

Solution of nonlinear second-order pantograph equations via differential transformation method

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Abstract—In this work, we successfully extended one-dimensional differential transform method (DTM), by presenting and proving some theorems, to solving nonlinear high-order multi-pantograph equations. This technique provides a sequence of functions which converges to the exact solution of the problem. Some examples are given to demonstrate the validity and applicability of the present method and a comparison is made with existing results.

Keywords—Nonlinear multi-pantograph equation; Delay differential equation; Differential transformation method; Proportional delay conditions; Closed form solution.

I. INTRODUCTION

THE nonlinear multi-pantograph equation reads

$$f(t, u(q_0t), u'(q_1t), u''(q_2t)) = 0, \quad t \in [t_0, T], \quad (1)$$

where $q_j \in (0, 1)$, for $j = 0, 1, 2$.

The pantograph type equations have been studied extensively owing to the numerous applications in which these equations arise. The name pantograph originated from the work of Ockendon and Tayler [1] on the collection of current by the pantograph head of an electric locomotive, this equations are appeared in modeling of various problems in engineering and sciences such as biology, economy, control and electrodynamics. For some applications of this equation we refer the interested reader to [2], [3], [4], [5], [6], [7], [8]. The linear form of Eq. (1) was studied by many authors numerically and analytically. Zhan-Hua Yu applied the Variational iteration method to solve the multi-pantograph delay equation [4]; Sezer et al. [5], [6] obtained the approximate solution of multi-pantograph equation using Taylor polynomials and A. Saadatmandi et al. [9] applied the Variational iteration method to solve the generalized pantograph equation.

In all previous work, the linear and variable coefficient form of pantograph equation was studied, In this work we consider the following two problems

Problem 1. Nonlinear pantograph equations with initial conditions:

$$\begin{aligned} f(t, u(q_0t), u'(q_1t), u''(q_2t)) &= 0, \quad t \in [t_0, T], \\ u(t_0) &= a, u'(t_0) = b, \end{aligned} \quad (2)$$

Problem 2. Nonlinear pantograph equations with boundary conditions:

$$\begin{aligned} f(t, u(q_0t), u'(q_1t), u''(q_2t)) &= 0, \quad t \in [t_0, T], \\ u(t_0) &= a, u(T) = b, \end{aligned} \quad (3)$$

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The method that is developed in this work depends on DTM, introduced by Zhou [10] in a study about electrical circuits. It is a semi-numerical-analytical technique that formulates Taylor series in a totally different manner. With this technique, the given differential equation and related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful for obtaining exact and approximate solutions of linear and nonlinear differential equations. There is no need for linearization or perturbations, large computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. It is possible to solve system of differential equations [11], [12], [13], differential-algebraic equations [14], difference equations [15], differential difference equations [16], partial differential equations [17], [18], [19], fractional differential equations [20], [21], pantograph equations [22], one-dimensional Volterra integral and integro-differential equations [23], [24], [25] by using this method.

The layout of the paper is as follows: In Section II, the differential transformation method (DTM) will be introduced. In Section III, some numerical results are given to clarify the method and a comparison is made with the existing results. Section IV is the brief conclusion of this paper. Finally some references are listed in the end. Note that we have computed the numerical results by Maple programming.

II. BASIC DEFINITIONS

The transformation of the k -th derivative of a function in one variable is as follows:

Definition.1. If $u(t) \in R$ can be expressed by Taylor's series about fixed point t_0 , then $u(t)$ can be represented as

$$u(t) = \sum_{k=0}^{\infty} \frac{u^{(k)}(t_0)}{k!} (t - t_0)^k. \quad (4)$$

If $u_n(t)$ is be the n -partial sums of a Taylor's series (4), then

$$u_n(t) = \sum_{k=0}^n \frac{u^{(k)}(t_0)}{k!} (t - t_0)^k + R_n(t). \quad (5)$$

where $u_n(t)$ is called the n -th Taylor polynomial for $u(t)$ about t_0 and $R_n(t)$ is remainder term.

If $U(k)$ is defined as

$$U(k) = \frac{1}{k!} \left[\frac{d^k u(t)}{dt^k} \right]_{t=t_0}, \quad (6)$$

where $k = 0, 1, \dots, \infty$, then Eq (4) reduce to

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^k. \quad (7)$$

and the n -partial sums of a Taylor's series (5) reduce to

$$u_n(t) = \sum_{k=0}^n U(k)(t-t_0)^k + R_n(t). \quad (8)$$

The $U(k)$ defined in Eq (6), is called the differential transform of function $u(t)$.

