Small Sample Bootstrap Confidence Intervals for Long-Memory Parameter

Josu Arteche, and Jesus Orbe

Abstract— The log periodogram regression is widely used in empirical applications because of its simplicity, since only a least squares regression is required to estimate the memory parameter, d, its good asymptotic properties and its robustness to misspecification of the short term behavior of the series. However, the asymptotic distribution is a poor approximation of the (unknown) finite sample distribution if the sample size is small. Here the finite sample performance of different nonparametric residual bootstrap procedures is analyzed when applied to construct confidence intervals. In particular, in addition to the basic residual bootstrap, the local and block bootstrap that might adequately replicate the structure that may arise in the errors of the regression are considered when the series shows weak dependence in addition to the long memory component. Bias correcting bootstrap to adjust the bias caused by that structure is also considered. Finally, the performance of the bootstrap in log periodogram regression based confidence intervals is assessed in different type of models and how its performance changes as sample size increases.

Keywords— bootstrap, confidence interval, log periodogram regression, long memory.

I. INTRODUCTION

Long memory processes have emerged as a useful tool to fill the gap between weakly dependent stationary processes and nonstationary integrated processes with a unit root. Long memory processes are characterized by a strong dependence such that the lag-j autocovariances γ_j decrease hyperbolically as $j \to \infty$

$$\gamma_j \sim G j^{2d-1}$$

for some finite constant G, d is the memory parameter and $a \sim b$ means that a/b tends in the limit to 1. Such processes are usually denoted $I(d)$. For $d > 0$, $\sum |\gamma_j| = \infty$ but stationarity is guaranteed as long as $d < 1/2$ and mean reversion holds for $d < 1$. It is also usually assumed that $d > -1/2$, which warrants invertibility.

Long memory can alternatively and equivalently be defined in the frequency domain. A stationary time series process has long memory if its spectral density function $f(\cdot)$ satisfies

$$f(\lambda) \sim C|\lambda|^{-2d} \quad \text{as} \quad \lambda \to 0, \tag{1}$$

for some positive finite constant C. Under positive long memory, which is the most common case in economic and financial series, the spectral density diverges at the origin at a rate governed by d. If $d > 1/2$ the process is not stationary and, by definition, the spectral density does not exist. However pseudo spectral density functions can be similarly defined (e.g. [1]) with a behavior as in (1).

One issue of main interest in these processes is the estimation of d. There is a large number of different procedures, parametric as maximum likelihood or the asymptotically equivalent Whittle estimation, semiparametric or local as the log periodogram regression, the local Whittle or the average periodogram and nonparametric such as the R/S. Perhaps the most popular is the log periodogram regression estimator (LPE hereafter) originally proposed by [2] and analyzed in detail in [3] and [4]. The LPE is widely used in empirical applications because of its simplicity, since only a least squares regression is required, its good asymptotic and finite samples properties and its robustness to misspecification of the short term behavior of the series. Taking logarithms of the local specification of the spectral density in (1), the LPE ($\hat{d}$) is obtained by least squares in the regression

$$\log I_j = a + dX_j + u_j, \quad j = 1, ..., m, \tag{2}$$

where $X_j = -2\log\lambda_j$, $a = \log C + c$, $c = 0.577216$ is Euler's constant, $I_j = (2\pi n)^{-1}|\sum_{t=1}^{n} x_t \exp(-it\lambda_j)|^2$ is the periodogram of the series x_t, $t = 1, .., n$, at Fourier frequency $\lambda_j = 2\pi j/n$, n is the sample size, $u_j = \log(I_j f(\lambda_j)^{-1}) - c$ and m represents the bandwidth, that is the number of

Authors are with Dpt. Econometría y Estadística, University of the Basque Country, Avda. Lehendakari Agirre 83, 48015 Bilbao, Spain (email: jesus.orbe@ehu.es)

frequencies used in the estimation. For the asymptotics, this bandwidth has to increase with n but at a slower rate such that the band of frequencies used in the estimation degenerates to zero and the local specification in (1) remains valid. [3] and [4] proved the consistency of $\hat{d}$ in the stationary and invertible region $-0.5 < d < 0.5$, and obtained its limit distribution

$$\sqrt{m}(\hat{d} - d) \xrightarrow{d} N\left(0, \frac{\pi^2}{24}\right). \tag{3}$$

Reference [1] showed that the consistency holds even in the nonstationary region $[0.5, 1)$ and the same limit distribution remains valid for $d \in [0.5, 0.75)$. Consistency is preserved in the unit root case $d = 1$ with a mixed normal limit distribution ([5]) but the LPE is inconsistent for $d > 1$ ([6]). For similarities with the local Whittle estimator, the asymptotic distribution of the LPE for $d \in [0.75, 1)$ is expected to be non normal and non pivotal depending on d ([7]).

In practice the choice of the bandwidth is crucial, a large m decreases the variance at the cost of a higher bias which can be extremely large in some situations, for example in the presence of some short term component such as those analyzed below. The choice of an optimal bandwidth is not a simple task. Some attempts have been made in [8], who proposes to estimate the bandwidth together with the rest of parameters by minimizing the contrast function, [9] who propose an adaptive LPE, and [10] with a plug in version of an optimal bandwidth in an asymptotic mean squared error sense. However, the performance of all these procedures is not very satisfactory and the results for a grid of bandwidths are usually shown in empirical applications.

The log-periodogram estimation of the memory parameter in economic series raises the problem of the small sample size since many economic time series consist of low frequency, monthly ([11]), quarterly ([12] and [13]) or even yearly ([12] and [13]) data. Furthermore, if the series shows a rich spectral behavior around the origin the bandwidth has to be low enough to avoid a large bias in the estimation of d ([14]). Also the strong seasonality in many quarterly and monthly economic series compels the use of a small bandwidth to avoid distorting influence of neighbouring seasonal spectral poles ([15]). As a result the number of frequencies used in the estimation is small and, as noted in [16], the asymptotic distribution in (3) is a poor approximation of the small sample distribution of $\hat{d}$. In this situation, the bootstrap could be a useful tool to make inference without relying on the asymptotic probability distribution.

The application of the bootstrap to approximate the distribution of some statistics of a long memory series x_t, $t = 1, 2, ..., n$, has primarily focus on generating bootstrap samples of the series to get the bootstrap distribution of a statistic $T(x_1, ..., x_n)$ (usually an estimator of d or a t-statistic). This has been done by a plug-in parametric bootstrap ([17]), by a pre-whitening and re-coloring bootstrap either in the time domain ([18]) or in the frequency domain ([19]), or bootstrapping directly the periodogram ([20]). In the LPE setup the bootstrap is carried out previously to the definition of the regression model (2) and the bootstrapped dependent variable is then the logarithm of the periodogram of the boot-

strap samples. Reference [21] proposed instead to bootstrap directly the residuals in the regression (2) avoiding in that way the necessity to deal with the temporal dependence of x_t with the corresponding computational savings and robustness against misspecification. We focus here in this last approach and analyze the implementation of different bootstraps on the coverage of confidence intervals.

