# Solution of First kind Fredholm Integral Equation by Sinc Function

Khosrow Maleknejad, Reza Mollapourasl,Parvin Torabi and Mahdiyeh Alizadeh,

*Abstract*—Sinc-collocation scheme is one of the new techniques used in solving numerical problems involving integral equations. This method has been shown to be a powerful numerical tool for finding fast and accurate solutions. So, in this paper, some properties of the Sinc-collocation method required for our subsequent development are given and are utilized to reduce integral equation of the first kind to some algebraic equations. Then convergence with exponential rate is proved by a theorem to guarantee applicability of numerical technique. Finally, numerical examples are included to demonstrate the validity and applicability of the technique.

*Keywords*—Integral equation, Fredholm type, Collocation method, Sinc approximation.

## I. INTRODUCTION

The purpose of this paper is to develop high order numerical methods for Fredholm integral equations of the first kind defined by

$$\int_a^b k(s,t)f(t)dt = g(s) \qquad -\infty < a \le s \le b < \infty \quad (1)$$

where $k(s,t)$ and $g(s)$ are known functions and $f(t)$ is the solution to be determined. This type of equations appear in many science and engineering fields, and in many cases, we can not solve this equation analytically to find an exact solution. So that, by using numerical methods we try to estimate a solution for this equation.

Numerical and theoretical methods for solving integro-differential and integral equations have been studied by many authors so far [1-9]. Some of them usually use techniques based on an expansion in terms of some basis functions or use some quadrature formulas, and the convergence rate of these methods are usually of polynomial order with respect to N, where N represents the number of terms of the expansion or the number of points of the quadrature formula. On the other hand, in [10] it is shown that if we use the Sinc method the convergence rate is $O(exp(-c\sqrt{N}))$ with some $c > 0$. Although this convergence rate is much faster than that of polynomial order.

So, in the present paper, we apply the Sinc-collocation method which has exponential approximation rate for solving Eq. (1). Our method consists of reducing the solution of Eq. (1) to a set of algebraic equations by expanding $f(t)$ as

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Sinc functions with unknown coefficients. The properties of the Sinc function are then utilized to evaluate the unknown coefficients [1,11-12]. In some papers such as [13-15] integral equation of first and second kind have been studied by some authors, but in some of these papers there is no error analysis which guarantees the convergence of the mentioned scheme. So, in this study a theorem is prepared to show convergence analysis of Sinc collocation method, then some numerical illustration examples are presented to show accuracy of this technique.

## II. PRELIMINARIES

In this section, we introduce the cardinal function and some of its properties. For this result $sinc(x)$ definition is followed by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \ne 0 \\ 1, & x = 0. \end{cases}$$

Now, for $h > 0$ and integer $k$, we define k'th Sinc function with step size h by

$$S(k,h)(x) = \frac{\sin(\pi(x-kh)/h)}{\pi(x-kh)/h}.$$

The Sinc approximation on the entire interval $(-\infty, \infty)$ is defined as

$$f(x) \approx \sum_{k=-N}^{N} f(kh)S(k,h)(x).$$

Now, the following Definition and Theorem will guarantee the approximation authority of Sinc functions on the real line.

**Definition 1.** Let $H^1(D_d)$ denote the family of all functions analytic in $D_d$ defined by

$$D_d = \{z \in \mathcal{C} : |Im(z)| < d\}$$

such that for $0 < \epsilon < 1$ , $D_d(\epsilon)$ is defined by

$$D_d(\epsilon) = \{z \in \mathcal{C} : |Im(z)| < d(1-\epsilon), |Re(z)| < \frac{1}{\epsilon}\}$$

then $N(f, D_d) < \infty$ with

$$N(f, D_d) = \lim_{\epsilon \to 0}(\int_{\partial D_d} |f(z)||dz|)$$

**Theorem 1.** Let $\alpha, \beta$ and $d$ as positive constants, that
1) $f \in H^1(D_d)$
2) $f$ decays exponentially on the real line such that