For simplicity assume that $t_0 = 0$, then solution (7) reduce to

$$u(t) = \sum_{k=0}^n U(k)t^k + R_n(t). \quad (9)$$

From the above definitions, it can be found that the concept of the one-dimensional differential transform is derived from the Taylor series expansion. The following theorems that can be deduced from Eqs. (6) and (7) are given below;

Theorem.1. Assume that $W(k)$, $U(k)$ and $V(k)$, are the differential transforms of the functions $w(t)$, $u(t)$ and $v(t)$, respectively, then

- (a) If $w(t) = u(t) \pm v(t)$, then $W(k) = U(k) \pm V(k)$.
- (b) If $w(t) = \lambda u(t)$, then $W(k) = \lambda U(k)$.
- (c) If $w(t) = \frac{d^m u(t)}{dt^m}$, then $W(k) = \frac{(k+m)!}{k!} U(k+m)$.
- (d) If $w(t) = u(t)v(t)$, then $W(k) = \sum_{\ell=0}^k U(\ell)V(k-\ell)$.
- (e) If $w(x) = x^m$ then

$$W(k) = \delta(k-m) = \begin{cases} 1 & k=m, \\ 0 & \text{otherwise} \end{cases}$$

Proof: See ([14], and their references).

Now we state the fundamental theorem of this paper.

Theorem.2. If $w(t) = u(qt)$, then $W(k) = q^k U(k)$.

Proof: From the definition II, we get

$$\frac{d^k}{dt^k} w(t) = \frac{d^k}{dt^k} [u(qt)] = q^k \frac{d^k}{d\tilde{t}^k} u(\tilde{t}),$$

where $\tilde{t} = qt$, therefore

$$\left[\frac{d^k}{dt^k} w(t) \right]_{t=t_0} = q^k \left[\frac{d^k}{d\tilde{t}^k} u(\tilde{t}) \right]_{\tilde{t}=t_0} = q^k k! U(k),$$

hence by (6)

$$W(k) = \frac{1}{k!} \left[\frac{d^k w(t)}{dt^k} \right]_{t=t_0} = q^k U(k).$$

Theorem.3. If $w(t) = u_1(q_1 t) u_2(q_2 t)$, then

$$W(k) = \sum_{\ell=0}^k q_1^\ell q_2^{k-\ell} U_1(\ell) U_2(k-\ell).$$

Proof: By using Leibnitz formula, we get

$$\begin{aligned} \frac{d^k}{dt^k} w(t) &= \frac{d^k}{dt^k} [u_1(q_1 t) u_2(q_2 t)] \\ &= \sum_{\ell=0}^k \binom{k}{\ell} \frac{d^\ell}{dt^\ell} [u_1(q_1 t)] \frac{d^{k-\ell}}{dt^{k-\ell}} [u_2(q_2 t)] \\ &= \sum_{\ell=0}^k \binom{k}{\ell} q_1^\ell \frac{d^\ell}{d\tilde{t}^\ell} u_1(\tilde{t}) q_2^{k-\ell} \frac{d^{k-\ell}}{d\hat{t}^{k-\ell}} u_2(\hat{t}), \end{aligned}$$

where $\tilde{t} = q_1 t$, and $\hat{t} = q_2 t$, therefore

$$\begin{aligned} \left[\frac{d^k}{dt^k} w(t) \right]_{t=t_0} &= \sum_{\ell=0}^k \binom{k}{\ell} q_1^\ell q_2^{k-\ell} \ell! (k-\ell)! U_1(\ell) U_2(k-\ell) \\ &= \sum_{\ell=0}^k k! q_1^\ell q_2^{k-\ell} U_1(\ell) U_2(k-\ell), \end{aligned}$$

then from (4), we get $W(k) = \sum_{\ell=0}^k q_1^\ell q_2^{k-\ell} U_1(\ell) U_2(k-\ell)$. ■

Theorem.4. If $w(t) = \frac{d^n}{dt^n} u_1(q_1 t) \frac{d^m}{dt^m} u_2(q_2 t)$, then

$$\begin{aligned} W(k) &= \sum_{\ell=0}^k q_1^{\ell+n} q_2^{k-\ell+m} \frac{(\ell+n)!(k-\ell+m)!}{\ell!(k-\ell)!} \times \\ &\quad U_1(\ell+n) U_2(k-\ell+m). \end{aligned}$$