As already mentioned, the LPE can be highly biased in the presence of some weak dependent component. Reference [22] shows that an autoregressive or moving-average component in an ARFIMA model can seriously distort the estimation of the memory parameter with a large bias. The source of the bias is the effect of these short memory components on the spectral behavior around frequency zero such that the approximation (1) is only reliable for frequencies very close to the origin. This weak dependence, not considered in the regression in (2), affects the behavior of u_j such that it shows some remaining structure. To try to capture this structure we consider a version of the local bootstrap of [23], applied also in a similar long memory context by [20], and compare its performance with the nonparametric residual and block bootstrap. Contrary to the local bootstrap that maintains the global structure of the residuals, the block bootstrap, designed for time dependent data, conserves the local or neighbouring structure by bootstrapping different overlapping blocks. We are also concerned with the effects of the bias in the LPE and evaluate the capacity of different bias corrections usually employed in the bootstrap literature, namely the Bias Corrected (BC) percentile of [24], the accelerated Bias Corrected (BC_a) percentile [25] and the Constant Bias Correcting (CBC) estimator of [26]. In addition the bootstrap-t method of [24], which implicitly includes a bias adjustment, is also examined.

The paper is organized as follows. Section II describes the different bootstrap procedures that we analyze in the Monte Carlo in Section III. Finally Section IV concludes.

II. DIFFERENT BOOTSTRAPS PROCEDURES IN LPE

The bootstrap, first introduced by [27], is an important tool for statistical inference to approximate standard errors of estimators, confidence intervals or p-values for test statistics, especially in complicated models. The implementation of the bootstrap relies on using the original sample as if it were a population to generate pseudo data. New samples, bootstrap samples, are then obtained resampling from it. Therefore, the bootstrap is usually interesting to approximate the asymptotic distribution of a statistic in complicated models or an unknown finite sample distribution when the asymptotic distribution does not resembles the finite sample counterpart.

Here the bootstrap to calculate confidence intervals for the LPE that improve standard confidence intervals based on the asymptotic distribution in a small sample size situation is used. We focus on a bootstrap in a regression context, using the LPE regression (2). There are two general bootstrap based regression methods: The *cases resampling* or *pairs bootstrap* considers the regressors as random covariates, changing from sample to sample, whereas the *model based resampling* or *residual bootstrap* takes the regressors as fixed. We use here a

nonparametric residual bootstrap since the regressor is based on non stochastic Fourier frequencies and we do not assume any probability distribution for the error term in the regression model, involving a nonparametric resampling of the residuals.

In this section the three nonparametric bootstrap procedures that are considered competitive to replicate the finite sample distribution of the LPE are described. In all of them, after obtaining the OLS estimated coefficients, the $\hat{u}_j$ residuals are rescaled to account for the leverages of the observations, since even when the errors are homoscedastic, the residuals have variances that depend on those leverages. Therefore the modified residuals are resampled.

A. Residual bootstrap (RB)

The appropriate performance of this simple bootstrap in the LPE context over other procedures, such as the wild bootstrap, has been discussed in [21]. The *residual bootstrap* for the log periodogram regression follows these steps:

1) Obtain the LPEs, $\hat{a}$, $\hat{d}$, by least squares in the regression (2) and the residuals $\hat{u}_j = \log I_j - \hat{a} - \hat{d}X_j$. Construct the modified residuals $\hat{v}_j = \hat{u}_j/(1 - h_j)^{1/2}$ ([28]), where $h_j = [X(X'X)^{-1}X']_{jj}$ and X is the matrix with columns the regressors in (2), that is 1 and X_j, ensuring in that way constant variances of $\hat{v}_j$ if the disturbances u_j were homoscedastic.

2) Resampling with replacement from the modified residuals $\hat{v}_j$, and giving equal probability $1/m$ to every residual, get B bootstrap samples $\hat{v}_{bj}^*$, $b = 1, 2, ..., B$ and $j = 1, ..., m$. Using the empirical distribution function of the residuals and based on model (2) we obtain the corresponding bootstrap dependent variable $\log I_{bj}^* = \hat{a} + \hat{d}X_j + \hat{v}_{bj}^*$.

3) Fit regression model (2) in each bootstrap sample to obtain the B bootstrap estimates $\hat{d}_b^*$, $b = 1, ..., B$.

B. Residual local bootstrap (RLB)

The RB implicitly assumes that the errors do not have any structure and their behavior approximate an iid sequence. This can be quite unrealistic, especially when the long memory series contains also a short memory component. We propose here a version of the local bootstrap ([23]) that tries to capture the structure of the errors by bootstrapping only in a neighborhood of each observation. It follows these steps:

1) Step 1 in the RB.
2) Select a resampling width $k_m \in \mathcal{N}$, $k_m \leq [m/2]$ for $[\cdot]$ denoting "the integer part of".
3) Define i.i.d. discrete random variables $S_1, ..., S_m$ taking values in the set $\{0, \pm 1, , ..., \pm k_m\}$ with equal probability $1/(2k_m + 1)$.
4) Generate B bootstrap series $\hat{v}_{bj}^* = \hat{v}_{|j+S_j|}$ if $|j + S_j| > 0$, $\hat{v}_{bj}^* = \hat{v}_1$ if $j + S_j = 0$ for $b = 1, 2, ..., B$, $j = 1, ..., m$.
5) Generate B bootstrap samples for the dependent variable $\log I_{bj}^* = \hat{a} + \hat{d}X_j + \hat{v}_{bj}^*$ for $b = 1, 2, ..., B$, $j = 1, ..., m$.
6) Fit the regression model (2) in each bootstrap sample to obtain the B bootstrap estimates $\hat{d}_b^*$, $b = 1, ..., B$.

C. Residual block bootstrap (RBB)

The local bootstrap attempts to conserve the global structure of the residuals by resampling locally in a neighborhood of each residual. On the contrary, the block bootstrap tries to maintain the local structure of the residuals by resampling blocks of observations. The idea of the block bootstrap is similar to the iid nonparametric bootstrap, both resampling observations with replacement. But instead of resampling single observations, the block bootstrap resamples blocks of consecutive observations. Different versions of block bootstrap have been proposed. We use here the moving blocks bootstrap proposed by [29] and [30], which has better properties than the version of non overlapping blocks ([31]).

1) Step 1 in the RB.
2) Select the block size l and obtain $m - l + 1$ possible overlapping blocks of consecutive modified residuals of length l.
3) Select (m/l) blocks resampling with replacement from the $m - l + 1$ overlapping blocks, that is, giving probability $1/(m - l + 1)$ to each overlapping block, and concatenating these blocks to obtain the bootstrap sample of modified residuals of size m.
4) Generate B bootstrap samples for the dependent variable $\log I_{bj}^* = \hat{a} + \hat{d}X_j + \hat{v}_{bj}^*$ for $b = 1, 2, ..., B$, $j = 1, ..., m$.
5) Fit the regression model (2) in each bootstrap sample to obtain the B bootstrap estimates $\hat{d}_b^*$, $b = 1, ..., B$.

These bootstrap techniques are used to construct confidence intervals trying to improve the coverage of confidence intervals based on the asymptotic distribution defined as

$$I_{(1-\alpha)} = \left(\hat{d} - z_{1-\frac{\alpha}{2}} \, \hat{se}(\hat{d}); \quad \hat{d} - z_{\frac{\alpha}{2}} \, \hat{se}(\hat{d}) \right)$$

where $\hat{se}(\hat{d})$ is the OLS estimate of the standard error and z_α indicate de $100 \cdot \alpha$th percentile of a $N(0, 1)$ distribution. The use of the OLS standard error $\hat{se}(\hat{d})$ instead of the asymptotic variance in (3) has proved to significantly improve the finite sample coverage probabilities.

For each of the three bootstrap resampling strategies we consider five different classes of bootstraps confidence intervals for the memory parameter: the percentile interval (P), the constant bias correction percentile interval (CBC), the bias corrected interval (BC), the accelerated bias corrected interval (BCa) and the bootstrap-t interval (b-t).