$$|f(x)| \le \alpha \, exp(-\beta|x|), \ \ \forall x \in R$$

then we have

$$\sup_{-\infty < x < \infty} \left| f(x) - \sum_{k=-N}^{N} f(kh)S(k,h)(x) \right| \leq$$

$$CN^{1/2} exp[-(\pi d\beta N)^{1/2}]$$

for some $C$ and step size $h$ is taken as

$$h = (\frac{\pi d}{\beta N})^{1/2}.$$

*Proof:* [11], [12] ∎

Let $t = \phi(z)$ denote a conformal map which maps the simply connected domain $D$ with boundary $\partial D$ onto a strip region $D_d$ such that

$$\phi((a,b)) = (-\infty, \infty), \quad \lim_{t \to a} \phi(t) = -\infty, \quad \lim_{t \to b} \phi(t) = \infty.$$

Now, in order to have the Sinc approximation on a finite interval $(a,b)$ conformal map is employed as follow

$$\phi(x) = \ln(\frac{x-a}{b-x}).$$

This map carries the eye-shaped complex domain

$$\left\{ z = x + iy : \left| arg(\frac{z-a}{b-z}) \right| < d \leq \frac{\pi}{2} \right\},$$

onto the infinite strip

$$D_d = \{ \mu = \alpha + \beta i : |\beta| < d < \frac{\pi}{2} \}.$$

The basis function on finite interval $(a,b)$ are given by

$$S(k,h) \circ \phi(x) = \frac{\sin(\pi(\phi(x) - kh)/h)}{\pi(\phi(x) - kh)/h},$$

also, Sinc function for interpolation points $x_k = kh$ is given by

$$S(k,h)(jh) = \delta_{kj}^{(0)} = \begin{cases} 1, & k = j; \\ 0, & k \neq j. \end{cases}$$

So, $S(k,h) \circ \phi(x)$ exhibits Kronecker delta behavior on the grid points

$$x_k = \phi^{-1}(kh) = \frac{a + b\,e^{kh}}{1 + e^{kh}}$$

and interpolation and quadrature formulas for $f(x)$ over $[a,b]$ are

$$f(x) \approx \sum_{k=-N}^{N} f(x_k)S(k,h) \circ \phi(x),$$

$$\int_a^b f(x)dx \approx h \sum_{k=-N}^{N} \frac{f(x_k)}{\phi'(x_k)}.$$

**Theorem 2.** Assume that, for a variable transformation $z = \phi^{-1}(\xi)$, the transformation function $f(\phi^{-1}(\xi))$ satisfies assumptions (1) and (2) in Theorem 1. with some $\alpha, \beta$ and $d$. Then we have

$$\sup_{a < x < b} \left| f(x) - \sum_{k=-N}^{N} f(\phi^{-1}(kh))S(k,h) \circ \phi(x) \right| \leq$$

$$CN^{1/2} exp[-(\pi d\beta N)^{1/2}]$$

for some $C$, where the step size $h$ is taken as

$$h = (\frac{\pi d}{\beta N})^{1/2}.$$

*Proof:* [11], [12] ∎

## III. SINC-COLLOCATION METHOD

Now, for solving integral equation of the first kind denoted by

$$\int_a^b k(s,t)f(t)dt = g(s) \qquad -\infty < a \leq s \leq b < \infty$$

with Sinc approximation, we need to chose a method to find unknown coefficients in this expansion. Collocation method is one of the projection methods that is used as follow:

By substituting Sinc approximation expansion of unknown function $f(t)$ in the above equation, we have

$$\int_a^b k(s,t) \left( \sum_{k=-N}^{N} f(\phi^{-1}(kh))S(k,h) \circ \phi(t) \right) dt = g(s)$$

then, define residual function as follow

$$R_N(s) = \int_a^b k(s,t) \left( \sum_{k=-N}^{N} f(\phi^{-1}(kh))S(k,h) \circ \phi(t) \right) dt - g(s).$$

