \beta, x \alpha, 1) \\
\varphi(Z) & =\varphi(n+z p, 1, z)=\left(\alpha^{-1}(n+z p) \beta, 1, \alpha^{-1} z\right)
\end{aligned}
$$

and $\varphi(l)=\left[1, \alpha^{-1} n \beta, p \beta\right]$. From (b) of Theorem 2.2, we have

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & =(a \alpha, b \alpha ; c \alpha, d \alpha) \\
& ={ }^{\sigma}(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(z^{-1} \alpha, b \alpha ; c \alpha, d \alpha\right) \\
& =^{\sigma} \quad\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=r_{\alpha^{-1}} \in \Lambda$.
If $l=[q, n, 1]$, then

$$
\begin{aligned}
\varphi(X) & =\varphi(1, x, q+x n)=\left(1, \beta^{-1} x \alpha, \beta^{-1}(q+x n)\right) \\
\varphi(Z) & =\varphi(z, 1, z q+n)=\left(\alpha^{-1} z \beta, 1, \alpha^{-1}(z q+n)\right)
\end{aligned}
$$

and $\varphi(l)=\left[\beta^{-1} q, \alpha^{-1} n, 1\right]$. From (c) of Theorem 2.2, we obtain

$$
\begin{aligned}
& (\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) \\
& =\left(\beta^{-1} a \alpha, \beta^{-1} b \alpha ; \beta^{-1} c \alpha, \beta^{-1} d \alpha\right)=^{\sigma}(a, b ; c, d) \\
& (\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) \\
& =\left(\beta^{-1} z^{-1} \alpha, \beta^{-1} b \alpha ; \beta^{-1} c \alpha, \beta^{-1} d \alpha\right)=^{\sigma}\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=l_{\beta} \circ r_{\alpha^{-1}} \in \Lambda$.
iv) Let $\varphi=\mathrm{I}_{2}$. If $l=[m, 1, k]$, then

$$
\begin{aligned}
\varphi(X)= & \varphi(x, x m+k, 1) \\
= & \left((x m+k)^{-1} x,(x m+k)^{-1}, 1\right), \\
& \text { where } x m+k \notin \mathbf{I} \\
\varphi(X)= & \varphi(x, x m+k, 1) \\
= & \left(1, x^{-1}, x^{-1}(x m+k)\right), \\
& \text { where } x m+k \in \mathbf{I} \text { and } x \notin \mathbf{I} \\
\varphi(X)= & \varphi(x, x m+k, 1) \\
= & (x, 1, x m+k), \text { where } x m+k \in \mathbf{I} \text { and } x \in \mathbf{I} \\
\varphi(Z)= & \varphi(1, m+z k, z) \\
= & \left((m+z k)^{-1},(m+z k)^{-1} z, 1\right), \\
& \text { where } m+z k \notin \mathbf{I} \\
\varphi(Z)= & \varphi(1, m+z k, z) \\
= & (1, z, m+z k), \text { where } m+z k \in \mathbf{I}
\end{aligned}
$$

ISSN: 2517-9934
Vol:3, No:11, 2009
and

$$
\begin{aligned}
\varphi(l) & =\left[-m k^{-1}, 1, k^{-1}\right], \text { where } k \notin \mathbf{I} \\
\varphi(l) & =\left[1,-k m^{-1}, m^{-1}\right], \text { where } k \in \mathbf{I} \text { and } m \notin \mathbf{I} \\
\varphi(l) & =[m, k, 1], \text { where } k \in \mathbf{I} \text { and } m \in \mathbf{I} .
\end{aligned}
$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $\left[-m k^{-1}, 1, k^{-1}\right]$ is as follows:

$$
\begin{aligned}
& (\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) \\
& =\left((a m+k)^{-1} a,(b m+k)^{-1} b ;\right. \\
& \left.(c m+k)^{-1} c,(d m+k)^{-1} d\right) \\
& =^{\sigma}(a, b ; c, d) \\
& (\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) \\
& =\left((m+z k)^{-1},(b m+k)^{-1} b ;\right. \\
& \left.(c m+k)^{-1} c,(d m+k)^{-1} d\right) \\
& ={ }^{\sigma}\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=i \circ r_{k^{-1}} \circ t_{-m} \circ i \in \Lambda$. From (b) of Theorem 2.2, the cross-ratio of the points of $\left[1,-k m^{-1}, m^{-1}\right]$ is as follows:

$$
\begin{aligned}
& (\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) \\
& =\left((a m+k)^{-1},(b m+k)^{-1} ;\right. \\
& \left.(c m+k)^{-1},(d m+k)^{-1}\right) \\
& ={ }^{\sigma}(a, b ; c, d) \\
& (\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) \\
& =\left((m+z k)^{-1} z,(b m+k)^{-1} ;\right. \\
& \left.(c m+k)^{-1},(d m+k)^{-1}\right) \\
& ={ }^{\sigma}\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=r_{m^{-1}} \circ t_{-k} \circ i \in \Lambda$. From (c) of Theorem 2.2, the cross-ratio of the points of $[m, k, 1]$ is as follows:

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(a^{-1}, b^{-1} ; c^{-1}, d^{-1}\right) \\
& ={ }^{\sigma}(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(z, b^{-1} ; c^{-1}, d^{-1}\right) \\
& ={ }^{\sigma}\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=i \in \Lambda$.
If $l=[1, n, p]$, then

$$
\begin{aligned}
\varphi(X)= & \varphi(x n+p, x, 1) \\
= & \left(x^{-1}(x n+p), x^{-1}, 1\right), \text { where } x \notin \mathbf{I} \\
\varphi(X)= & \varphi(x n+p, x, 1) \\
= & \left(1,(x n+p)^{-1},(x n+p)^{-1} x\right), \\
& \text { where } x \in \mathbf{I} \text { and } x n+p \notin \mathbf{I} \\
\varphi(X)= & \varphi(x n+p, x, 1) \\
= & (x n+p, 1, x), \text { where } x \in \mathbf{I} \text { and } x n+p \in \mathbf{I} \\
\varphi(Z)= & \varphi(n+z p, 1, z)=(n+z p, z, 1)
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi(l) & =\left[p^{-1}, 1,-n p^{-1}\right], \text { where } p \notin \mathbf{I} \\
\varphi(l) & =[1, p, n], \text { where } p \in \mathbf{I} .
\end{aligned}
$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $\left[p^{-1}, 1,-n p^{-1}\right]$ is as follows:

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D))= & \left(a^{-1}(a n+p), b^{-1}(b n+p) ;\right. \\
& \left.c^{-1}(c n+p), d^{-1}(d n+p)\right) \\
& ={ }^{\sigma}(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D))= & \left(n+z p, b^{-1}(b n+p) ;\right. \\
& \left.c^{-1}(c n+p), d^{-1}(d n+p)\right) \\
& ={ }^{\sigma}\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=i \circ r_{p^{-1}} \circ t_{-n} \in \Lambda$. From (b) of Theorem 2.2, the cross-ratio of the points of $[1, p, n]$ is as follows:

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(a^{-1}, b^{-1} ; c^{-1}, d^{-1}\right) \\
& ={ }^{\sigma}(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(z, b^{-1} ; c^{-1}, d^{-1}\right) \\
& ={ }^{\sigma}\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=i \in \Lambda$.
If $l=[q, n, 1]$, then

$$
\begin{aligned}
\varphi(X) & =\varphi(1, x, q+x n) \\
& =\left(x^{-1}, x^{-1}(q+x n), 1\right), \text { where } x \notin \mathbf{I} \\
\varphi(X) & =\varphi(1, x, q+x n) \\
& =(1, q+x n, x), \text { where } x \in \mathbf{I} \\
\varphi(Z) & =\varphi(z, 1, z q+n)=(z, z q+n, 1)
\end{aligned}
$$

and $\varphi(l)=[q, 1, n]$. In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $[q, 1, n]$ is as follows:

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & = \\
& \left.=a^{-1}, b^{-1} ; c^{-1}, d^{-1}\right) \\
& =(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(z, b^{-1} ; c^{-1}, d^{-1}\right) \\
& ={ }^{\sigma} \quad\left(z^{-1}, b ; c, d\right),
\end{aligned}
$$

where $\sigma=i \in \Lambda$.
Consequently, by considering other all cases we get

$$
\begin{aligned}
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) & =(a, b ; c, d) \\
(\varphi(Z), \varphi(B) ; \varphi(C), \varphi(D)) & =\left(z^{-1}, b ; c, d\right) \\
(\varphi(A), \varphi(Z) ; \varphi(C), \varphi(D)) & =\left(a, z^{-1} ; c, d\right) \\
(\varphi(A), \varphi(B) ; \varphi(Z), \varphi(D)) & =\left(a, b ; z^{-1}, d\right) \\
(\varphi(A), \varphi(B) ; \varphi(C), \varphi(Z)) & =\left(a, b ; c, z^{-1}\right)
\end{aligned}
$$

for every collineation $\varphi$. Combining the last result and the result of (1), the proof is completed.

Remark 3.2: In the present paper we show that the collineations $\mathrm{L}_{u}, \mathrm{~F}_{u}, \mathrm{~S}_{\alpha, \beta}$, and $\mathrm{I}_{2}$, given in [8], preserve crossratio. A paper related to the result that the other collineations of [8] ( $\mathrm{T}_{u, v}, \mathrm{I}_{1}, \mathrm{~F}$ and $\left.\mathrm{G}_{u}\right)$ preserve cross-ratio, is under review.

## References

[1] A. Akpinar, B. Celik and S. Ciftci, Cross-ratios and 6 -figures in some Moufang-Klingenberg planes. Bulletin of the Belgian Math. Soc.-Simon Stevin 15(2008), 49-64.
[2] C.A. Baker, N.D. Lane and J.W. Lorimer. A coordinatization for Moufang-Klingenberg planes. Simon Stevin 65(1991), 3-22.
[3] J.L. Bell. The art of the intelligible: An elementary survey of mathematics in its conceptual development. Kluwer Acad. Publishers, The Netherland, 2001.
[4] A. Blunck. Cross-ratios in Moufang planes. J. Geometry 40(1991), 2025.
[5] A. Blunck. Projectivities in Moufang-Klingenberg planes. Geom. Dedicata 40(1991), 341-359.
[6] A. Blunck. Cross-ratios over local alternative rings. Res. Math. 19 (1991), 246-256.
[7] A. Blunck. Cross-ratios in Moufang-Klingenberg planes. Geom. Dedicata 43(1992), 93-107.
[8] B. Celik, A. Akpinar and S. Ciftci. 4-Transitivity and 6-figures in some Moufang-Klingenberg planes. Monatshefte für Mathematik 152(2007), 283-294.
[9] S. Ciftci and B. Celik. On the cross-ratios of points and lines in Moufang planes. J. Geometry 71(2001), 34-41.
[10] J.C. Ferrar. Cross-ratios in projective and affine planes. in Plaumann, P and Strambach, K., Geometry - von Staudt's Point of View (Proceedings Bad Windsheim, 1980), Reidel, Dordrecht, (1981) 101-125.
[11] D.R. Hughes and F.C. Piper. Projective planes. Springer, New York, 1973.
[12] G. Pickert. Projektive Ebenen. Springer, Berlin, 1955.
[13] R.D. Schafer. An introduction to nonassociative algebras. Dover Publications, New York, 1995.
14] F.W. Stevenson. Projective planes. W.H. Freeman Co., San Francisco, 1972.


[^0]:    Süleyman Ciftci, Atilla Akpinar and Basri Celik are with the Uludag University, Department of Mathematics, Faculty of Arts and Science, Bursa-TURKEY, email: sciftci@uludag.edu.tr, aakpinar@uludag.edu.tr, basri@uludag.edu.tr