1) The basic percentile method (P), proposed by [27], considers the existence of some monotonic transformation of the parameter d, $\phi = g(d)$, verifying

$$\hat{\phi} - \phi \sim N(0, \sigma^2) \qquad (4)$$

has the advantage of its simplicity because it does not require knowledge of the parameters defining the (possibly unknown) distribution of the statistic of interest. This method does not require of knowing of a σ parameter or the normalizing function of the parameter ϕ. The $(1 - \alpha)$ percentile interval first is calculated for $\hat{\phi}$ and then transform this back to the $\hat{d}$ scale The $(1 - \alpha)$ percentile interval is defined as

$$I_{(1-\alpha)} = \left(\hat{d}_{((B+1)(\frac{\alpha}{2}))}^* \quad ; \quad \hat{d}_{((B+1)(1-\frac{\alpha}{2}))}^* \right).$$

where the $\hat{d}^*_{(j)}$ denotes the jth ordered value of the bootstrap estimates of d. So we estimate the $\alpha/2$ percentile by the $(B+1)(\frac{\alpha}{2})$ ordered value of $\hat{d}^*$. We choose a B value such that $(B+1)(\frac{\alpha}{2})$ is an integer.

For the percentile method to work well we need $\hat{d}$ to be an unbiased estimator of the memory parameter. But this does not usually happen. Next we consider different alternatives that try to handle this possible bias.

2) Reference [26] proposed a method for reducing the finite sample bias of consistent estimators using a pre-bootstrap estimation of the bias. The constant bias correcting (CBC) estimator is obtained as $\tilde{d} = \hat{d} - \hat{b}$ where $\hat{b}$ is a bootstrap estimate of the finite sample bias of $\hat{d}$. This bias correction is adequate when the bias function does not depend on d, which is the case, at least asymptotically, for the LPE in many long memory models [4]. The bias is estimated in a prior bootstrap as

$$\hat{b} = \frac{1}{B}\sum_{b=1}^{B}\hat{d}^*_b - \hat{d}$$

where $\hat{d}^*_b$ is the LPE obtained in the bootstrap sample b out of B replications. With this correction the confidence interval is obtained as in the basic percentile with the estimates of d in each bootstrap replication corrected with the bias estimate $\hat{b}$

$$I_{(1-\alpha)} = \left(\tilde{d}^*_{((B+1)(\frac{\alpha}{2}))} \quad ; \quad \tilde{d}^*_{((B+1)(1-\frac{\alpha}{2}))}\right).$$

In fact, for each bootstrap replication a bootstrap bias correction should be applied resulting in a double bootstrap. However, this approach would be computationally infeasible in our Monte Carlo and we instead use the same bias estimate for every bootstrap replication in a bootstrap after bootstrap procedure ([32]).

3) In order to improve the coverage probability of the basic percentile interval [24] introduced the bias-corrected (BC) percentile. This method, as in the basic percentile one, based on the existence of some monotonic function $\phi = g(d)$, but in this case considering the possibility of bias by introducing a bias parameter k_0 in the distribution of the statistic of interest.

$$\hat{\phi} - \phi \sim N(-k_0\sigma, \sigma^2) \Rightarrow \frac{\hat{\phi}-\phi}{\sigma} \sim N(-k_0, 1)$$

The confidence interval is then constructed as

$$I_{(1-\alpha)} = \left(\hat{d}^*_{((B+1)(\frac{\tilde{\alpha}}{2}))} \quad ; \quad \hat{d}^*_{((B+1)(1-\frac{\tilde{\alpha}}{2}))}\right),$$

where

$$\frac{\hat{\alpha}}{2} = \Phi\left(2k_0 + z_{\frac{\alpha}{2}}\right) \quad \text{and} \quad 1 - \frac{\hat{\alpha}}{2} = \Phi\left(2k_0 + z_{1-\frac{\alpha}{2}}\right),$$

Φ is the standard normal cumulative distribution function and k_0 is the bias-correction parameter that can be estimated as

$$\hat{k}_0 = \Phi^{-1}\left(\frac{\sharp\{\hat{d}^* < \hat{d}\}}{B}\right)$$

where $\sharp\{\hat{d}^* < \hat{d}\}$ represents the number of bootstrap estimates $\hat{d}^*$ smaller than $\hat{d}$. This method improves the performance of the percentile method in non symmetric situations. However if the distribution of $\hat{d}^*$ is symmetric about $\hat{d}$, then $k_0 = 0$ and P and BC confidence intervals are the same.

4) The accelerated bias-corrected (BC_a) percentile method of [25] accounts also for some unknown monotone transformation $\phi = g(d)$, some unknown bias factor k_0 and some unknown skewness or acceleration correction factor s so that

$$\hat{\phi} - \phi \sim N(\phi - k_0\sigma(\phi), \sigma^2(\phi))$$

where, now, instead of considering a constant σ, we have the possibility of changing with ϕ, $\sigma(\phi) = 1 + s\phi$, such that bias and variance can depend on it. The BC_a confidence interval is defined as

$$I_{(1-\alpha)} = \left(\hat{d}^*_{((B+1)(\frac{\hat{\alpha}}{2}))} \quad ; \quad \hat{d}^*_{((B+1)(1-\frac{\hat{\alpha}}{2}))}\right),$$

where

$$\frac{\tilde{\alpha}}{2} = \Phi\left(k_0 + \frac{k_0 + z_{\frac{\alpha}{2}}}{1 - s\left(k_0 + z_{\frac{\alpha}{2}}\right)}\right)$$

and

$$1 - \frac{\tilde{\alpha}}{2} = \Phi\left(k_0 + \frac{k_0 + z_{1-\frac{\alpha}{2}}}{1 - s\left(k_0 + z_{1-\frac{\alpha}{2}}\right)}\right).$$

If the shape, or skewness, of the probability distribution of $\hat{d}$ does not change when d varies, the acceleration parameter s takes a value of zero and this confidence interval will be equal to the BC confidence interval. In addition, if the k_0 parameter is zero we are in the basic percentile case.

Although the bias constant k_0 is estimated as in the BC, the acceleration parameter s is not easy to estimate and different ways have been proposed. In this paper, we use the estimate of [33], adequate for regression models like (2)

$$\hat{s} = -\frac{1}{6m^{7/2}\hat{\sigma}_u^3 S_{xx}^{3/2}}\sum_{j=1}^{m}\hat{u}_j^3\sum_{j=1}^{m}X_j^3$$

where $\hat{\sigma}_u^2 = \frac{\sum_{j=1}^{m}\hat{u}_j^2}{m}$ and $S_{xx} = \sum_{j=1}^{m}(X_j - \overline{X})^2$, $\overline{X} = \frac{\sum_{j=1}^{m}X_j}{m}$.

5) The percentile-t or bootstrap-t method ([24]) is based on a given studentized pivot, in this case: $t = \frac{\hat{d}-d}{\hat{se}(\hat{d})}$. Applying the percentile method to the t statistic the estimates of the required percentiles are obtained. The resulting $(1-\alpha)$ confidence interval is

$$I = \left(\hat{d} - \hat{se}(\hat{d})t^*_{((B+1)(1-\frac{\alpha}{2}))}; \hat{d} - \hat{se}(\hat{d})t^*_{((B+1)(\frac{\alpha}{2}))}\right)$$

where the bootstrapped t statistics are $t^* = \frac{\hat{d}^*-\hat{d}}{\hat{se}^*(\hat{d}^*)}$. A bias correction is implicit in the definition of the t^* statistic. In a general context, the main disadvantage of this method is the necessity of an estimate of the

standard error of the parameter. However, in this case, the estimate of the standard error of the parameter is easily obtained as

$$\hat{se}(\hat{d}) = \left(\frac{\hat{\sigma}_u^2}{\sum_{j=1}^m (X_j - \overline{X})^2} \right)^{1/2}.$$

For a more detailed description of these and others bootstrap resample procedures and confidence intervals see, for example, [34], [33] or [35].