So, to find $f(\phi^{-1}(kh))$ in Sinc approximation expansion, there are some techniques such as projection methods like Galerkin and collocation [16-18]. In this study, collocation method which has less computations than Galerkin is applied with some collocation points in interval $[a,b]$ for residual function as follows

$$R_N(s_i) = 0; \qquad i = -N, -N+1, ..., N-1, N$$

$$s_i = \phi^{-1}(ih) = \frac{a + b\,e^{ih}}{1 + e^{ih}}; \quad i = -N, -N+1, ..., N-1, N$$

So that

$$\int_a^b k(s_i,t) \left( \sum_{k=-N}^{N} f(\phi^{-1}(kh))S(k,h) \circ \phi(t) \right) dt = g(s_i)$$

Then integral equation of the first kind is converted to system of linear algebraic equations $A_N X = b_N$ where

$$A_N = \left[ \int_a^b k(s_i,t)S(k,h) \circ \phi(t)dt \right]_{k=-N}^{N},$$

$$X^T = \left[ f(\phi^{-1}(kh)) \right]_{k=-N}^{N},$$

$$b_N = [g(s_i)], \qquad i = -N, -N+1, ..., N-1, N.$$

Now, for evaluating matrix elements of algebraic equations we have

$$\int_a^b k(s_i,t)S(k,h) \circ \phi(t)dt \approx h \sum_{j=-N}^{N} \frac{k(s_i,t_j)S(k,h) \circ \phi(t_j)}{\phi'(t_j)},$$

where

$$t_j = \phi^{-1}(jh) = \frac{a + b\,e^{jh}}{1 + e^{jh}}; \quad j = -N, -N+1, ..., N-1, N.$$

## IV. CONVERGENCE ANALYSIS

In this section, we discuss about convergence of Sinc-Collocation for Fredholm integral equation of the first kind. For this result, consider the following theorem.

**Theorem 3.** In Eq. (1) assume that, $k(s,t)$ is continuous on square $[a,b]^2$, and let

$$\phi^{-1}(\xi) = \frac{a + b\,e^\xi}{1 + e^\xi}$$

such that, $f(\phi^{-1}(\xi))$ satisfies in assumptions (1) and (2) in Theorem 1. where $f$ is solution of integral equation. Also let

$$P_N^{num}(f)(t) = \sum_{k=-N}^{N} f^{num}(\phi^{-1}(kh))S(k,h) \circ \phi(t)$$

as Sinc approximation of $f$ with step size $h$ and $f^{num}(\phi^{-1}(kh))$ are unknown coefficients which will be determined by solving system of algebraic equation $A_N X = b_N$. Now, if $A_N$ is nonsingular then

$$\|f - P_N^{num}(f)\|_\infty \le [c_1 + c_3\|A_N^{-1}\|_\infty N]N^{1/2}\exp\{-c_2 N^{1/2}\}$$

for some positive constant $c_1, c_2, c_3$.

*Proof:* Let

$$f(t) \simeq P_N^{num}(f)(t) = \sum_{k=-N}^{N} f^{num}(\phi^{-1}(kh))S(k,h) \circ \phi(t)$$

where $f^{num}(\phi^{-1}(kh))$ are unknown coefficients which are found by solving linear system of equations [19]. Also, assume

$$P_N(f)(t) = \sum_{k=-N}^{N} f(\phi^{-1}(kh))S(k,h) \circ \phi(t)$$

and in this approximation $f(\phi^{-1}(kh))$ is the exact value of $f$ at $\phi^{-1}(kh)$. If substitute $P_N^{num}(f)(t)$ as an approximation of $f(t)$ in Eq. (1) then