Proof: Analogously from to previous Theorems we get

$$\begin{aligned} \frac{d^k}{dt^k} w(t) &= \frac{d^k}{dt^k} \left[\frac{d^n}{dt^n} u_1(q_1 t) \frac{d^m}{dt^m} u_2(q_2 t) \right] \\ &= \sum_{\ell=0}^k \binom{k}{\ell} \frac{d^\ell}{dt^\ell} \left[\frac{d^n}{dt^n} u_1(q_1 t) \right] \frac{d^{k-\ell}}{dt^{k-\ell}} \left[\frac{d^m}{dt^m} u_2(q_2 t) \right] \\ &= \sum_{\ell=0}^k \binom{k}{\ell} q_1^{\ell+n} \frac{d^{\ell+n}}{d\tilde{t}^{\ell+n}} u_1(\tilde{t}) q_2^{k-\ell+m} \frac{d^{k-\ell+m}}{d\hat{t}^{k-\ell+m}} u_2(\hat{t}), \end{aligned}$$

where $\tilde{t} = q_1 t$, and $\hat{t} = q_2 t$, therefore

$$\begin{aligned} \left[\frac{d^k}{dt^k} w(t) \right]_{t=t_0} &= \sum_{\ell=0}^k \binom{k}{\ell} [q_1^{\ell+n} (\ell+n)! U_1(\ell+n)] \times \\ &\quad [q_2^{k-\ell+m} (k-\ell+m)! U_2(k-\ell+m)] \\ &= \sum_{\ell=0}^k \frac{k! (\ell+n)! (k-\ell+m)!}{\ell! (k-\ell)!} q_1^{\ell+n} q_2^{k-\ell+m} \times \\ &\quad U_1(\ell+n) U_2(k-\ell+m), \end{aligned}$$

then from (6), we get

$$\begin{aligned} W(k) &= \sum_{\ell=0}^k q_1^{\ell+n} q_2^{k-\ell+m} \frac{(\ell+n)! (k-\ell+m)!}{\ell! (k-\ell)!} \times \\ &\quad U_1(\ell+n) U_2(k-\ell+m). \end{aligned}$$

III. APPLICATIONS AND NUMERICAL EXAMPLES

This section is devoted to computational results. We applied the method presented in this paper and solved four prototype examples. Example.1 and Example.2 are initial value pantograph equation and Examples.3 and 4 are

boundary value pantograph equations. In these examples, we first obtain a recurrence systems for the differential transform of nonlinear pantograph equation and solve it by programming in MAPLE environment. These examples are chosen such that there exist exact solutions for them.

Example.1. Consider the following nonlinear second-order pantograph equation

$$u''(\frac{t}{2}) + \left[u'(\frac{t}{2})\right]^2 - \frac{1}{4}u(\frac{t}{2}) - \frac{1}{4}u(t) = 0, \quad t \in [0, 1], \quad (10)$$

subject to initial conditions $u(0) = u'(0) = 1$.

By applying differential transformation method on nonlinear pantograph equation (10), for $k = 0, 1, 2, \dots, N$, we get

$$\begin{aligned} & \frac{(k+2)!}{k!} \left(\frac{1}{2}\right)^{k+2} U(k+2) \\ & + \sum_{\ell=0}^k \left(\frac{1}{2}\right)^{k+2} (\ell+1)(k-\ell+1) U(\ell+1) U(k-\ell+1) \\ & - \frac{1}{4} \left(\frac{1}{2}\right)^k U(k) - \frac{1}{4} U(k) = 0, \end{aligned} \quad (11)$$

and from initial conditions $u(0) = u'(0) = 1$, we get $U(0) = U(1) = 1$, respectively, where $U(k)$ is the differential transform of $u(t)$.

By taking $N = 5$, the following system is obtained:

$$\begin{aligned} & \frac{1}{2} U(2) - \frac{1}{4} = 0, \\ & \frac{3}{4} U(3) + \frac{1}{2} U(2) - \frac{3}{8} = 0, \\ & \frac{3}{4} U(4) + \frac{3}{8} U(3) + \frac{1}{4} U(2)^2 - \frac{5}{16} U(2) = 0, \\ & \frac{5}{8} U(5) + \frac{1}{4} U(4) + \frac{3}{8} U(2) U(3) - \frac{9}{32} U(3) = 0 \\ & \frac{15}{32} U(6) + \frac{5}{32} U(5) + \frac{1}{4} U(2) U(4) + \frac{9}{64} U(3)^2 - \frac{17}{64} U(4) = 0, \\ & \frac{21}{64} U(7) + \frac{3}{32} U(6) + \frac{5}{32} U(2) U(5) + \frac{3}{16} U(3) U(4) \\ & - \frac{33}{128} U(5) = 0, \end{aligned} \quad (12)$$

Solving the above system and using the inverse transformation rule (9), we get the following series solution

$$u(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 + \frac{1}{5040}t^7,$$

The closed form of above solution, when $N \rightarrow \infty$ is $u(t) = e^t$, which is exactly the same as the exact solution.