III. MONTE CARLO SIMULATION STUDY

The performance of the bootstrap in LPE based confidence intervals is assessed in three different type of models:

- **Model 1**: $(1 - 0.9L)(1 - L)^d x_t = \varepsilon_{1t}$
- **Model 2**: $(1 - 0.3L)(1 - L)^d x_t = \varepsilon_{1t}$
- **Model 3**: $x_t = \pi^{-1}(1 - L)^{-d} \varepsilon_{1t} + \varepsilon_{2t}$

where L is the lag operator ($Lx_t = x_{t-1}$), ε_{1t} and ε_{2t} are independent standard normal series and $d \in (0, 0.4, 0.8)$. For $d = 0$ the series are short memory such that the spectral density function is positive, bounded and continuos at every frequency. The value $d = 0.4$ corresponds to a stationary long memory series with a spectral density diverging at the origin. For $d = 0.8$ the series is nonstationary and mean reverting. Note that in this case the asymptotic distribution in (3) does not apply and the LPE, although consistent, has a nonnormal limit distribution that depends on d (non pivotal).

The first two models belong to the ARFIMA class and have a spectral (pseudospectral in the nonstationary case) density function

$$f(\lambda) = \frac{1}{2\pi} \frac{[2\sin\left(\frac{\lambda}{2}\right)]^{-2d}}{|1 - \phi e^{-i\lambda}|^2} \sim \frac{1}{2\pi(1 - \phi)^2} |\lambda|^{-2d} \quad \text{as} \ \ \lambda \to 0$$

for $\phi = 0.9, 0.3$ in Models 1 and 2 respectively. Both include an $AR(1)$ short memory component with moderate (Model 2) and high (Model 1) dependence that gives rise to a bias in the LPE if a large bandwidth is used, especially in Model 1. Model 3 is a long memory series perturbed by an added noise with a long run noise to signal ratio π^2. These models have gained recently great interest in the econometric modeling since it encompasses many economic and financial series (e.g. [36] and [37]). In Models 1 and 2 the asymptotic bias of the LPE does not depend on d. However, in Model 3 the asymptotic bias is a function of d ([38]) such that the CBC should perform worse. The bias in this class of models is also very high if a large bandwidth is used.

Since the bootstrap is essentially beneficial with a low sample size, we only consider $n = 128$, which is comparable to the number of observations in many economic series as those analyzed in the next section. For each model three bandwidths are considered $m = 5$, 10 and 20. For the local bootstrap we use different resampling widths $k_m = 2$ (for $m = 5$), $k_m = 2, 4$ (for $m = 10$) and $k_m = 2, 4, 8$ (for $m = 20$) and the block bootstrap is analyzed with blocks of length 5 (for $m = 10$) and 5, 10 (for $m = 20$), which are similar to the lengths of the blocks in the local bootstrap. Since the results are very sensitive to the choice of the bandwidth

we also consider the plug-in optimal bandwidth proposed by [10] and defined as $m^* = \hat{C} n^{4/5}$ for

$$\hat{C} = \left(\frac{27}{128\pi^2} \right)^{1/5} \hat{K}^{-2/5}$$

where $\hat{K}$ is obtained as the third coefficient in an ordinary linear regression of $\log I_j$ on $(1, -2\log\lambda_j, \lambda_j^2/2)$ for $j = 1, 2, ..., An^\delta$, with $4/5 < \delta < 1$ and A an arbitrary constant. Following [10], we use $\delta = 6/7$ and $A = 0.25$. Note that this optimal bandwidth is only consistent for Models 1 and 2 in the stationary region, but we use it also in the rest of cases for illustrative purposes. In practice m^* is obtained as the median of the optimal bandwidths in 1000 series generated in each model. The use of the median instead of the mean avoids the distorting effect of extreme cases. We get in this way $m^* = 12$, 13 and 12 for Models 1, 2 and 3 respectively. The optimal bandwidth is quite robust to different values of d (for large d the optimal bandwidth differs at most one unity from the corresponding optimal bandwidth for low d) and we use the same m^* for all d. Due to the poor performance of the RBB, only the RB and RLB (with $k_m = 2$, 4 and 6) are analyzed for m^*. The number of bootstraps is $B = 999$ which is large enough for the calculus of confidence intervals ([34]). The number of simulations is 1000.

TABLE I

LPE 95% CONFIDENCE INTERVALS COVERAGE FOR $m = 5$

	AR(1)	p=0.9		AR(1)	p=0.3		SPN		
	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8
Asym									
	72.8	72.8	76.9	85.9	85.5	86	86.3	79.4	85
	1.790	1.756	1.560	1.766	1.761	1.704	1.814	1.770	1.717
RB									
P	70.5	71.4	75.2	84.7	84.9	85.4	85.3	78.1	83.9
	1.734	1.702	1.520	1.714	1.713	1.657	1.758	1.718	1.663
CBC	71	72.3	75.6	84.7	85.3	85.8	85.4	78.5	84.7
	1.769	1.737	1.553	1.749	1.749	1.690	1.792	1.752	1.697
BC	70.9	70.7	75.1	83.9	84.9	85.3	85	77.8	83.5
	1.725	1.693	1.511	1.704	1.706	1.648	1.749	1.709	1.653
BCa	70.9	71.4	75.3	84.3	84.7	85.4	85.2	77.8	84.1
	1.740	1.706	1.525	1.719	1.717	1.662	1.763	1.722	1.670
b-t	91.8	88.5	90	95.2	94.8	95.3	95.8	89.8	94.2
	2.849	2.802	2.481	2.813	2.806	2.709	2.890	2.818	2.728
RLB(2)									
P	66	68.6	73.6	90.5	90.1	91.3	90.6	84.7	90.2
	1.668	1.636	1.443	1.631	1.631	1.579	1.657	1.638	1.600
CBC	70.6	70.7	72.4	89.2	89.2	89.9	90	84.2	90
	1.813	1.781	1.574	1.783	1.789	1.727	1.815	1.790	1.739
BC	62.5	64	64.2	80	80.6	79.3	79.9	75.2	79.9
	1.561	1.524	1.355	1.536	1.518	1.463	1.548	1.519	1.487
BCa	61.5	63.4	64.4	80.8	80	79.3	79.5	75.8	79.6
	1.559	1.527	1.356	1.537	1.518	1.465	1.552	1.524	1.488
b-t	89.4	87.3	89.1	95.5	94.4	94.6	95.7	89.6	93.3
	2.714	2.665	2.383	2.698	2.668	2.574	2.763	2.686	2.609

SPN stands for the Signal Plus Noise process in Model 3. RB and RLB(k_m) denote the residual bootstrap and the residual local bootstrap with resampling width k_m. In each cell the first number is coverage frequency in percentages over 1000 simulations and the number below it is the average length of the interval. Asym, P, CBC, BC, BCa and b-t denote the confidence intervals based on the asymptotic distribution, the basic percentile, the constant bias correcting percentile, the bias corrected, the accelerated bias corrected and the bootstrap-t respectively.

Tables I-IV show the coverage frequencies in percentage (first number in each cell) and the average length of the interval (under the frequencies) of confidence intervals for a 95% nominal confidence level over 1000 Monte Carlo replications with bandwidths $m = 5$, 10, 20 and m^*. The following conclusions can be extracted:

- The bootstrap confidence intervals clearly beats the asymptotic distribution with better coverage frequencies.