$$g(s) = \int_a^b k(s,t)P_N^{num}(f)(t)dt \qquad (2)$$

but if use $P_N(f)(t)$ then obtain

$$\hat{g}(s) = \int_a^b k(s,t)P_N(f)(t)dt. \qquad (3)$$

Now, by converting Eq. (2) to linear system then by solving this system we have

$$[f^{num}(\phi^{-1}(kh))]_{k=-N}^N = A_N^{-1}[g(s_i)]_{i=-N}^N$$

but, by applying numerical scheme which was discussed in previous section to Eq. (3) we have

$$[f(\phi^{-1}(kh))]_{k=-N}^N = A_N^{-1}[\hat{g}(s_i)]_{i=-N}^N.$$

So that

$$\sup_{k\in S_N}|f^{num}(\phi^{-1}(kh)) - f(\phi^{-1}(kh))| \le \|A_N^{-1}\|\sup_{i\in S_N}|g(s_i) - \hat{g}(s_i)| \qquad (4)$$

where $S_N$ is all integers belong to $[-N, N]$.

Now, we have

$$\int_a^b k(s,t)P_N(f)(t)dt = g(s) - \int_a^b k(s,t)[f(t) - P_N(f)(t)]dt$$

then let

$$\hat{g}(s) = g(s) - \int_a^b k(s,t)[f(t) - P_N(f)(t)]dt$$

so that we have

$$\sup_{i\in S_N}|\hat{g}(s_i) - g(s_i)| = \sup_{s\in[a,b]}|\int_a^b k(s,t)[f(t) - P_N(f)(t)]dt|$$
$$\le (b-a)\sup_{t,s\in[a,b]}|k(s,t)|\,\|f - P_N(f)\|$$

Since $k(s,t)$ is continuous on $[a,b]^2$ so that let

$$M = \sup_{t,s\in[a,b]}|k(s,t)|$$

also regarding to Theorem (2) we have

$$\|f - P_N(f)\| \le c_1 N^{1/2}\exp\{-c_2 N^{1/2}\}$$

so,

$$\sup_{i\in S_N}|g(s_i) - \hat{g}(s_i)| \le (b-a)Mc_1 N^{1/2}\exp\{-c_2 N^{1/2}\}. \qquad (5)$$

Finally, by substituting Eq. (5) in Eq. (4) we can derive

$$\sup_{k\in S_N}|f^{num}(\phi^{-1}(kh)) - f(\phi^{-1}(kh))| \le$$
$$c_3\|A_N^{-1}\|N^{1/2}\exp\{-c_2 N^{1/2}\}.$$

Also, we need to determine a bound for $\|P_N(f)(t) - P_N^{num}(f)(t)\|$ hence

$$\sup_{t\in[a,b]}|P_N(f)(t) - P_N^{num}(f)(t)|$$
$$= \sup_{t\in[a,b]}\left|\sum_{k=-N}^N [f(\phi^{-1}(kh)) - f^{num}(\phi^{-1}(kh))]S(k,h)\circ\phi(t)\right|$$
$$\le \|f(\phi^{-1}(kh)) - f^{num}(\phi^{-1}(kh))\|\sup_{t\in[a,b]}\sum_{k=-N}^N |S(k,h)\circ\phi(t)|$$
$$\le \|A_N^{-1}\|c_3 N^{1/2}\exp\{-c_2 N^{1/2}\}\sup_{t\in[a,b]}\sum_{k=-N}^N |S(k,h)\circ\phi(t)|$$

Also, in [10]

$$\sup_{t\in[a,b]}\sum_{k=-N}^N |S(k,h)\circ\phi(t)| \le \frac{2}{\pi}\{3 + \log(N)\}$$

for sufficiently large $N$ so, it is possible to replace $\frac{2}{\pi}\{3 + \log(N)\}$ by $N$ so that

$$\sup_{t\in[a,b]}|P_N(f)(t) - P_N^{num}(f)(t)| \le c_3\|A_N^{-1}\|N^{3/2}\exp\{-c_2 N^{1/2}\}$$