Example.2. Consider the following nonlinear second-order pantograph equation

$$u''(\frac{t}{2}) + \frac{1}{2} \left[u'(\frac{t}{2})\right]^2 - \frac{7}{8}u(t) = \frac{1}{4}, \quad t \in [0, 1], \quad (13)$$

subject to initial conditions $u(0) = 2$, and $u'(0) = 0$.

By applying differential transformation method on nonlinear pantograph equation (13), for $k = 0, 1, 2, \dots, N$, we get

$$\begin{aligned} & \frac{(k+2)!}{k!} \left(\frac{1}{2}\right)^{k+2} U(k+2) \\ & + \frac{1}{2} \sum_{\ell=0}^k \left(\frac{1}{2}\right)^{k+2} (\ell+1)(k-\ell+1) U(\ell+1) U(k-\ell+1) \\ & - \frac{7}{8} U(k) = \frac{1}{4} \delta(k), \end{aligned} \quad (14)$$

and from initial conditions $u(0) = 2$, and $u'(0) = 0$, we get $U(0) = 2$, and $U(1) = 0$, respectively, where $U(k)$ is the differential transform of $u(t)$.

By taking $N = 5$, the following system is obtained:

$$\begin{aligned} & 2U(2) - \frac{7}{4} = \frac{1}{4}, \\ & 6U(3) = 0, \\ & 12U(4) - \frac{1}{8}U(2)^2 - \frac{7}{8}U(2) = 0, \\ & 20U(5) - \frac{3}{16}U(3)U(2) - \frac{7}{8}U(3) = 0, \\ & 30U(6) - \frac{1}{8}U(4)U(2) - \frac{9}{128}U(3)^2 - \frac{7}{8}U(4) = 0, \\ & 42U(7) - \frac{5}{64}U(5)U(2) - \frac{3}{32}U(4)U(3) - \frac{7}{8}U(5) = 0, \end{aligned} \quad (15)$$

Solving the above system and using the inverse transformation rule (9), we get the following series solution

$$u(t) = 2 + t^2 + \frac{1}{12}t^4 + \frac{1}{360}t^6,$$

For $N \rightarrow \infty$, the closed form of above solution is $u(t) = e^t + e^{-t}$, which is exactly the same as the exact solution.

Example.3. Consider the following nonlinear second-order pantograph equation

$$u''(t) - \frac{8}{3}u'(\frac{t}{2})u(t) - 8t^2u(\frac{t}{2}) = -\frac{4}{3} - \frac{22}{3}t - 7t^2 - \frac{5}{3}t^3, \quad (16)$$

for $t \in [0, 1]$ and subject to boundary conditions $u(0) = 1$, and $u(1) = 1$.

By applying differential transformation method on nonlinear pantograph equation (16), for $k = 0, 1, 2, \dots, N$, we get

$$\begin{aligned} & \frac{(k+2)!}{k!} U(k+2) - \frac{8}{3} \sum_{\ell=0}^k \left(\frac{1}{2}\right)^{\ell+1} (\ell+1) U(\ell+1) U(k-\ell) \\ & - 8 \sum_{\ell=0}^k \left(\frac{1}{2}\right)^{k-\ell} \delta(\ell-2) U(k-\ell) = -\frac{4}{3} \delta(k) - \frac{22}{3} \delta(k-1) \\ & - 7 \delta(k-2) - \frac{5}{3} \delta(k-3), \end{aligned} \quad (17)$$

and the differential transform version of boundary conditions $u(0) = 1$, and $u(1) = 0$, are respectively,

$$U(0) = 1, \quad \sum_{k=0}^N U(k) = 1, \quad (18)$$

where $U(k)$ is the differential transform of $u(t)$.