TABLE II

LPE 95% CONFIDENCE INTERVALS COVERAGE FOR $m = 10$

	AR(1) p=0.9			AR(1) p=0.3			SPN		
	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8
Asym									
	34.7	38.5	47	92.2	89.6	91	89.9	80.2	82.3
	1.097	1.068	0.907	1.085	1.097	1.053	1.086	1.099	1.084
RB									
P	37.7	40.5	49	92.4	90.1	90.6	90.5	79.8	81.4
	1.102	1.073	0.916	1.086	1.100	1.052	1.090	1.100	1.089
CBC	37.1	40.4	48.9	92.5	90.5	91	90.7	79.9	80.8
	1.104	1.074	0.917	1.088	1.102	1.054	1.092	1.102	1.090
BC	39.1	43.5	51.5	92.5	90.5	91.3	90	78.4	80
	1.102	1.073	0.916	1.087	1.101	1.053	1.090	1.099	1.090
BCa	39	43.4	51.3	92.5	90.4	91.3	90	78.5	80.1
	1.102	1.073	0.915	1.086	1.101	1.053	1.090	1.099	1.089
b-t	50	53.6	62.6	95.2	94.4	95.2	93.7	84.4	85.7
	1.297	1.263	1.073	1.285	1.300	1.246	1.286	1.304	1.283
RLB(2)									
P	25.8	28.8	36.7	88.9	85.6	87	87.3	75.3	76.5
	0.954	0.929	0.792	0.929	0.948	0.895	0.937	0.939	0.915
CBC	26.2	30.7	38	87.6	82.9	84.8	84.9	75.1	75.6
	0.967	0.940	0.803	0.942	0.960	0.906	0.952	0.953	0.927
BC	26.1	30.6	38.1	83.7	79.8	81.7	80.8	70.8	71.3
	0.947	0.920	0.793	0.924	0.942	0.892	0.932	0.935	0.908
BCa	26	30.6	38.1	83.7	79.8	81.6	80.8	70.9	71.4
	0.947	0.920	0.794	0.923	0.941	0.892	0.932	0.935	0.907
b-t	44.5	48.3	57.2	93.4	92.7	93.6	93.5	83.4	83.5
	1.275	1.229	1.086	1.236	1.261	1.190	1.259	1.255	1.227
RLB(4)									
P	28.8	32.8	42	93.6	91.8	93	92.8	81.5	81.6
	1.044	1.018	0.862	1.018	1.040	0.990	1.021	1.032	1.018
CBC	33.7	37.3	44.3	90.1	87.2	88.6	88.1	78.4	80.4
	1.053	1.025	0.867	1.032	1.051	1.000	1.035	1.044	1.027
BC	34.3	38.4	44.1	86.1	82.6	86.1	82.3	74.8	76.4
	1.030	1.006	0.856	1.005	1.027	0.974	1.008	1.019	1.001
BCa	34.2	38.1	44	86.1	82.5	86	82.4	74.8	76.4
	1.030	1.006	0.856	1.004	1.027	0.974	1.008	1.018	1.000
b-t	47	52.8	64	97	96	97.2	96.4	86.6	86.4
	1.314	1.278	1.118	1.284	1.310	1.249	1.292	1.300	1.288
RBB(5)									
P	21	24.8	31.7	76	74.2	76.7	73.8	60.9	62.6
	0.783	0.761	0.646	0.752	0.773	0.741	0.750	0.759	0.770
CBC	21.8	25.1	31.2	76.2	74.6	75.5	75.3	63.2	64.3
	0.777	0.758	0.644	0.749	0.772	0.738	0.749	0.760	0.765
BC	20.8	24.3	30.6	73.9	71.7	73.9	72.7	59	60.5
	0.740	0.721	0.614	0.717	0.739	0.705	0.712	0.724	0.735
BCa	20.7	24.3	30.6	73.9	71.5	73.9	72.7	58.9	60.5
	0.740	0.722	0.614	0.717	0.739	0.705	0.712	0.725	0.735
b-t	28.5	30.5	38.9	79.6	77.2	79.6	76.5	63.9	64.4
	0.912	0.885	0.757	0.871	0.890	0.849	0.865	0.865	0.893

SPN stands for the Signal Plus Noise process in Model 3. RB, RLB(k_m) and RBB(l) denote the residual bootstrap, the residual local bootstrap with resampling width k_m and the residual block bootstrap with block length l. In each cell the first number is coverage frequency in percentages over 1000 simulations and the number below it is the average length of the interval. Asym, P, CBC, BC, BCa and b-t denote the confidence intervals based on the asymptotic distribution, the basic percentile, the constant bias correcting percentile, the bias corrected, the accelerated bias corrected and the bootstrap-t respectively.