finally,

$$\|f(t) - P_N^{num}(f)(t)\|$$
$$\le \|f(t) - P_N(f)(t)\| + \|P_N(f)(t) - P_N^{num}(f)(t)\|$$
$$\le c_1 N^{1/2}\exp\{-c_2 N^{1/2}\} + c_3\|A_N^{-1}\|N^{3/2}\exp\{-c_2 N^{1/2}\}$$
$$= N^{1/2}\exp\{-c_2 N^{1/2}\}[c_1 + c_3\|A_N^{-1}\|N],$$

and proof of this theorem is completed. ∎

## V. NUMERICAL EXAMPLES

Aim of this section is to show efficiency and accuracy of numerical method which is discussed in previous sections. So that, Eq. (1) is discredited by Sinc-collocation method and converted to system of algebraic equations and then it is solved to find a numerical solution for Eq. (1).

**Example1.** In this example, let

$$\int_0^{\frac{\pi}{2}} (\sin(s-t)+\cos(t-s))f(t)dt = 0.048\cos(s)+0.2853\sin(s),$$

where the exact solution is $f(t) = \sin(t)(1 - \sin(t))$. That is easy to show that this exact solution satisfies assumptions (1) and (2) in Theorem 1. for $\alpha = \beta = 1$, also let $d = \frac{\pi}{2}$, so step size for Sinc function is $h = \frac{\pi}{(2N)^{\frac{1}{2}}}$.

Now let

$$E(N) = \max_{-N \le j \le N} |f(t_j) - P_N^{num}(f)(t_j)|$$

where $t_j$'s are collocation nodes, and assume

$$Cond(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

where $A$ is coefficient matrix in algebraic system of equations. Then Numerical results are shown for different values of $N$ in Table 1.

| N | E(N) | Cond(A) |
|---|---|---|
| 5 | $1.2 \times 10^{-2}$ | 11 |
| 10 | $3.6 \times 10^{-3}$ | 23 |
| 15 | $2.4 \times 10^{-4}$ | 38 |
| 20 | $1.1 \times 10^{-4}$ | 42 |
| 25 | $3.1 \times 10^{-5}$ | 57 |

TABLE I
NUMERICAL RESULTS FOR EXAMPLE 1

**Example 2.** In this example, we have

$$\int_0^1 (\sin(t)t^3 + s^2)f(t)dt = 0.021 + 0.167s^2$$

and the exact solution is $f(t) = t(1 - t)$. This is clear that $f(t)$ satisfies assumption (1) and (2) is Theorem 1. Also, let

$$e(t) = |f(t) - P_N^{num}(f)(t)|, \quad \forall t \in [a, b]$$

as absolute error function. Error function has been drawn for different values of $N$ and results are shown in Figures 1, 2.
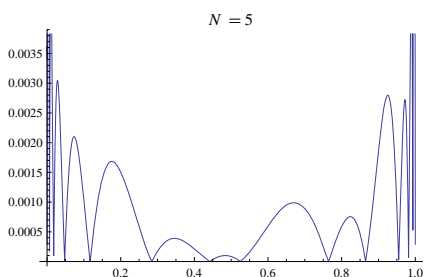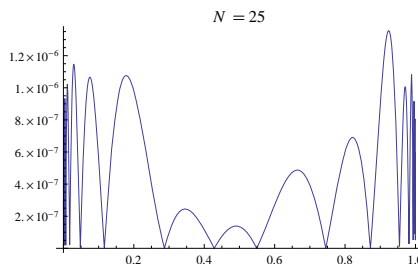


Fig. 1. Error Function for $N = 5$ .



Fig. 2. Error Function for $N = 25$ .

## CONCLUSION

Properties of the Sinc-collocation method are utilized to reduce the computation of this problem to some algebraic equation and then get the numerical results with high accuracy and little computational efforts. Our method is shown to be of good convergence, easy to program. So, we expect our method can be extended to the nonlinear Fredholm and Volterra type equations. This is left for our next paper.

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