By taking $N = 5$, the following system is obtained from differential transform version (17):

$$\begin{aligned} & 2U(2) - \frac{4}{3}U(1) + \frac{4}{3} = 0, \\ & 6U(3) - \frac{4}{3}U(1)^2 - \frac{4}{3}U(2) + \frac{22}{3} = 0, \\ & 12U(4) - \frac{8}{3}U(1)U(2) - U(3) - 1 = 0, \\ & 20U(5) - \frac{7}{3}U(1)U(3) - \frac{4}{3}U(2)^2 - \frac{2}{3}U(4) - 4U(1) + \frac{5}{3} = 0, \\ & 30U(6) - 2U(1)U(4) - \frac{7}{3}U(2)U(3) - \frac{5}{12}U(5) - 2U(2) = 0, \\ & 42U(7) - \frac{7}{4}U(1)U(5) - 2U(2)U(4) - U(3)^2 - \frac{1}{4}U(6) \\ & - U(3) = 0, \end{aligned} \quad (19)$$

and from differential transform version of boundary conditions (18), we get

$$\begin{aligned} & U(0) = 1, \\ & U(0) + U(1) + U(2) + U(3) + U(4) + U(5) = 1, \end{aligned} \quad (20)$$

Solving the system (19) and differential transform version of boundary condition (20), simultaneously, and using the inverse transformation rule (5), we get the following series solution $u(t) = 1 + t - t^3$. Note that for $N > 5$ we evaluate

the same solution, which is the exact solution of Eq.(16).

Example.4. As an application of present method, consider the following nonlinear integro-differential equation with proportional delay in kernel

$$u'(t) + \left(\frac{1}{2}t - 2\right)u(t) - 2 \int_0^t u\left(\frac{s}{2}\right)^2 ds = 1, \quad (21)$$

for $t \geq 0$ and subject to initial condition $u(0) = 0$.

By substituted $t = 0$, in integro-differential equation (21), we get $u'(0) = 1$, then the initial condition will be

$$\begin{aligned} u(0) &= 0, \\ u'(0) &= 1, \end{aligned} \quad (22)$$

Taking derivatives of Eq. (21), we get

$$u''(t) + \left(\frac{1}{2}t - 2\right)u'(t) + \frac{1}{2}u(t) - 2u\left(\frac{t}{2}\right)^2 = 0, \quad (23)$$

Then the integro-differential equation (21) subject to initial condition $u(0) = 0$, reduced to pantograph equation (23) subject to initial conditions $u(0) = 0$, and $u'(0) = 1$.

By applying differential transformation method on nonlinear pantograph equation (23), for $k = 0, 1, 2, \dots, N$, we get

$$\begin{aligned} \frac{(k+2)!}{k!} U(k+2) + \sum_{\ell=0}^k \left[\frac{1}{2} \delta(\ell-1) - 2\delta(\ell) \right] U(k-\ell) \\ + \frac{1}{2} U(k) - 2 \sum_{\ell=0}^k \left(\frac{1}{2} \right)^k U(\ell) U(k-\ell) = 0, \end{aligned} \quad (24)$$

and the differential transform version of initial conditions (22) will be

$$\begin{aligned} U(0) &= 0, \\ U(1) &= 1, \end{aligned} \quad (25)$$

where $U(k)$ is the differential transform of $u(t)$.

By taking $N = 5$, the following system is obtained from differential transform version (24):

$$\begin{aligned} 2U(2) - 2 &= 0, \\ 6U(3) - 4U(2) + 1 &= 0, \\ 12U(4) - 6U(3) + \frac{3}{2}U(2) - \frac{1}{2} &= 0, \\ 20U(5) - 8U(4) + 2U(3) - \frac{1}{2}U(2) &= 0, \\ 30U(6) - 10U(5) + \frac{5}{2}U(4) - \frac{1}{4}U(3) - \frac{1}{8}U(2)^2 &= 0, \\ 42U(7) - 12U(6) + 3U(5) - \frac{1}{8}U(4) - \frac{1}{8}U(2)U(3) &= 0, \end{aligned} \quad (26)$$

Solving the system (26), and using the inverse transformation rule (5), we get the following series solution

$$u(t) = t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + \frac{1}{24}t^5 + \frac{1}{120}t^6 + \frac{1}{720}t^7,$$

The closed form of above solution, when $N \rightarrow \infty$ is

$$u(t) = te^t,$$

which is exactly the same as the exact solution of integro-differential equation (21).

IV. CONCLUSION

In this paper, we have shown that the differential transformation method can be used successfully for solving the nonlinear IVPs and BVPs for pantograph equation and as application for nonlinear integro-differential equation with proportional delay. New theorems are introduced with their proofs and as application some examples are carried out. This method is simple and easy to use and solves the problem without any need for discretizing the variables. Also, the method we present here can be further expanded to solve integro-multi-pantograph-differential equations and partial-multi-pantograph-differential equations and the coupled systems of them for future studies.

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