TABLE III

LPE 95% CONFIDENCE INTERVALS COVERAGE FOR $m = 20$

	AR(1) p=0.9			AR(1) p=0.3			SPN		
	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8
Asym									
	2.8	4	20.1	90.8	89.6	87.7	92.9	57.6	46.8
	0.696	0.678	0.522	0.687	0.690	0.676	0.694	0.693	0.689
RB									
P	3.6	5.6	22.9	92.7	91.1	89.4	93.1	55.1	44.1
	0.703	0.685	0.531	0.691	0.696	0.681	0.700	0.697	0.696
CBC	3.7	5.8	22.8	92.4	90.8	89.4	93.2	55.1	43.6
	0.703	0.685	0.531	0.691	0.696	0.681	0.700	0.697	0.696
BC	4.7	7.4	25.2	92.6	91	90.4	93.1	52.4	40.4
	0.705	0.687	0.532	0.693	0.698	0.684	0.702	0.699	0.698
BCa	4.7	7.4	25.2	92.6	91	90.4	93.1	52.4	40.4
	0.705	0.687	0.532	0.693	0.698	0.684	0.702	0.699	0.698
b-t	5.1	7.6	25.6	94.5	92.5	92	94.3	57.9	47
	0.749	0.729	0.561	0.738	0.742	0.727	0.746	0.744	0.741
RLB(2)									
P	4	6	22.2	83.4	81	79	88.7	48	34.9
	0.625	0.610	0.492	0.594	0.598	0.576	0.595	0.596	0.597
CBC	4.2	6.4	22.6	83.3	80.8	77.7	88.2	48	35.4
	0.629	0.614	0.496	0.598	0.602	0.578	0.600	0.600	0.598
BC	4.5	6.6	23.3	81	78.6	76	86.3	44.4	31.8
	0.625	0.613	0.496	0.597	0.601	0.579	0.599	0.599	0.600
BCa	4.5	6.6	23.3	81	78.6	76	86.3	44.4	31.8
	0.625	0.613	0.496	0.597	0.601	0.579	0.599	0.599	0.600
b-t	5.7	7.7	25.5	90.1	87.9	84.7	92.7	53.5	43.1
	0.712	0.698	0.584	0.672	0.681	0.653	0.676	0.673	0.678
RLB(4)									
P	5.2	6.8	25.4	87.5	84.7	83.2	92.3	53.7	37.5
	0.677	0.656	0.531	0.644	0.650	0.626	0.647	0.648	0.650
CBC	5.6	7.7	24.2	87.2	85	82.1	90.7	53.6	39.5
	0.677	0.659	0.532	0.646	0.651	0.627	0.649	0.650	0.648
BC	7.3	9.2	26.2	85.1	82.2	79.3	87.9	50.2	38.8
	0.676	0.657	0.532	0.645	0.650	0.625	0.649	0.647	0.644
BCa	7.3	9.2	26.2	85.1	82.2	79.3	87.9	50.2	38.8
	0.676	0.657	0.532	0.645	0.650	0.625	0.648	0.647	0.644
b-t	6.2	9.1	31.1	92.9	92.1	89.5	95.1	58.5	45.2
	0.764	0.746	0.621	0.723	0.733	0.705	0.729	0.727	0.728
RLB(8)									
P	4.2	6	26.4	90.1	88.8	86.5	95.1	54.8	35.9
	0.698	0.676	0.543	0.677	0.681	0.659	0.680	0.680	0.683
CBC	6.2	8.1	25.7	88.8	86.4	84.4	91.9	56.5	44.5
	0.695	0.672	0.541	0.675	0.679	0.660	0.681	0.680	0.679
BC	10.1	11.4	27.5	86.3	84.8	81.5	88.4	52.9	44.6
	0.700	0.680	0.542	0.679	0.683	0.659	0.678	0.678	0.665
BCa	10.1	11.4	27.5	86.3	84.8	81.5	88.4	52.9	44.6
	0.700	0.680	0.542	0.679	0.683	0.659	0.678	0.678	0.665
b-t	4.7	7.6	32	94.2	92.9	92.8	96.7	58.9	38.3
	0.763	0.747	0.611	0.739	0.746	0.724	0.746	0.745	0.745
RBB(5)									
P	2.1	3.4	16.1	82.5	81.4	78.2	86.1	41.8	32.9
	0.568	0.560	0.437	0.562	0.557	0.549	0.564	0.562	0.574
CBC	2.4	3.2	17	82.4	80.4	78.6	86	41.7	33.2
	0.567	0.559	0.437	0.560	0.557	0.550	0.564	0.562	0.573
BC	2.9	3.8	17.9	82.4	81	79	85.7	40.9	31.8
	0.568	0.559	0.438	0.562	0.557	0.549	0.564	0.562	0.574
BCa	2.9	3.8	17.9	82.4	81	79	85.7	40.9	31.8
	0.568	0.559	0.438	0.562	0.557	0.549	0.564	0.562	0.574
b-t	3.2	4.7	20.5	85	83.8	80.6	87.7	44.3	35.3
	0.612	0.603	0.476	0.599	0.598	0.585	0.597	0.598	0.615
RBB(10)									
P	1.9	2.1	13.5	74.1	71.1	68.6	77.8	33.9	21.5
	0.481	0.474	0.381	0.471	0.472	0.462	0.474	0.476	0.485
CBC	2	2.7	14.8	75.7	72.7	70.6	79.9	35.9	27.8
	0.486	0.481	0.387	0.476	0.478	0.467	0.480	0.482	0.490
BC	1.8	2.4	13.7	74.3	69.5	67.9	76.9	31.5	22.9
	0.462	0.456	0.366	0.455	0.453	0.446	0.456	0.460	0.461
BCa	1.8	2.4	13.7	74.3	69.5	67.9	76.9	31.5	22.9
	0.462	0.456	0.366	0.455	0.453	0.446	0.456	0.460	0.461
b-t	2.2	2.7	18.1	77.7	73.1	71.4	79.7	34.2	22.9
	0.512	0.507	0.419	0.500	0.501	0.486	0.496	0.502	0.516

The block bootstrap performs worse than the local and basic residual based bootstraps for any block length. We should not forget that the RBB is designed for time series to maintain the original time dependence in the bootstrapped resamples whereas the regression model (2) is defined in the frequency domain and there is not any time or serial dependence on it.

- The different bias correction techniques are only slightly beneficial in Model 1 with a large bandwidth where the bias of the LPE is especially large. The BC and BCa give better results in terms of coverage frequencies than the CBC and the basic P. However the bootstrap-t generally overcomes all the others even in these highly biased situations.
- The choice of the bandwidth is crucial. The best results are obtained with a low bandwidth when there is a highly dependent short memory component (Model 1) or in the presence of an additive noise (Model 3 with $d > 0$) and with a larger bandwidth for models with low dependent short memory component (Model 2) or in the white noise case (Model 3 with $d = 0$). Especially harmful is the use of a large bandwidth ($m = 20$) in Model 1, with very low coverage frequencies.

- The performance of the local bootstrap depends on the choice of the resampling width k_m. Reference [20] suggested a value of $k_m = 1$ or 2. These values can be too small when the short memory component is of lesser importance and a larger k_m gives better results in these cases. An excessively large k_m can however be harmful in those cases where the estimator is subject to a large bias as in Models 1 and 3 ($d > 0$) with a large bandwidth. Thus a larger k_m should be chosen when the bias component is low. In this situation a large bandwidth should also be used. Then, as a rule of thumb, a larger k_m can be chosen when the optimal bandwidth m^* is large and a low m^* should be accompanied by a small k_m. For

TABLE IV

LPE 95% CONFIDENCE INTERVALS COVERAGE FOR OPTIMAL BANDWIDTH m^*

	AR(1)	p=0.9		AR(1)	p=0.3		SPN		
	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8
Asym									
	22.6	25.4	39.6	91.5	92.4	89.9	92.3	74.2	74.5
	0.971	0.946	0.762	0.904	0.908	0.883	0.904	0.905	0.928
RB									
P	26.9	28.6	41.8	91.9	92.4	90.7	92.5	73	73.8
	0.977	0.952	0.773	0.908	0.914	0.887	0.910	0.909	0.937
CBC	27.1	28.6	41.5	92.1	93	91	92.4	72.9	74
	0.978	0.952	0.774	0.909	0.914	0.887	0.911	0.909	0.937
BC	29.3	31.1	43.6	91.8	92.7	90.9	91.9	71.1	71.1
	0.979	0.952	0.774	0.911	0.916	0.889	0.911	0.910	0.939
BCa	29.2	30.9	43.6	91.9	92.6	91	91.9	71.2	71.2
	0.978	0.951	0.774	0.911	0.916	0.888	0.911	0.910	0.939
b-t	35.2	38.6	51.3	94.3	95.3	93.7	94.9	76.8	77.5
	1.109	1.077	0.871	1.018	1.025	0.995	1.019	1.020	1.049
RLB(2)									
P	17.2	20.6	35	86.5	86.7	83.2	88.7	66.7	67.3
	0.840	0.826	0.675	0.785	0.783	0.757	0.785	0.764	0.802
CBC	17.7	21	34.9	83.9	86.2	82	87	66.4	66.8
	0.847	0.835	0.683	0.793	0.790	0.765	0.793	0.770	0.809
BC	17.8	21.1	35.1	81.3	82.8	79.8	83.8	63	62
	0.838	0.826	0.680	0.788	0.787	0.758	0.786	0.766	0.804
BCa	17.8	21.1	35.1	81.3	82.8	79.8	83.8	63	62
	0.838	0.826	0.680	0.788	0.787	0.758	0.786	0.766	0.804
b-t	31.1	32.9	47.4	92.5	92.8	90.3	92.5	73.3	73.9
	1.059	1.040	0.886	0.975	0.971	0.939	0.972	0.943	1.001
RLB(4)									
P	19.5	23.7	39.8	91.1	91.5	88.3	92.7	72.6	70.8
	0.919	0.906	0.736	0.854	0.856	0.827	0.858	0.842	0.874
CBC	21.8	24.8	38	87.8	89.8	85.9	90.2	71.9	71.6
	0.922	0.910	0.738	0.858	0.862	0.832	0.863	0.847	0.878
BC	23.2	27.7	40.7	83.9	85.2	82.2	86.8	68	67.8
	0.910	0.897	0.735	0.850	0.853	0.822	0.852	0.838	0.864
BCa	23.1	27.7	40.7	83.8	85.2	82.1	86.8	68	67.9
	0.910	0.897	0.734	0.850	0.853	0.822	0.852	0.837	0.864
b-t	34.4	37.5	53.4	95.2	96	93.6	95.5	78.2	77.8
	1.117	1.101	0.927	1.027	1.030	0.993	1.029	1.009	1.054
RLB(6)									
P	19.9	24.5	40.5	92.7	94.6	91.1	94.3	75.7	70.8
	0.949	0.933	0.755	0.883	0.884	0.859	0.883	0.876	0.903
CBC	25.3	28.1	40.2	88.3	90.6	86.4	90.5	73.2	71.9
	0.954	0.938	0.758	0.888	0.888	0.862	0.888	0.881	0.907
BC	28.1	31	41.9	84.4	86.2	83	86.8	69.2	69.9
	0.938	0.924	0.752	0.878	0.878	0.852	0.876	0.867	0.886
BCa	27.8	31	41.8	84.4	86.2	83	86.8	69.2	69.9
	0.938	0.924	0.752	0.878	0.879	0.852	0.876	0.867	0.886
b-t	31.5	36.7	55.1	96.5	96.8	95	96.5	79.3	76.4
	1.107	1.093	0.910	1.021	1.027	0.993	1.024	1.017	1.055

SPN stands for the Signal Plus Noise process in Model 3. RB and RLB(k_m) denote the residual bootstrap and the residual local bootstrap with resampling width k_m. In each cell the first number is coverage frequency in percentages over 1000 simulations and the number below it is the average length of the interval. Asym, P, CBC, BC, BCa and b-t denote the confidence intervals based on the asymptotic distribution, the basic percentile, the constant bias correcting percentile, the bias corrected, the accelerated bias corrected and the bootstrap-t respectively.

TABLE V

BEST RESULTS FOR COVERAGE FREQUENCIES WITH ASYMPTOTIC DISTRIBUTION AND BOOTSTRAP-t

	AR(1)	p=0.9		AR(1)	p=0.3		SPN		
	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8	d=0	d=0.4	d=0.8
Asym									
m	5	5	5	10	m^*	10	20	10	5
cov	72.8	72.8	76.9	92.2	92.4	91	92.9	80.2	85
ampl	1.790	1.756	1.560	1.085	0.908	1.053	0.694	1.099	1.717
RB									
m	5	5	5	10	5	10	m^*	5	5
cov	91.8	88.5	90	95.2	94.8	95.2	94.9	89.8	94.2
ampl	2.849	2.802	2.481	1.285	2.806	1.246	1.019	2.818	2.728
RLB									
m	5	5	5	m^*	5	m^*	20	5	5
k_m	2	2	2	4	2	6	4	2	2
cov	89.4	87.3	89.1	95.2	94.4	95	95.1	89.6	93.3
ampl	2.714	2.665	2.383	1.027	2.668	0.993	0.729	2.686	2.609

SPN stands for the Signal Plus Noise process in Model 3. Asym, RB and RLB denote the confidence intervals based on the asymptotic distribution, the residual bootstrap-t and the residual local bootstrap-t with resampling width k_m.

the optimal bandwidths in table IV we found that a value around $k_m = 4$ is adequate.

- The optimal bandwidth of [10] is obtained by minimizing an asymptotic approximation of the mean squared error of the LPE but need not give the best coverage frequencies. This is the case in Models 1 and 3 where the bias component is especially large and better coverage frequencies are achieved with a lower bandwidth than the optimal m^* in Table IV.
- Overall the basic and local residual bootstrap-t give the best performances. Table V displays the outcome obtained with the asymptotic distribution and the RB and RLB bootstrap-t with the values of m and the resampling width k_m that give the best coverage frequencies. Note that the optimal bandwidth m^* does not generally correspond to the best performance. The improvements of the bootstrap over the asymptotic distribution are significant. The local bootstrap gives similar coverages to the basic residual bootstrap but with narrower intervals.

Although in the LPE context bootstrap is specially beneficial with low sample sizes, it is also interesting to analyze how its performance changes as sample size increases, in order to learn about the asymptotic behaviour. Fig. 1 shows the coverage frequencies, in percentages over 1000 replications, of 95% confidence intervals based on the asymptotic distribution and the basic and local residual bootstrap-t in Model 1 with sample sizes $n = 64, 128, 256, 512$ and 2048 and bandwidths of the closest integer number larger than $n^{0.4}$, $m = 6, 7, 10, 13$ and 22 respectively. These bandwidths are lower than the values given in the proposal by [10] -around 7, 12, 20, 31 and 38- and are adequate for purposes of comparison. The resampling width is $k_m = 2$ for $n = 64, 128$, $k_m = 4$ for $n = 256, 512$ and $k_m = 8$ for $n = 2048$. In all cases the coverage approaches the nominal 95% confidence level as the sample size increases, with the bootstrap coverages always closer to the nominal confidence level. The lengths of the intervals (not reported but available upon request) decrease as expected with sample size and, as before, are larger for the bootstrap proposals, indicating that confidence intervals wider than the asymptotic intervals are required to approximate the nominal confidence level.

IV. CONCLUSION

This paper shows the improvements of some residuals based nonparametric bootstrap strategies over the asymptotic distribution of the LPE in the construction of confidence intervals with a small sample size. It is noteworthy the crucial role played by the choice of the bandwidth. The coverage frequencies and length of the confidence interval vary significantly with m and an appropriate m should be selected as a first step. Whereas the performance of the RBB is quite poor, the RB and the RLB bootstrap-t seems to perform well with an appropriate selection of the resampling width. We have proposed a rule of thumb for approximate selection of the resampling width of the RLB linked to the optimal bandwidth estimation of [10], a high optimal bandwidth requires a high
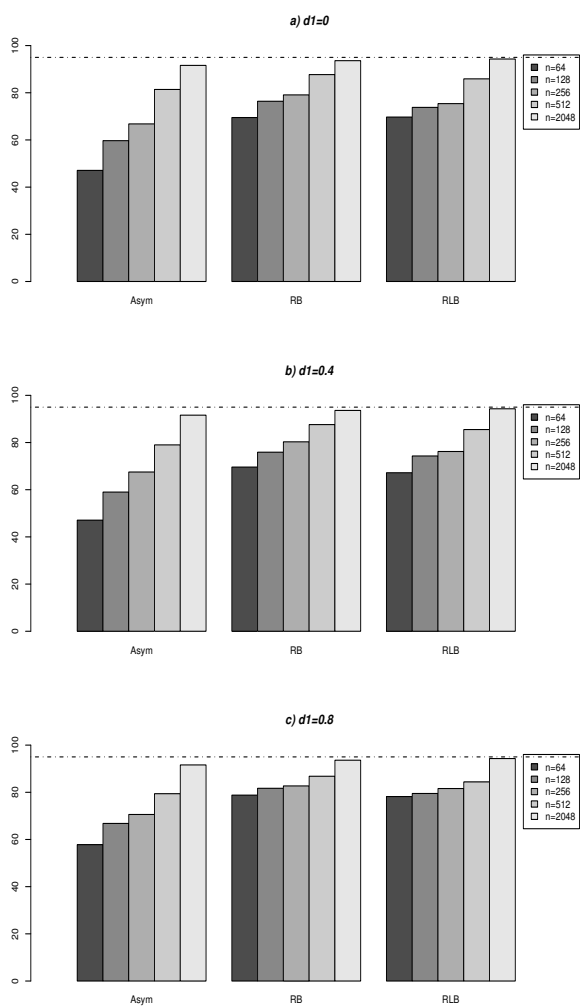
Fig. 1. Coverage frequencies with different sample sizes

resampling width. The advantage of using the RLB over the RB is the reduction of the length of the confidence intervals without significantly affecting the coverage.

Our analysis has focused on the basic LPE, which is the most popular method of estimation of the memory parameter. There have been recently further refinements either in a linear regression setup or in a nonlinear regression approach. For example [39] proposed a bias reduced LPE by including linearly extra regressors that account for the weak dependent components. This extension can be applied to ARFIMA models such as Models 1 and 2 in our Monte Carlo, but not to Model 3. In the perturbed long memory case, a somewhat similar refinement has been considered by [38] and [36] who proposed to include extra regressors to account for the added noise. The resulting regression model is non linear in this case. The bootstraps here proposed could be useful also in these cases since they are nonparametric and based on residuals.

Further analysis in these setups is however required.

REFERENCES

[1]   Velasco, C., 1999. Non stationary log-periodogram regression. *J. Econometrics* 91, 325-371.
[2]   Geweke, J. and Porter-Hudak, S., 1983. The estimation and application of long-memory time series models. *J. Time Ser. Anal.* 4, 221-238.
[3]   Robinson, P.M., 1995. Log-periodogram regression of time series with long-range dependence. *Ann. Statist.* 23, 1048-1072.
[4]   Hurvich, C.M., Deo, R. and Brodsky, J., 1998. The mean squared error of Geweke and Porter-Hudak's estimator of the memory parameter in a long-memory time series. *J. Time Ser. Anal.* 19, 19-46.
[5]   Phillips, P.C.B., 2007. Unit root log periodogram regression. *J. Econometrics* 138, 104-124.
[6]   Kim, C.S. and Phillips, P.C.B., 2006. Log periodogram regression: The nonstationary case. Cowles Foundation Discussion Paper No. 1587.
[7]   Phillips, P.C.B. and Shimotsu, K., 2004. Local Whittle estimation in nonstationary and unit root cases. *Ann. Statist.* 32, 656-692.
[8]   Arteche, J., 2004, Gaussian Semiparametric Estimation in Long Memory in Stochastic Volatility and Signal Plus Noise Models. *J. Econometrics* 119, 131-154.
[9]   Giraitis, L., Robinson, P.M. and Samarov, A., 2000. Adaptive semiparametric estimation of the memory parameter. *J. Multiv. Anal.* 72, 183-207.
[10] Hurvich, C.M., and Deo, R.S., 1999. Plug-in selection of the number of frequencies in regression estimates of the memory parameter of a long-memory time series. *J. Time Ser. Anal.* 20, 331-341.
[11] Hassler, U. and Wolters, J., 1995. Long memory in inflation rates: International evidence. *J. Business Econ. Stat.* 13, 37-45.
[12] Diebold, F.X. and Rudebush, G., 1989. Long memory and persistence in aggregate output. *J. Monet. Econ.* 24, 189-209.
[13] Diebold, F.X. and Rudebush, G., 1991. Is consumption too smooth: Long memory and the Deaton paradox. *Rev. Econ. Statist.* 73, 1-9.
[14] Sowell, F., 1992a. Maximum likelihood estimation of stationary univariate fractionally integrated time series models. *J. Econometrics* 53, 165-188.
[15] Arteche, J. and Robinson, P.M., 2000. Semiparametric inference in seasonal and cyclical long memory processes. *J. Time Ser. Anal.* 21, 1-27.
[16] Sowell, F., 1992b. Modeling long-run behaviour with the fractional ARIMA model. *J. Monet. Econ.* 29, 277-302.
[17] Andrews, D.W.K., Lieberman, O. and Marmer, V., 2006. Higher-order improvements of the parametric bootstrap for long-memory time series. *J. Econometrics* 133, 673-702.
[18] Andersson, M.K. and Gredenhoff, M.P., 1998. Robust testing for fractional integration using the bootstrap. Working Paper Series in Economics and Finance 218, Stockholm School of Economics, Sweden.
[19] Hidalgo, J., 2003. An alternative bootstrap to moving blocks for time series regression models. *J. Econometrics* 117, 369-399.
[20] Silva, E.M., Franco, G.C., Reisen, V.A. and Cruz, F.R.B., 2006. Local bootstrap approaches for fractional differential parameter estimation in ARFIMA models. *Comput. Statist. Data Anal.* 51, 1002-1011.
[21] Arteche, J. and Orbe, J., 2005. Bootstrapping the log-periodogram regression. *Econ. Letters* 86, 70-85.
[22] Agiakloglou, C., Newbold, P. and Wohar, M., 1993. Bias in an estimator of the fractional difference parameter. *J. Time Ser. Anal.* 14, 235-246.
[23] Paparoditis, E. and Politis, D., 1999. The local bootstrap for periodogram statistics. *J. Time Ser. Anal.* 20, 193-222.
[24] Efron, B., 1982. The jackknife, the bootstrap, and other resampling plans. Volume 38 of CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM.
[25] Efron, B., 1987. Better bootstrap confidence intervals. *J. Amer. Statistical Assoc.* 82, 171-200.
[26] Mackinnon, J.G. and Smith, A.A., 1998. Approximate bias correction in econometrics. *J. Econometrics* 85, 205-230.

[27] Efron, B., 1979. Bootstrap methods: Another look at the jackknife. *Ann. Statist.* 7, 1-26.

[28] Davidson, R. and MacKinnon, J.G., 2006. Bootstrap methods in econometrics. In: Mills, T.C. and Patterson, K.D. (eds.), *Palgrave Handbooks of Econometrics: Volume 1, Econometric Theory*. Palgrave Macmillan, 812-838.

[29] Künsch, H.R., 1989. The jackknife and the bootstrap for general stationary observations. *Ann. Statist.* 17, 1217-1261.

[30] Liu, R.Y. and Singh, K., 1992. Moving blocks jackknife and bootstrap capture weak dependence. In: LePage, R. and Billard, L.(Eds.) *Exploring the limits of bootstrap*. Wiley, New York, 225-248.

[31] Lahiri, S.N., 1999. Theoretical comparisons of block bootstrap methods. *Ann. Statist.* 27, 386-404.

[32] Kilian, L., 1998. Small sample confidence intervals for impulse response functions. *Rev. Econ. Statist.* 80, 218-230.

[33] Shao, J. and Tu, D., 1995. The jackknife and bootstrap. Springer Verlag: New York.

[34] Efron, B. and Tibshirani, R.J., 1993. An introduction to the bootstrap. Chapman and Hall: New York.

[35] Davison, A.C. and Hinkley, D.V., 1997. Bootstrap methods and their application. Cambridge University Press: Cambridge.

[36] Arteche, J., 2006. Semiparametric estimation in perturbed long memory series. *Comput. Statist. Data Anal.* 51, 2118-2141.

[37] Hurvich, C.M., Moulines, E. and Soulier, P., 2005. Estimating long memory in volatility. *Econometrica* 73, 1283-1328.

[38] Sun, Y. and Phillips, P.C.B., 2003. Nonlinear log-periodogram regression for perturbed fractional processes. *J. Econometrics* 115, 355-389.

[39] Andrews, D.W.K. and Guggenberger, P., 2003. A bias-reduced log-periodogram regression estimator for the long-memory parameter. *Econometrica* 71, 675-712.